lowest order approximation $\pi^{(0)}$. The quantity

$$\lim_{\Lambda^2\to\infty}\int_0^{\Lambda^2}\frac{d\kappa^2}{\kappa^2}\pi^{(0)}(\kappa^2)\to\eta\left[-\frac{5}{3}+2\ln\left(\frac{\Lambda}{m}\right)\right]$$

is logarithmically divergent and remains so when regulated by means of the standard conditions[2] $\sum_i C_i=\sum_i C_i m_i^2=0$, where $C_0=1$ and m_0 denotes the electron's mass. It should be noted, however, that these relations represent the minimum number of such conditions required to ensure the vanishing of the photon mass integral. There is nothing to prevent the adoption of the further condition $\sum_i C_i \ln(m_i/m_0)=P$, where P is zero or any selected finite number. The use of such an "extended" regularization procedure requires an extra regulating field and changes the values of the coefficients C_i, for $i\geqslant 1$ (they are now logarithmically divergent with the regulating masses), but in no way alters the results of the lowest order vacuum polarization calculation. Regularizing in this manner, the value of this renormalization integral is proportional to the arbitrary number P.

This discussion should, of course, not be regarded as exact in any way, or even correct; but rather as merely a kind of plausibility argument. Certainly, any precise statements concerning the effect of regularization on the magnitude of the renormalization constants must await the explicit demonstration of a consistent regularization procedure for the coupled Heisenberg fields.

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Moment of Inertia of Superfluid Many-Fermion Systems*

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The effects of possible superfluidity on the cranking moment of a large many-fermion system moving under periodic boundary conditions are investigated within the framework of the theory of superconductivity recently formulated by Bogolyubov. The Hamiltonian is initially subjected to Bogolyubov's general unitary transformation. The collective excitations of the fermions are then considered in the usual pair approximation; the appropriate cranking terms are linear in the boson pair operators. On performing a unitary transformation which transforms away these linear terms, one obtains an expression for the moment of inertia of the system which includes both the effects of possible superfluidity and collective excitation. This expression, by virtue of its being stationary with respect to arbitrary variations in the amplitude associated with the latter unitary transformation, is then utilized as a variational principle for the moment of inertia. For the normal state, the result previously obtained by the author, that the moment of inertia has the rigid value, is rederived in more compact form. For the superfluid state, one finds that collective excitations effect a marked increase in the superfluid moment at intermediate coupling strengths although the resulting moment is still quite small compared to the rigid value. In contrast to the normal state case, where particle-hole pairs play a major role, this increase is almost entirely due to excitations consisting of particle-pairs or hole-pairs. The precise magnitude of the apparent resonance in the moment produced by the *d*-wave part of the cranking interaction is dependent to some extent on the features of the particle-particle potential which leads to the superfluid state. Variational expressions for the moment are exhibited for both Yukawa and delta-function shell potentials. These results are identical in charged and neutral Fermi systems. A calculation of the cranking moment at finite temperatures is presented in an Appendix along with an interpretation of it in terms of Bardeen's two-fluid model of superconductivity.

I. INTRODUCTION

In a previous work by the author[1] (hereafter referred to as I), some of the consequences of particle-particle interaction on the cranking moment of a large many-fermion system moving under periodic boundary conditions were investigated. In particular, it was shown that the shift in the rigid moment of inertia due to collective excitations consisting in mass-renormalized particle-hole pairs could be obtained exactly without recourse to the usual perturbation theory.[2] Further, this shift was found to vanish exactly, although, initially, it had seemed likely that pair excitations would furnish the major contribution to such an interaction shift. One noted that stability requirements[3,4] in I re-

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[1] R. M. Rockmore, Phys. Rev. **116**, 469 (1959).

[2] The effect of interparticle forces in the lowest order of perturbation theory has recently been investigated by R. Amado and K. Brueckner, Phys. Rev. **115**, 778 (1959).

[3] K. Sawada and R. Rockmore, Phys. Rev. **116**, 1618 (1959); A. E. Glassgold, W. Heckrotte, and K. M. Watson, Ann. Phys. **6**, 1 (1959).

[4] N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau, Inc., New York, 1959).

stricted consideration to repulsive potentials or more generally to those potentials which forbade the existence of a superfluid state.[5] However, it has been demonstrated[6] that the superfluidity of a many-fermion system (which is characterized by the presence of an energy gap in the spectrum of elementary excitations) leads to important consequences for the thermodynamic and electromagnetic behavior of such a system at low temperatures. It seems reasonable then, to inquire whether the pair correlations responsible for superfluidity may also produce significant changes in the moment of inertia of a many-body fermion system and to calculate in some suitable model what these and the additional effects of collective excitation may be. The present paper is devoted to answering this question.

In what follows, we have suitably generalized the method utilized in I, that of the equivalent Hamiltonian,[7] so as to obtain an expression for the cranking moment of a system of fermions in a cubic box, which includes both the effects of superfluidity and collective excitation. In Sec. II of this paper we develop this generalization of I in terms of the formulation of the theory of superconductivity recently given by Bogolyubov.[4] The Hamiltonian for the collective excitations of the fermions derived in Sec. II is next subjected to a unitary transformation in Sec. III which yields an expression for the moment of inertia and an integral equation which determines this unitary transformation. Evaluation of the moment is dependent upon solution of the integral equation; the difficulties associated with the latter are obviated through the formulation of a variational principle for the cranking moment. In Secs. IV and V, the results for the normal state and superfluid state cases, respectively, are discussed in detail. Additional remarks appear in Sec. VI. Although our principal concern is with the cranking moment at the absolute zero of temperature, we also present in an Appendix a simple, but instructive, treatment of the superconducting moment of inertia at finite temperature.

II. DERIVATION OF THE HAMILTONIAN FOR COLLECTIVE EXCITATIONS

In the notation of second quantization, the dynamical system of fermions moving under periodic boundary conditions is characterized by a Hamiltonian of the form[8]

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1+\mathcal{H}_{\text{int}}, \tag{1a}$$

$$\mathcal{H}_0=\sum_{\mathbf{k}\sigma}\epsilon_{\mathbf{k}}c_{\mathbf{k}\sigma}{}^\dagger c_{\mathbf{k}\sigma}, \tag{1b}$$

$$\mathcal{H}_1=\frac{1}{2\Omega}\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}(\mathbf{k}\neq\mathbf{k}')}\sum_{\sigma\sigma'}\langle-\mathbf{k}'+\mathbf{q},\mathbf{k}'|v|-\mathbf{k}+\mathbf{q},\mathbf{k}\rangle \times c_{-\mathbf{k}'+\mathbf{q}\sigma}{}^\dagger c_{-\mathbf{k}+\mathbf{q}\sigma}c_{\mathbf{k}'\sigma'}{}^\dagger c_{\mathbf{k}\sigma'}, \tag{1c}$$

$$\mathcal{H}_{\text{int}}=-\omega\Big\{\sum_{r\neq0}\sum_{\mathbf{k}\sigma}L_{\mathbf{k}+\mathbf{r},\mathbf{k}}c_{\mathbf{k}+\mathbf{r}\sigma}{}^\dagger c_{\mathbf{k}\sigma} -\sum_{s\neq0}\sum_{\mathbf{k}\sigma}L_{\mathbf{k}+\mathbf{s},\mathbf{k}}c_{\mathbf{k}+\mathbf{s}\sigma}{}^\dagger c_{\mathbf{k}\sigma}\Big\}, \tag{1d}$$

where Ω is the quantization volume (a cubic box of side L), ω is the angular frequency for rotation about the z axis, and the energy, $\epsilon_{\mathbf{k}}$, measured relative to the Fermi surface, is given by

$$\epsilon_{\mathbf{k}}=\frac{k^2}{2M^*}-\frac{k_F{}^2}{2M^*}; \tag{2}$$

M^* and k_F denote the effective mass and Fermi momentum, respectively. The matrix element of the z component of the angular momentum operator L_Z is given by[9]

$$\langle\mathbf{k}+\mathbf{r}|L_Z|\mathbf{k}\rangle\equiv L_{\mathbf{k}+\mathbf{r},\mathbf{k}}=[(-1)^{\Delta l}/i](k_y/r), \tag{3a}$$

$$\langle\mathbf{k}+\mathbf{s}|L_Z|\mathbf{k}\rangle\equiv L_{\mathbf{k}+\mathbf{s},\mathbf{k}}=[(-1)^{\Delta m}/i](k_x/s). \tag{3b}$$

As in reference 4 we introduce new fermion operators α, through the unitary transformation,

$$\begin{aligned}\alpha_{\mathbf{k}0}&=u_{\mathbf{k}}c_{\mathbf{k}\uparrow}-v_{\mathbf{k}}c_{-\mathbf{k}\downarrow}{}^\dagger\\ \alpha_{\mathbf{k}1}&=u_{\mathbf{k}}c_{-\mathbf{k}\downarrow}+v_{\mathbf{k}}c_{\mathbf{k}\uparrow}{}^\dagger,\end{aligned} \tag{4}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are real functions which satisfy the relation

$$u_{\mathbf{k}}{}^2+v_{\mathbf{k}}{}^2=1,$$

and are symmetric with respect to the transformation $\mathbf{k}\to-\mathbf{k}$. As a result of the substitution,

$$\begin{aligned}c_{\mathbf{k}\uparrow}&=u_{\mathbf{k}}\alpha_{\mathbf{k}0}+v_{\mathbf{k}}\alpha_{\mathbf{k}1}{}^\dagger\\ c_{-\mathbf{k}\downarrow}&=u_{\mathbf{k}}\alpha_{\mathbf{k}1}-v_{\mathbf{k}}\alpha_{\mathbf{k}0}{}^\dagger,\end{aligned} \tag{4$'$}$$

the cranking interaction, $\mathcal{H}_{\text{int}}$, takes the form

$$\begin{aligned}\mathcal{H}_{\text{int}}=-\omega\Big\{&\sum_{r\neq0}\sum_{\mathbf{k}}[L_{\mathbf{k}+\mathbf{r},\mathbf{k}}(u_{\mathbf{k}+\mathbf{r}}u_{\mathbf{k}}+v_{\mathbf{k}}v_{\mathbf{k}+\mathbf{r}})\\ &\times(\alpha_{\mathbf{k}+\mathbf{r},0}{}^\dagger\alpha_{\mathbf{k}0}-\alpha_{\mathbf{k}1}{}^\dagger\alpha_{\mathbf{k}+\mathbf{r},1})\\ &+L_{\mathbf{k}+\mathbf{r},\mathbf{k}}(u_{\mathbf{k}+\mathbf{r}}v_{\mathbf{k}}-v_{\mathbf{k}+\mathbf{r}}u_{\mathbf{k}})\\ &\times(\alpha_{\mathbf{k}+\mathbf{r},0}{}^\dagger\alpha_{\mathbf{k}1}{}^\dagger-\alpha_{\mathbf{k}+\mathbf{r},1}\alpha_{\mathbf{k}0})]\\ &-\sum_{s\neq0}\sum_{\mathbf{k}}[L_{\mathbf{k}+\mathbf{s},\mathbf{k}}(u_{\mathbf{k}+\mathbf{s}}u_{\mathbf{k}}+v_{\mathbf{k}}v_{\mathbf{k}+\mathbf{s}})\\ &\times(\alpha_{\mathbf{k}+\mathbf{s},0}{}^\dagger\alpha_{\mathbf{k}0}-\alpha_{\mathbf{k}1}{}^\dagger\alpha_{\mathbf{k}+\mathbf{s},1})\\ &+L_{\mathbf{k}+\mathbf{s},\mathbf{k}}(u_{\mathbf{k}+\mathbf{s}}v_{\mathbf{k}}-v_{\mathbf{k}+\mathbf{s}}u_{\mathbf{k}})\\ &\times(\alpha_{\mathbf{k}+\mathbf{s},0}{}^\dagger\alpha_{\mathbf{k}1}{}^\dagger-\alpha_{\mathbf{k}+\mathbf{s},1}\alpha_{\mathbf{k}0})]\Big\}.\end{aligned} \tag{5}$$

Then associating the boson operators $\beta_{\mathbf{p}}(\mathbf{k}),\beta_{\mathbf{p}}{}^\dagger(\mathbf{k})$ with the respective fermion operator pairs, $\alpha_{\mathbf{k}1}\alpha_{\mathbf{k}+\mathbf{p},0}$,

[5] L. N. Cooper, R. L. Mills, and A. M. Sessler, Phys. Rev. **114**, 1377 (1959).

[6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[7] K. Sawada, Phys. Rev. **106**, 372 (1957); G. Wentzel, Phys. Rev. **108**, 1593 (1957); R. M. Rockmore, Phys. Rev. **114**, 941 (1959). See also reference 4 where this method is termed the "method of approximate second quantization."

[8] We use the system of units in which $\hbar=1$. Our notation follows closely that of I and reference 4.

[9] See I and reference 2 for details; the notation is that of I.

$\alpha_{\mathbf{k}+\mathbf{p},0}{}^\dagger\alpha_{\mathbf{k}1}{}^\dagger$ *in all possible ways*, one obtains

$$\mathcal{H}_1'=\sum_{\mathbf{kk'p}(\mathbf{k}\neq\mathbf{k'})}[A_{\mathbf{p}}(\mathbf{k},\mathbf{k}')\beta_{\mathbf{p}}{}^\dagger(\mathbf{k})\beta_{\mathbf{p}}(\mathbf{k}')+\tfrac{1}{2}B_{\mathbf{p}}(\mathbf{k},-\mathbf{k}')\times\{\beta_{\mathbf{p}}{}^\dagger(\mathbf{k})\beta_{-\mathbf{p}}{}^\dagger(-\mathbf{k}')+\beta_{-\mathbf{p}}(-\mathbf{k}')\beta_{\mathbf{p}}(\mathbf{k})\}], \quad (6)$$

where[10]

$$\begin{aligned}A_{\mathbf{p}}(\mathbf{k},\mathbf{k}')=&(1/\Omega)\\&\times(\langle\mathbf{k}+\mathbf{p},\mathbf{k}'|v|\mathbf{k},\mathbf{k}'+\mathbf{p}\rangle-\langle\mathbf{k}+\mathbf{p},\mathbf{k}'|v|\mathbf{k}'+\mathbf{p},\mathbf{k}\rangle)\\&\times(u_{\mathbf{k}+\mathbf{p}}v_{\mathbf{k}}u_{\mathbf{k}'+\mathbf{p}}v_{\mathbf{k}'}+u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{p}}u_{\mathbf{k}'}v_{\mathbf{k}'+\mathbf{p}})\\&+(1/\Omega)\langle\mathbf{k}+\mathbf{p},-\mathbf{k}|v|\mathbf{k}'+\mathbf{p},-\mathbf{k}'\rangle\\&\times(u_{\mathbf{k}+\mathbf{p}}u_{\mathbf{k}'+\mathbf{p}}u_{\mathbf{k}}u_{\mathbf{k}'}+v_{\mathbf{k}'+\mathbf{p}}v_{\mathbf{k}+\mathbf{p}}v_{\mathbf{k}'}v_{\mathbf{k}})\\&+(1/\Omega)\langle\mathbf{k}+\mathbf{p},-\mathbf{k}'-\mathbf{p}|v|\mathbf{k},-\mathbf{k}'\rangle\\&\times(u_{\mathbf{k}+\mathbf{p}}v_{\mathbf{k}}v_{\mathbf{k}'+\mathbf{p}}u_{\mathbf{k}'}+u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{p}}v_{\mathbf{k}'}u_{\mathbf{k}'+\mathbf{p}}),\end{aligned}\quad (7)$$

and

$$\begin{aligned}B_{\mathbf{p}}(\mathbf{k},-\mathbf{k}')=&(1/\Omega)(\langle\mathbf{k}+\mathbf{p},-\mathbf{k}'-\mathbf{p}|v|\mathbf{k},-\mathbf{k}'\rangle\\&-\langle\mathbf{k}+\mathbf{p},-\mathbf{k}'-\mathbf{p}|v|-\mathbf{k}',\mathbf{k}\rangle)\\&\times(u_{\mathbf{k}'+\mathbf{p}}v_{\mathbf{k}'}u_{\mathbf{k}+\mathbf{p}}v_{\mathbf{k}}+u_{\mathbf{k}'}v_{\mathbf{k}'+\mathbf{p}}u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{p}})\\&+(1/\Omega)\langle\mathbf{k}+\mathbf{p},\mathbf{k}'|v|\mathbf{k},\mathbf{k}'+\mathbf{p}\rangle\\&\times(u_{\mathbf{k}'}v_{\mathbf{k}'+\mathbf{p}}u_{\mathbf{k}+\mathbf{p}}v_{\mathbf{k}}+u_{\mathbf{k}'+\mathbf{p}}v_{\mathbf{k}'}u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{p}})\\&-(1/\Omega)\langle\mathbf{k}+\mathbf{p},-\mathbf{k}|v|-\mathbf{k}',\mathbf{k}'+\mathbf{p}\rangle\\&\times(u_{\mathbf{k}'+\mathbf{p}}u_{\mathbf{k}'}v_{\mathbf{k}}v_{\mathbf{k}+\mathbf{p}}+u_{\mathbf{k}+\mathbf{p}}u_{\mathbf{k}}v_{\mathbf{k}'}v_{\mathbf{k}'+\mathbf{p}}).\end{aligned}\quad (8)$$

Note the explicit appearance of the exchange terms among the matrix elements of v in (7) and (8). The quantities $B_{\mathbf{p}}(\mathbf{k},-\mathbf{k}')$, $A_{\mathbf{p}}(\mathbf{k},\mathbf{k}')$ are real and symmetric under the interchange $\mathbf{k}\leftrightarrow\mathbf{k}'$,[11] i.e.,

$$B_{\mathbf{p}}(\mathbf{k},-\mathbf{k}')=B_{\mathbf{p}}(\mathbf{k}',-\mathbf{k}), \quad (9a)$$

$$A_{\mathbf{p}}(\mathbf{k},\mathbf{k}')=A_{\mathbf{p}}(\mathbf{k}',\mathbf{k}). \quad (9b)$$

In terms of the pair operators $\beta_{\mathbf{p}}(\mathbf{k}),\beta_{\mathbf{p}}{}^\dagger(\mathbf{k})$, one has for the "principal part" of the cranking interaction,

$$\mathcal{H}_{\mathrm{int}}'=-\omega\{\sum_{r\neq0}\sum_{\mathbf{k}}[L_{\mathbf{k}+\mathbf{r},\mathbf{k}}(u_{\mathbf{k}+\mathbf{r}}v_{\mathbf{k}}-v_{\mathbf{k}+\mathbf{r}}u_{\mathbf{k}})\times(\beta_{\mathbf{r}}{}^\dagger(\mathbf{k})-\beta_{-\mathbf{r}}(\mathbf{k}+\mathbf{r}))]-(r\leftrightarrow s)\}. \quad (10)$$

The Hamiltonian for collective excitation is finally completed by adding to (6) and (10) the self-energy of the pairs,

$$\mathcal{H}_0'=\sum_{\mathbf{kp}}E_{\mathbf{kp}}\beta_{\mathbf{p}}{}^\dagger(\mathbf{k})\beta_{\mathbf{p}}(\mathbf{k}), \quad (11)$$

where

$$E_{\mathbf{kp}}=E_{\mathbf{k}+\mathbf{p}}+E_{\mathbf{k}}. \quad (12)$$

The dispersion law for single particles,[4,6]

$$E_{\mathbf{k}}=(\epsilon_{\mathbf{k}}{}^2+I^2)^{\frac{1}{2}}, \quad (13)$$

where I denotes the energy gap [we shall take the energy gap to be constant[12] with $I=I(k_F)$], obtains in the superconducting state and goes over to the normal law $E_{\mathbf{k}}=|\epsilon_{\mathbf{k}}|$ when $I=0$. Moreover the functions $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ which, in the superconducting state are given by[4]

$$u_{\mathbf{k}}=[\tfrac{1}{2}(1+\epsilon_{\mathbf{k}}/E_{\mathbf{k}})]^{\frac{1}{2}}, \quad (14a)$$

$$v_{\mathbf{k}}=[\tfrac{1}{2}(1-\epsilon_k/E_k)]^{\frac{1}{2}} \quad (14b)$$

take the values

$$\begin{aligned}u_{\mathbf{k}}&=1\quad(k>k_F)\\&=0\quad(k<k_F),\end{aligned}\quad (15a)$$

$$\begin{aligned}v_{\mathbf{k}}&=0\quad(k>k_F)\\&=1\quad(k<k_F),\end{aligned}\quad (15b)$$

when the energy gap goes to zero.

Thus we have *for excitations consisting of particle-hole pairs only*,[13]

$$\begin{aligned}[\mathcal{H}_0']_{I=0}&=\sum_{\substack{\mathbf{kp}\\(|\mathbf{k}+\mathbf{p}|>k_F;\,k<k_F)}}\omega_{\mathbf{kp}}{}^*c_{\mathbf{p}}{}^\dagger(\mathbf{k};\uparrow)c_{\mathbf{p}}(\mathbf{k};\uparrow)\\&\quad-\sum_{\substack{\mathbf{kp}\\(|\mathbf{k}+\mathbf{p}|<k_F;\,k>k_F)}}\omega_{\mathbf{kp}}{}^*c_{\mathbf{p}}{}^\dagger(-\mathbf{k}-\mathbf{p};\downarrow)c_{\mathbf{p}}(-\mathbf{k}-\mathbf{p};\downarrow)\\&=\sum_{\substack{\mathbf{kp}\sigma\\(|\mathbf{k}+\mathbf{p}|>k_F;\,k<k_F)}}\omega_{\mathbf{kp}}{}^*c_{\mathbf{p}}{}^\dagger(\mathbf{k};\sigma)c_{\mathbf{p}}(\mathbf{k};\sigma),\end{aligned}$$

together with[14]

$$\begin{aligned}[\mathcal{H}_1']_{I=0}&=\frac{1}{2\Omega}\sum_{\substack{\mathbf{kk'p}\sigma\sigma'\\(|\mathbf{k}+\mathbf{p}|>k_F;\,k<k_F)}}v_1(\mathbf{k},\mathbf{k}';\mathbf{p})\\&\quad\times[c_{\mathbf{p}}{}^\dagger(\mathbf{k};\sigma)c_{\mathbf{p}}(\mathbf{k}';\sigma')+c_{\mathbf{p}}{}^\dagger(\mathbf{k}';\sigma')c_{\mathbf{p}}(\mathbf{k};\sigma)]\\&\quad+\frac{1}{2\Omega}\sum_{\substack{\mathbf{kk'p}\sigma\sigma'\\(|\mathbf{k}+\mathbf{p}|>k_F;\,k<k_F)}}v_1(\mathbf{k},\mathbf{k}'+\mathbf{p};\mathbf{p})\\&\quad\times[c_{\mathbf{p}}(\mathbf{k};\sigma)c_{-\mathbf{p}}(-\mathbf{k}';\sigma')+c_{-\mathbf{p}}{}^\dagger(-\mathbf{k}';\sigma')c_{\mathbf{p}}{}^\dagger(\mathbf{k};\sigma)],\end{aligned}$$

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[10] For the superconducting state we make the usual assumption of translational invariance in momentum space (at least for small $\mathbf{p}$) and set $\langle\mathbf{k}_1\mathbf{k}_2|v|\mathbf{k}_3\mathbf{k}_4\rangle=v(\mathbf{k}_4,\mathbf{k}_2)=v(\mathbf{k}_4-\mathbf{k}_2)$. [See, for example, G. Rickayzen, Phys. Rev. **115**, 795 (1959).] Then, it is easy to show that

$$A_{\mathbf{p}}(\mathbf{k},\mathbf{k}')=(1/\Omega)v(\mathbf{k}'+\mathbf{p},\mathbf{k}')M(\mathbf{k}+\mathbf{p},\mathbf{k})M(\mathbf{k}'+\mathbf{p},\mathbf{k}')+(1/\Omega)v(\mathbf{k},\mathbf{k}')L(\mathbf{k}+\mathbf{p},\mathbf{k}'+\mathbf{p})L(\mathbf{k},\mathbf{k}'),$$

$$B_{\mathbf{p}}(\mathbf{k},-\mathbf{k}')=(1/\Omega)v(\mathbf{k}'+\mathbf{p},\mathbf{k}')M(\mathbf{k}+\mathbf{p},\mathbf{k})M(\mathbf{k}'+\mathbf{p},\mathbf{k}')-(1/\Omega)v(\mathbf{k},-\mathbf{k}'-\mathbf{p})M(\mathbf{k}+\mathbf{p},\mathbf{k}')M(\mathbf{k}'+\mathbf{p},\mathbf{k}),$$

where

$$L(\mathbf{k},\mathbf{k}')=u_{\mathbf{k}}u_{\mathbf{k}'}-v_{\mathbf{k}}v_{\mathbf{k}'},$$
$$M(\mathbf{k},\mathbf{k}')=u_{\mathbf{k}}v_{\mathbf{k}'}+u_{\mathbf{k}'}v_{\mathbf{k}}.$$

[11] Relation (9a) follows from the symmetries

$$\langle\mathbf{a},\mathbf{b}|v|\mathbf{c},\mathbf{d}\rangle=\langle-\mathbf{b},-\mathbf{a}|v|-\mathbf{d},-\mathbf{c}\rangle,$$
$$\langle\mathbf{a},\mathbf{b}|v|\mathbf{c},\mathbf{d}\rangle=\langle\mathbf{d},\mathbf{c}|v|\mathbf{b},\mathbf{a}\rangle;$$

relation (9b) follows from the hermiticity of $\mathcal{H}_1'$.

[12] See the relevant discussion in Sec. III of reference 5.

[13] See Eq. (25) of I where the sums over $\mathbf{k}$ and $\mathbf{k}'$ do *not* include spin.

[14] See Eq. (26) of I.

$$[\mathcal{H}_{\text{int}}']_{I=0} = -\omega\{\sum_{r\neq 0} \sum_{\substack{\mathbf{k}\sigma \\ (|\mathbf{k}+\mathbf{r}|>k_F;\, k<k_F)}} L_{\mathbf{k}+\mathbf{r},\mathbf{k}} \times[c_{\mathbf{r}}^\dagger(\mathbf{k};\sigma)-c_{-\mathbf{r}}(\mathbf{k}+\mathbf{r};\sigma)]-(r\leftrightarrow s)\},$$

where

$$\begin{aligned}[\beta_{\mathbf{p}}(\mathbf{k})]_{I=0} &= c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}+\mathbf{p}\uparrow} = c_{\mathbf{p}}(\mathbf{k};\uparrow), \quad (|\mathbf{k}+\mathbf{p}|>k_F;\, k<k_F)\\ &= c_{-\mathbf{k}-\mathbf{p}\downarrow}^\dagger c_{-\mathbf{k}\downarrow} = c_{\mathbf{p}}(-\mathbf{k}-\mathbf{p};\downarrow), \quad (|\mathbf{k}+\mathbf{p}|<k_F;\, k>k_F).\end{aligned}$$

Hence it is apparent that the present investigation contains our previous work as a special case, namely the zero gap limit. (However, it will be of some advantage to treat the normal and superfluid state cases somewhat differently in the following sections.)

Considerable ease in the further manipulation of

$$\mathcal{H}_{\text{collective excitation}} = \mathcal{H}_0' + \mathcal{H}_1' + \mathcal{H}_{\text{int}}', \tag{16}$$

is gained by rewriting it in terms of the canonical variables,

$$\varphi_{\mathbf{p}}(\mathbf{k}) = (2E_{\mathbf{kp}})^{-\frac{1}{2}}[\beta_{\mathbf{p}}(\mathbf{k})+\beta_{-\mathbf{p}}^\dagger(-\mathbf{k})], \tag{17a}$$

$$\pi_{\mathbf{p}}(\mathbf{k}) = i(E_{\mathbf{kp}}/2)^{\frac{1}{2}}[\beta_{\mathbf{p}}^\dagger(\mathbf{k})-\beta_{-\mathbf{p}}(-\mathbf{k})], \tag{17b}$$

which satisfy the usual commutation relations

$$[\varphi_{\mathbf{p}}(\mathbf{k}),\varphi_{\mathbf{p}'}(\mathbf{k}')] = [\pi_{\mathbf{p}}(\mathbf{k}),\pi_{\mathbf{p}'}(\mathbf{k}')] = 0, \tag{18a}$$

$$[\varphi_{\mathbf{p}}(\mathbf{k}),\pi_{\mathbf{p}'}(\mathbf{k}')] = i\delta_{\mathbf{pp}'}\delta_{\mathbf{kk}'}. \tag{18b}$$

The additional symmetries

$$\begin{aligned}\varphi_{\mathbf{p}}^\dagger(\mathbf{k}) &= \varphi_{-\mathbf{p}}(-\mathbf{k}),\\ \pi_{\mathbf{p}}^\dagger(\mathbf{k}) &= \pi_{-\mathbf{p}}(-\mathbf{k}),\end{aligned} \tag{19}$$

follow on inspection. After substituting (17a), (17b) in (16), and making use of the relations

$$\begin{aligned}A_{\mathbf{p}}(\mathbf{k},\mathbf{k}') &= A_{-\mathbf{p}}(-\mathbf{k}',-\mathbf{k}),\\ B_{\mathbf{p}}(\mathbf{k},\mathbf{k}') &= B_{\mathbf{p}}(-\mathbf{k}',-\mathbf{k}),\end{aligned} \tag{20}$$

one finally obtains

$$\begin{aligned}\mathcal{H}_{\text{collective excitation}} =& \tfrac{1}{2}\sum_{\mathbf{kp}}\{\pi_{\mathbf{p}}^\dagger(\mathbf{k})\pi_{\mathbf{p}}(\mathbf{k})+E_{\mathbf{kp}}^2\varphi_{\mathbf{p}}^\dagger(\mathbf{k})\varphi_{\mathbf{p}}(\mathbf{k})+E_{\mathbf{kp}}\}\\ &+\tfrac{1}{2}\sum_{\mathbf{kk}'\mathbf{p}(\mathbf{k}\neq\mathbf{k}')}(E_{\mathbf{kp}}E_{\mathbf{k}'\mathbf{p}})^{\frac{1}{2}}[A_{\mathbf{p}}(\mathbf{k},\mathbf{k}')\\ &\times\{\varphi_{\mathbf{p}}^\dagger(\mathbf{k})\varphi_{\mathbf{p}}(\mathbf{k}')+(E_{\mathbf{kp}}E_{\mathbf{k}'\mathbf{p}})^{-1}\pi_{\mathbf{p}}^\dagger(\mathbf{k}')\pi_{\mathbf{p}}(\mathbf{k})\}\\ &+\tfrac{1}{2}B_{\mathbf{p}}(\mathbf{k},-\mathbf{k}')\{(\varphi_{\mathbf{p}}^\dagger(\mathbf{k})\varphi_{\mathbf{p}}(\mathbf{k}')+\varphi_{\mathbf{p}}(\mathbf{k})\varphi_{\mathbf{p}}^\dagger(\mathbf{k}'))\\ &-(E_{\mathbf{kp}}E_{\mathbf{k}'\mathbf{p}})^{-1}(\pi_{\mathbf{p}}^\dagger(\mathbf{k})\pi_{\mathbf{p}}(\mathbf{k}')+\pi_{\mathbf{p}}(\mathbf{k})\pi_{\mathbf{p}}^\dagger(\mathbf{k}'))\}]\\ &+\omega\sum_{\mathbf{k}}\left\{\sum_{r\neq 0}(-1)^{\Delta l}\left(\frac{k_y}{r}\right)\left(\frac{2}{E_{\mathbf{kr}}}\right)^{\frac{1}{2}}(u_{\mathbf{k}+\mathbf{r}}v_{\mathbf{k}}-v_{\mathbf{k}+\mathbf{r}}u_{\mathbf{k}})\right.\\ &\times\pi_{\mathbf{r}}(\mathbf{k})-\sum_{s\neq 0}(-1)^{\Delta m}\left(\frac{k_x}{s}\right)\left(\frac{2}{E_{\mathbf{ks}}}\right)^{\frac{1}{2}}\\ &\left.\times(u_{\mathbf{k}+\mathbf{s}}v_{\mathbf{k}}-v_{\mathbf{k}+\mathbf{s}}u_{\mathbf{k}})\pi_{\mathbf{s}}(\mathbf{k})\right\}.\end{aligned} \tag{21}$$

III. VARIATIONAL PRINCIPLE FOR THE MOMENT OF INERTIA

As in I, we need only consider

$$\mathcal{H}_x = \mathcal{H}_{0x}' + \mathcal{H}_{1x}' + \mathcal{H}_{\text{int }x}', \tag{22a}$$

where

$$\mathcal{H}_{0x}' = \tfrac{1}{2}\sum_{\mathbf{kr}}\pi_{\mathbf{r}}^\dagger(\mathbf{k})\pi_{\mathbf{r}}(\mathbf{k}), \tag{22b}$$

$$\begin{aligned}\mathcal{H}_{1x}' = \tfrac{1}{4}\sum_{\mathbf{kk}'\mathbf{r}(\mathbf{k}\neq\mathbf{k}')}&(E_{\mathbf{kr}}E_{\mathbf{k}'\mathbf{r}})^{-\frac{1}{2}}\\ &\times[A_{\mathbf{r}}(\mathbf{k},\mathbf{k}')-B_{\mathbf{r}}(\mathbf{k},-\mathbf{k}')]\\ &\times[\pi_{\mathbf{r}}^\dagger(\mathbf{k})\pi_{\mathbf{r}}(\mathbf{k}')+\pi_{\mathbf{r}}^\dagger(\mathbf{k}')\pi_{\mathbf{r}}(\mathbf{k})],\end{aligned} \tag{22c}$$

$$\begin{aligned}\mathcal{H}_{\text{int }x}' = \omega\sum_{r\neq 0}\sum_{\mathbf{k}}(-1)^{\Delta l}\left(\frac{k_y}{r}\right)\left(\frac{2}{E_{\mathbf{kr}}}\right)^{\frac{1}{2}}&\\ \times(u_{\mathbf{k}+\mathbf{r}}v_{\mathbf{k}}-v_{\mathbf{k}+\mathbf{r}}u_{\mathbf{k}})\pi_{\mathbf{r}}(\mathbf{k}).&\end{aligned} \tag{22d}$$

One assumes the analogous unitary transformation,[1]

$$U_x = \exp[-i\omega\sum_{\mathbf{pr}'}f_{\mathbf{r}'}(\mathbf{p})\varphi_{\mathbf{r}'}(\mathbf{p})], \tag{23}$$

with $f_{\mathbf{r}}(\mathbf{p}) = f_{-\mathbf{r}}(-\mathbf{p})$; the requirement,

$$U_x^\dagger\mathcal{H}_x U_x = \mathcal{H}_{0x}' + \mathcal{H}_{1x}' - \tfrac{1}{2}\mathcal{I}_x\omega^2, \tag{24}$$

yields an expression for the moment of inertia,

$$\begin{aligned}\mathcal{I}_x = 2\sum_{r\neq 0}\sum_{\mathbf{k}}(-1)^{\Delta l}\left(\frac{k_y}{r}\right)\left(\frac{2}{E_{\mathbf{kr}}}\right)^{\frac{1}{2}}(u_{\mathbf{k}+\mathbf{r}}v_{\mathbf{k}}-u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{r}})f_{\mathbf{r}}(\mathbf{k})&\\ -\sum_{\mathbf{kr}}\{f_{\mathbf{r}}^2(\mathbf{k})+\sum_{\mathbf{k}'(\mathbf{k}'\neq\mathbf{k})}(E_{\mathbf{kr}}E_{\mathbf{k}'\mathbf{r}})^{-\frac{1}{2}}&\\ \times[A_{\mathbf{r}}(\mathbf{k},\mathbf{k}')-B_{\mathbf{r}}(\mathbf{k},-\mathbf{k}')]f_{\mathbf{r}}(\mathbf{k})f_{\mathbf{r}}(\mathbf{k}')\},&\end{aligned} \tag{25}$$

together with an integral equation for $f_{\mathbf{r}}(\mathbf{k})$,

$$\begin{aligned}f_{\mathbf{r}}(\mathbf{k})+\sum_{\mathbf{k}'}(E_{\mathbf{kr}}E_{\mathbf{k}'\mathbf{r}})^{-\frac{1}{2}}[A_{\mathbf{r}}(\mathbf{k},\mathbf{k}')-B_{\mathbf{r}}(\mathbf{k},-\mathbf{k}')]f_{\mathbf{r}}(\mathbf{k}')&\\ -(-1)^{\Delta l}\left(\frac{k_y}{r}\right)\left(\frac{2}{E_{\mathbf{kr}}}\right)^{\frac{1}{2}}(u_{\mathbf{k}+\mathbf{r}}v_{\mathbf{k}}-u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{r}})=0.&\end{aligned} \tag{26}$$

Introduction of the auxiliary function,

$$F_{\mathbf{r}}(\mathbf{k}) = (-1)^{\Delta l}(2E_{\mathbf{kr}})^{-\frac{1}{2}}f_{\mathbf{r}}(\mathbf{k}), \tag{27}$$

produces further simplification. One then has

$$\begin{aligned}\mathcal{I}_x = 4\sum_{\mathbf{kr}}(k_y/r)(u_{\mathbf{k}+\mathbf{r}}v_{\mathbf{k}}-u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{r}})F_{\mathbf{r}}(\mathbf{k})&\\ -2\sum_{\mathbf{kr}}\{E_{\mathbf{kr}}[F_{\mathbf{r}}(\mathbf{k})]^2+\sum_{\mathbf{k}'(\mathbf{k}'\neq\mathbf{k})}&\\ \times[A_{\mathbf{r}}(\mathbf{k},\mathbf{k}')-B_{\mathbf{r}}(\mathbf{k},-\mathbf{k}')]F_{\mathbf{r}}(\mathbf{k}')F_{\mathbf{r}}(\mathbf{k})\},&\end{aligned} \tag{25'}$$

where

$$\begin{aligned}E_{\mathbf{kr}}F_{\mathbf{r}}(\mathbf{k})+\sum_{\mathbf{k}'}[A_{\mathbf{r}}(\mathbf{k},\mathbf{k}')-B_{\mathbf{r}}(\mathbf{k},-\mathbf{k}')]F_{\mathbf{r}}(\mathbf{k}')&\\ -(k_y/r)(u_{\mathbf{k}+\mathbf{r}}v_{\mathbf{k}}-u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{r}})=0.&\end{aligned} \tag{26'}$$

We will also find the following compact expression for $\mathcal{I}_x$ useful,

$$\mathcal{I}_x=2\sum_{\mathbf{kr}} (k_y/r)(u_{\mathbf{k+r}}v_{\mathbf{k}}-u_{\mathbf{k}}v_{\mathbf{k+r}})F_{\mathbf{r}}(\mathbf{k}); \qquad (25'')$$

it follows from the equation of motion for $F_{\mathbf{r}}(\mathbf{k})$ [Eq. (26′)].

In the case of a nonzero gap, exact solution of the integral equation (26′) for $F_{\mathbf{r}}(\mathbf{k})$ presents a formidible problem. The strong possibility of resonant effects in $F_{\mathbf{r}}(\mathbf{k})$ in the superfluid state case[15] {This is indicated by the fact that the homogeneous integral equation,[16]

$$E_{\mathbf{kr}}F_{\mathbf{r}}^{(0)}(\mathbf{k})+\sum_{\mathbf{k}'}[A_{\mathbf{r}}(\mathbf{k},\mathbf{k}')-B_{\mathbf{r}}(\mathbf{k},-\mathbf{k}')]F_{\mathbf{r}}^{(0)}(\mathbf{k}')=0, \qquad (26'')$$

goes over at $r=0$ into the familiar linearized integral equation,[17]

$$\mathfrak{F}(\mathbf{k})+\tfrac{1}{2}\sum_{\mathbf{k}'}\frac{v(\mathbf{k},\mathbf{k}')}{E_{k'}}\mathfrak{F}(\mathbf{k}')=0, \qquad (26''')$$

which is the criterion for the existence of a superstate. [We have set $2E_kF_0^{(0)}(\mathbf{k})=\mathfrak{F}(\mathbf{k})$.]}, argues, moreover, against solution of (26′) by iterative or other approximate methods. We shall instead determine the moment of inertia from a variational principle. Such a principle is furnished by (25′), which is easily shown to be stationary with respect to small variations in $F_{\mathbf{r}}(\mathbf{k})$, i.e.,

$$\frac{\delta\mathcal{I}_x}{\delta F_{\mathbf{r}}(\mathbf{k})}=\frac{\delta}{\delta F_{\mathbf{r}}(\mathbf{k})}\Bigg\{4\sum_{\mathbf{k}'r}\left(\frac{k_y'}{r}\right)(u_{\mathbf{k}'+\mathbf{r}}v_{k'}-u_{k'}v_{\mathbf{k}'+\mathbf{r}})F_{\mathbf{r}}(\mathbf{k}')-2\sum_{\mathbf{k}'r}[E_{\mathbf{kr}}(F_{\mathbf{r}}(\mathbf{k}))^2+\sum_{\mathbf{k}''(\mathbf{k}''\neq\mathbf{k}')}F_{\mathbf{r}}(\mathbf{k}')\times(A_{\mathbf{r}}(\mathbf{k}',\mathbf{k}'')-B_{\mathbf{r}}(\mathbf{k}',-\mathbf{k}''))F_{\mathbf{r}}(\mathbf{k}'')]\Bigg\}=0. \qquad (27)$$

This stationary property is a consequence of (a) the symmetry of the kernel, $A_{\mathbf{r}}(\mathbf{k},\mathbf{k}')-B_{\mathbf{r}}(\mathbf{k},-\mathbf{k}')$, under the transformation $\mathbf{k}\leftrightarrow\mathbf{k}'$, and (b) the equation of motion for $F_{\mathbf{r}}(\mathbf{k})$ [Eq. (26′)]. [(26′) is, however, neither a maximum nor a minimum.]

IV. THE NORMAL MOMENT

Here we derive results appropriate to the normal state [$I(k_F)=0$], which is characterized by $u_{\mathbf{k}}=\theta(k-k_F)$, $v_{\mathbf{k}}=\theta(k_F-k)$.[18] Although this case is treated fully in I, comparison with the discussion given in the previous section, shows the procedure employed there[19] in the determination of $\mathcal{I}_x$ to be completely equivalent to the variational principle (Sec. III above). Consequently, it seems worthwhile to prove *explicitly* that the Ansatz assumed for $F_{\mathbf{r}}(\mathbf{k})$ in I is indeed the solution of the equation of motion in the normal state. (In the course of our discussion, it will become apparent that the relevant sums over momenta, evaluation of which occupies the major part of reference 2, are trivial.)

We have in the zero gap limit,[20]

$$(\mathcal{I}_x)_{\text{normal}}=2\sum_{\mathbf{kr}}(k_y/r)F_{\mathbf{r}}(\mathbf{k}), \quad (|\mathbf{k}+\mathbf{r}|>k_F;k<k_F) \qquad (28)$$

and

$$\omega_{\mathbf{kr}}^*F_{\mathbf{r}}(\mathbf{k})+\sum_{\mathbf{k}'(k'<k_F,|\mathbf{k}'+\mathbf{r}|>k_F)}\times[v_1(\mathbf{k}',\mathbf{k};\mathbf{r})-v_1(-\mathbf{k}'-\mathbf{r},\mathbf{k};\mathbf{r})]F_{\mathbf{r}}(\mathbf{k}')-(k_y/r)=0, \quad (|\mathbf{k}+\mathbf{r}|>k_F;k<k_F). \qquad (29)$$

{Our specialization to spin-independent potentials permits us to make use of the normalization convention[21]

$$\sum_{\mathbf{k}}\theta(k_F-k)=N, \qquad (30)$$

where

$$\sum_{\mathbf{k}}\rightarrow\frac{\Omega}{(2\pi)^3}\int k^2dkd\omega_k,$$

and thus, to omit spin sums; otherwise, one would obtain, for example,

$$(\mathcal{I}_x)_{\text{normal}}=2\sum_{\mathbf{kr}}(k_y/r)[2F_{\mathbf{r}}(\mathbf{k})]=2\sum_{\mathbf{k}\sigma\mathbf{r}}(k_y/r)F_{\mathbf{r}}(\mathbf{k}).\}$$

If the general Ansatz,

$$F_{\mathbf{r}}(\mathbf{k})=(k_y/r)[\phi(k)/\omega_{\mathbf{kr}}^*], \qquad (31)$$

is substituted into (29), one obtains an integral equation for $\phi(k)$,

$$\phi(k)=1-\sum_{\mathbf{k}'}(k_y'/k_y)[\phi(k')/\omega_{\mathbf{k}'\mathbf{r}}^*]\times[v_1(\mathbf{k}',\mathbf{k};\mathbf{r})-v_1(-\mathbf{k}'-\mathbf{r},\mathbf{k};\mathbf{r})], \quad (k,k'<k_F;\ |\mathbf{k}+\mathbf{r}|,\ |\mathbf{k}'+\mathbf{r}|>k_F). \qquad (32)$$

For small r (i.e., in the neighborhood of the Fermi surface), one makes the reasonable approximations,

$$\phi(k),\phi(k')\simeq\phi(k_F)\equiv\phi, \qquad (33)$$

$$\begin{aligned} v_1(\mathbf{k}',\mathbf{k};\mathbf{r})&\rightarrow v_1(\mathbf{k}',\mathbf{k};0)|_{k,k'=k_F},\\ v_1(-\mathbf{k}'-\mathbf{r},\mathbf{k};\mathbf{r})&\rightarrow v_1(-\mathbf{k}',\mathbf{k};0)|_{k,k'=k_F}, \end{aligned} \qquad (34)$$

together with the customary decomposition into spherical harmonics,[2]

$$v_1(\mathbf{k}',\mathbf{k};0)|_{k,k'=k_F}=\sum\nolimits_l v_l(k_F^2,k_F^2)P_l(\hat{k}\cdot\hat{k}'), \qquad (35)$$

[15] This has been noted by K. Sawada (private communication).
[16] $\mathbf{k}$ and $\mathbf{k}+\mathbf{r}$ lie on the same side of the Fermi surface as do $\mathbf{k}'$ and $\mathbf{k}'+\mathbf{r}$.
[17] See Eq. (12) of reference 5.
[18] $\theta(x)=1,\ x>0$; $\theta(x)=0,\ x<0$.
[19] Eq. (47) of I clearly epitomizes the variational method.
[20] We have used the relation $F_{\mathbf{r}}(\mathbf{k})=F_{-\mathbf{r}}(\mathbf{k}+\mathbf{r})$ in deriving (28) from the zero gap limit of (26″).
[21] See Eq. (13) of reference 2.

where $(\hat{k})_x = x$ and $(\hat{k})_y = (1-x^2)^{\frac{1}{2}} \sin\varphi$. It is readily seen that[22]

$$\sum_{\mathbf{k}'(k'<k_F;\ |\mathbf{k}'+\mathbf{r}|>k_F)} \rightarrow \frac{\Omega k_F^2}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^1 dx \int_{k_F - rx}^{k_F} dk, \quad (36)$$

and hence,

$$\phi = \left[1 + \frac{k_F M^* \Omega}{(2\pi)^3} \sum_l v_l(k_F^2, k_F^2) \times \int_0^{2\pi} d\varphi' \int_0^1 dx' \frac{(\hat{k}')_y}{(\hat{k})_y} (1-(-1)^l) P_l(\hat{k}\cdot\hat{k}')\right]^{-1}$$
$$= \left[1 + \frac{M^* k_F \Omega}{6\pi^2} v_1(k_F^2, k_F^2)\right]^{-1}. \quad (37)$$

From the relation[23]

$$\frac{\Omega}{6\pi^2} k_F v_1(k_F^2, k_F^2) = -\frac{1}{k_F}\left(\frac{dV}{dk}\right)_{k=k_F} = \frac{1}{M} - \frac{1}{M^*}, \quad (38)$$

it then follows that $\phi = M/M^*$. Note also that evaluation of the rigid moment,

$$(\mathcal{I}_x)_{\text{rigid}} = 2 \sum_{r\neq 0} \sum_{\mathbf{k}(k<k_F;\ |\mathbf{k}+\mathbf{r}|>k_F)} \left(\frac{k_y}{r}\right)^2 \frac{1}{\omega_{kr}}, \quad (39)$$

becomes, on application of (36), the trivial calculation,

$$(\mathcal{I}_x)_{\text{rigid}} = 2 \sum_{r\neq 0} \frac{1}{r^2} \frac{\Omega k_F^2}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^1 dx \int_{k_F-rx}^{k_F} \times [k_F^2(1-x^2)\sin^2\varphi] \frac{M}{k_F r x}$$
$$= \frac{NML^2}{6}, \quad (40)$$

where

$$\sum_{r\neq 0} \frac{1}{r^2} = 2\left(\frac{L^2}{4\pi^2}\right)\frac{\pi^2}{6}. \quad (41)$$

V. THE SUPERFLUID MOMENT

We shall calculate here the superfluid moment by means of the variational principle described in Sec. III. Our discussion is limited to two tractable two-body interactions: (a) a spherically-symmetric delta-function shell potential, (b) a Yukawa shell potential. (Both potentials are assumed to be zero everywhere in momentum space except in a thin spherical shell at the Fermi surface.) Let us consider the class of trial functions,

$$F_{\mathbf{r}}(\mathbf{k}) = \phi g_{\mathbf{r}}(\mathbf{k}), \quad (42)$$

where $g_{\mathbf{r}}(\mathbf{k})$ is a known function and ϕ a variationally determined parameter. Then,

$$(\mathcal{I}_x/4) = \phi \sum_{\mathbf{k}r} (k_y/r)(u_{\mathbf{k}+\mathbf{r}} v_{\mathbf{k}} - v_{\mathbf{k}+\mathbf{r}} u_{\mathbf{k}}) g_{\mathbf{r}}(\mathbf{k}) - \tfrac{1}{2}\phi^2 \sum_{\mathbf{k}\mathbf{k}'r} g_{\mathbf{r}}(\mathbf{k}) \{E_{\mathbf{k}\mathbf{r}} \delta_{\mathbf{k}\mathbf{k}'} + [A_{\mathbf{r}}(\mathbf{k},\mathbf{k}') - B_{\mathbf{r}}(\mathbf{k}, -\mathbf{k}')]\} g_{\mathbf{r}}(\mathbf{k}'). \quad (43)$$

The condition, $\partial(\mathcal{I}_x/4)/\partial\phi = 0$, leads to

$$\phi = \frac{\sum_{\mathbf{k}r} (k_y/r)(u_{\mathbf{k}+\mathbf{r}} v_{\mathbf{k}} - v_{\mathbf{k}+\mathbf{r}} u_{\mathbf{k}}) g_{\mathbf{r}}(\mathbf{k})}{\sum_{\mathbf{k}\mathbf{k}'r} g_{\mathbf{r}}(\mathbf{k}) \{E_{\mathbf{k}\mathbf{r}} \delta_{\mathbf{k}\mathbf{k}'} + [A_{\mathbf{r}}(\mathbf{k},\mathbf{k}') - B_{\mathbf{r}}(\mathbf{k}, -\mathbf{k}')]\} g_{\mathbf{r}}(\mathbf{k}')}; \quad (44)$$

on substituting (44) into (43), there results

$$(\mathcal{I}_x)_{\text{var}} = 2 \sum_{\mathbf{k}r} (k_y/r)(u_{\mathbf{k}+\mathbf{r}} v_{\mathbf{k}} - u_{\mathbf{k}} v_{\mathbf{k}+\mathbf{r}}) g_{\mathbf{r}}(\mathbf{k}) \phi, \quad (45)$$

which is seen to be the compact expression for $\mathcal{I}_x$ [Eq. (26″)] with $F_{\mathbf{r}}(\mathbf{k})$ replaced by $g_{\mathbf{r}}(\mathbf{k})\phi$.[24] Let us then take

$$g_{\mathbf{r}}(\mathbf{k}) = \left(\frac{k_y}{r}\right) \frac{(u_{\mathbf{k}+\mathbf{r}} v_{\mathbf{k}} - v_{\mathbf{k}+\mathbf{r}} u_{\mathbf{k}})}{E_{\mathbf{k}\mathbf{r}}}; \quad (46)$$

it follows that

$$(\mathcal{I}_x)_{\text{superfluid}} = (\mathcal{I}_x^0)_{\text{superfluid}} [1/(1+R)], \quad (47)$$

where

$$R = \frac{1}{(\mathcal{I}_x^0)_{\text{superfluid}}} \times \left\{ 2 \sum_{\mathbf{k}\mathbf{k}'r} \left(\frac{k_y k_y'}{r^2}\right) [A_{\mathbf{r}}(\mathbf{k},\mathbf{k}') - B_{\mathbf{r}}(\mathbf{k}, -\mathbf{k}')] \times \frac{(u_{\mathbf{k}+\mathbf{r}} v_{\mathbf{k}} - u_{\mathbf{k}} v_{\mathbf{k}+\mathbf{r}})(u_{\mathbf{k}'+\mathbf{r}} v_{\mathbf{k}'} - u_{\mathbf{k}'} v_{\mathbf{k}'+\mathbf{r}})}{E_{\mathbf{k}\mathbf{r}} E_{\mathbf{k}'\mathbf{r}}} \right\} \quad (48)$$

and

$$(\mathcal{I}_x^0)_{\text{superfluid}} = 2 \sum_{\mathbf{k}r} \left(\frac{k_y}{r}\right)^2 \frac{(u_{\mathbf{k}+\mathbf{r}} v_{\mathbf{k}} - u_{\mathbf{k}} v_{\mathbf{k}+\mathbf{r}})^2}{E_{\mathbf{k}\mathbf{r}}}. \quad (49)$$

Note that[25]

$$(\mathcal{I}_x^0)_{\text{superfluid}} \lesssim \left(\frac{2\pi}{LP_0}\right)\left(\frac{k_F}{P_0}\right) \frac{1}{P_0^2} \left(\frac{3\rho}{\pi}\right)^{\frac{2}{3}} \frac{M^*}{M} (\mathcal{I}_x)_{\text{rigid}}, \quad (50)$$

where

$$P_0^2 = 2M^* I, \quad (51)$$

and

$$\rho = N/L^3.$$

[22] See M. Gell-Mann and K. Brueckner, Phys. Rev. **106**, 364 (1957).

[23] See Sec. III of reference 2.

[24] Compare with the procedure in I.

[25] See, for example, K. Yosida, Progr. Theoret. Phys. (Kyoto) **21**, 731 (1959), where the approximations made here are fully discussed.

Since $(\mathcal{I}_x^0)_{\text{superfluid}}=(1/L)\times\text{const.}\times(\mathcal{I}_x)_{\text{rigid}}$, we see that the superfluid moment $(\mathcal{I}_x^0)_{\text{superfluid}}$ is very much smaller than the rigid moment in the limit $N\to\infty$, $\Omega\to\infty$, with the ratio $N/\Omega=\rho$ fixed (the limit of large systems at constant density which is the case under consideration in this paper). For attractive interactions [for which $v_2(k_F^2,k_F^2)$ is nonzero], the quantity R defined in (48) will be found to be negative. [Note that since both $(\mathcal{I}_x)_{\text{superfluid}}$ and $(\mathcal{I}_x^0)_{\text{superfluid}}$ are positive-definite finite quantities, R is bounded by $R>-1$.] In demonstrating the resonant character of the ratio $(1+R)^{-1}$, we will make use of the criterion for superfluidity[26] which is taken to be satisfied by the S-wave component of the two-body interaction, $v_0(k_F^2,k_F^2)$. *For small r*, and in the neighborhood of the Fermi surface, one has[27]

$$[A_r(\mathbf{k},\mathbf{k}')-B_r(\mathbf{k},-\mathbf{k}')]_{l=2}=(1/\Omega)v_2(k_F^2,k_F^2), \quad (52)$$

and hence,

$$R\simeq\frac{1}{(\mathcal{I}_x^0)_{\text{superfluid}}}\left\{2\sum_{\mathbf{k}\mathbf{k}'r}\left(\frac{k_yk_y'}{r^2}\right)\frac{1}{\Omega}v_2(k_F^2,k_F^2)P_2(\hat{k}\cdot\hat{k}')\right.$$
$$\left.\times\frac{1}{4}\frac{I^2}{E_\mathbf{k}^3E_{\mathbf{k}'}^3}\frac{k_xk_x'r^2}{M^{*2}}\right\}. \quad (53)$$

We may also write

$$(\mathcal{I}_x^0)_{\text{superfluid}}=\sum_r 1\cdot\frac{N}{12M^*I^2}(3\pi^2\rho)^{\frac{2}{3}}$$
$$\cong\sum_r 1\cdot\frac{N}{12M^*I^2}(3\pi^2\rho)^{\frac{2}{3}}$$
$$\times\left(-\frac{1}{4\pi^2}v_0(k_F^2,k_F^2)k_FM^*\int_\Delta\frac{d\epsilon'}{E_{\mathbf{k}'}}\right), \quad (54)$$

where we have used the criterion for superfluidity.[26] [The integration in (54) is to be taken over a spherical shell of thickness Δ at the Fermi surface.] By straightforward manipulation, one obtains finally,

$$R\simeq-\frac{12}{25\mathcal{J}}\left(\frac{v_2}{v_0}\right), \quad (55)$$

where

$$\mathcal{J}=\int_0^{\Delta/2I}\frac{dZ}{(1+Z^2)^{\frac{1}{2}}}=\ln\left(\frac{\Delta}{2I}+[(\Delta/2I)^2+1]^{\frac{1}{2}}\right). \quad (56)$$

Since $(\Delta/2I)=\sinh[1/N(0)(-v_0)]$, where

$$N(0)=k_FM^*\Omega/2\pi^2,$$

one sees that the value of $\mathcal{J}$ is closely tied to the coupling strength.[6]

[26] See reference 25, Eq. (2.4).

[27] See footnote 10; we have used the relation $[L(\mathbf{k},\mathbf{k}')]^2+[M(\mathbf{k},\mathbf{k}')]^2=1$.

(a) Spherically-Symmetric Delta-Function Shell-Interaction[28]

For the contact interaction $v(\mathbf{r}_{12})=-V_0\delta(r_{12}-a)\times\theta(\Delta/2-|\epsilon_\mathbf{k}|)$, with $V_0>0$ one has

$$R\simeq-\frac{12}{5\mathcal{J}}\left[\frac{j_2(k_Fa)}{j_0(k_Fa)}\right]^2. \quad (55a)$$

(b) Yukawa Shell-Interaction

For the Yukawa interaction, $v(\mathbf{r}_{12})=-V_0(e^{-\mu r_{12}}/r_{12})\times\theta(\Delta/2-|\epsilon_\mathbf{k}|)$, there results[29]

$$R\simeq-\frac{12}{5\mathcal{J}}[Q_2(1+\mu^2/2k_F^2)/Q_0(1+\mu^2/2k_F^2)]. \quad (55b)$$

The possibility of a resonance for intermediate values of the coupling[6] is more apparent in the first case than in the second. It seems likely, in any event, that the resulting moment will still be quite small compared with the rigid value, with the collective excitations preserving the superfluid character of the moment.

VI. CONCLUDING REMARKS

Although the results obtained in Sec. V are qualitative and undoubtedly model-dependent, they indicate that collective excitations may be expected to produce additional effects on the cranking moment of a many-fermion system when the conditions for superfluidity are met.[30] In the case of fermions moving under periodic boundary conditions, it appears that while the effects of collective excitation are reduced in the weak coupling limit, they lead to the possibility of a marked resonance in the superfluid moment at intermediate coupling strengths, this enhancement being almost entirely due to excitations consisting of quasiparticle or hole pairs. (Note that our results are identical for charged and neutral Fermi systems.) The possibility of such effects does not appear to have been discussed in recent theories of the superfluid nucleus.[31]

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[28] We wish to thank Dr. K. Sawada for suggesting this tractable shell interaction.

[29] We have used the expansion

$$\frac{1}{x-y}=\sum_{l=0}^{\infty}(2l+1)P_l(y)Q_l(x)$$

in deriving (55b).

[30] In the model considered above, these additional effects arise mainly from excitations consisting of pairs of particles or holes with small net momentum.

[31] V. G. Soloviev, Nuclear Phys. **9**, 655 (1959); S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat-fys. Medd. **31**, No. 11 (1959).

APPENDIX

Cranking Moment at Finite Temperatures

In the calculation of the cranking moment of a superfluid many-body fermion system at finite temperatures, we shall find the notation and results of reference 6 (particularly those of Sec. IV of that paper) most convenient. It was shown there that a single-particle scattering operator of the form

$$U=\sum_{\mathbf{k}\mathbf{k}'\sigma} B_{\mathbf{k}'\mathbf{k}} c_{\mathbf{k}'\sigma}{}^{\dagger} c_{\mathbf{k}\sigma}, \tag{A.1}$$

gives rise to the second-order energy,

$$\Delta E^{(2)}=\sum_f \frac{|(\Psi_f|\sum_{\mathbf{k}\mathbf{k}'\sigma} B_{\mathbf{k}'\mathbf{k}} c_{\mathbf{k}'}{}^{\dagger} c_{\mathbf{k}}|\Psi_i)|^2}{W_i-W_f}$$
$$=-\sum_{\mathbf{k}\mathbf{k}'} |B_{\mathbf{k}'\mathbf{k}}|^2 L(\epsilon',\epsilon), \tag{A.2}$$

where

$$L(\epsilon',\epsilon)=\frac{1}{2}\left(\frac{(1-2f)E-(1-2f')E'}{\epsilon^2-\epsilon'^2}\right)$$
$$+\frac{1}{2}\left(\frac{\epsilon\epsilon'+I^2}{EE'}\right)\left(\frac{(1-2f)E'-(1-2f')E}{\epsilon^2-\epsilon'^2}\right) \tag{A.3}$$

for the case,[32] $B_{\mathbf{k}'\mathbf{k}}=-B_{-\mathbf{k},-\mathbf{k}'}$. This case is appropriate here since for the cranking interaction, one has

$$B_{\mathbf{k}'\mathbf{k}}=B_{\mathbf{k}+\mathbf{p},\mathbf{k}}(\delta_{\mathbf{pr}}-\delta_{\mathbf{ps}}),$$

where

$$B_{\mathbf{k}+\mathbf{p},\mathbf{k}}=-\omega L_{\mathbf{k}+\mathbf{p},\mathbf{k}}, \tag{A.4}$$

and

$$L_{\mathbf{k}+\mathbf{r},\mathbf{k}}=\frac{(-1)^{\Delta l}}{i}\left(\frac{k_y}{r}\right)=\frac{(-1)^{\Delta l}}{i}\left(\frac{k_y}{(k_x+r)-k_x}\right)$$
$$=-L_{-\mathbf{k},-(\mathbf{k}+\mathbf{r})}. \tag{A.5}$$

From the relation,

$$\Delta E^{(2)}=-\omega^2\sum_{\mathbf{kprs}} |L_{\mathbf{k}+\mathbf{p},\mathbf{k}}(\delta_{\mathbf{pr}}-\delta_{\mathbf{ps}})|^2 L(\epsilon_{\mathbf{k}+\mathbf{p}},\epsilon_{\mathbf{k}})$$
$$=-\omega^2\sum_{\mathbf{k}r}\frac{k_y^2}{r^2}L(\epsilon_{\mathbf{k}+\mathbf{r}},\epsilon_{\mathbf{k}})-\omega^2\sum_{\mathbf{k}s}\frac{k_x^2}{s^2}L(\epsilon_{\mathbf{k}+\mathbf{s}},\epsilon_{\mathbf{k}})$$
$$=-\tfrac{1}{2}\mathcal{I}(T)\omega^2, \tag{A.6}$$

[32] Case II of Sec. IV, reference 6.

it follows that[33]

$$\mathcal{I}_x(T)=2\sum_{\mathbf{k}r}\frac{k_y^2}{r^2}L(\epsilon_{\mathbf{k}+\mathbf{r}},\epsilon_{\mathbf{k}}). \tag{A.7}$$

Since the principal contribution to $\mathcal{I}_x(T)$ comes from small r, we make the appropriate approximations,

$$E_{\mathbf{k}+\mathbf{r}}=E_{\mathbf{k}}+\delta E_{\mathbf{k}},$$

where

$$\delta E_{\mathbf{k}}\simeq(1/2E_{\mathbf{k}})(\epsilon_{\mathbf{k}+\mathbf{r}}{}^2-\epsilon_{\mathbf{k}}{}^2),$$
$$f_{\mathbf{k}+\mathbf{r}}=f_{\mathbf{k}}+\delta f_{\mathbf{k}}, \tag{A.8}$$

where

$$\delta f_{\mathbf{k}}\simeq-\beta\delta E_{\mathbf{k}} f_{\mathbf{k}}(1-f_{\mathbf{k}}), \tag{A.9}$$

$$L(\epsilon_{\mathbf{k}+\mathbf{r}},\epsilon_{\mathbf{k}})\simeq[E_{\mathbf{k}}\delta f_{\mathbf{k}}/(-E_{\mathbf{k}}\delta E_{\mathbf{k}})]=\beta f_{\mathbf{k}}(1-f_{\mathbf{k}}). \tag{A.10}$$

Thus we have,

$$\mathcal{I}_x(T)\simeq 2\sum_{\mathbf{k}r}\frac{k_y^2}{r^2}\beta f_{\mathbf{k}}(1-f_{\mathbf{k}})$$
$$\simeq 2\sum_{r\neq 0}\frac{1}{r^2}\beta\sum_{\mathbf{k}} k_y^2 f_{\mathbf{k}}(1-f_{\mathbf{k}}). \tag{A.11}$$

(A.11) may be further simplified by making use of the relation,

$$\sum_{\mathbf{k}} k^2 f_{\mathbf{k}}(1-f_{\mathbf{k}})=-\frac{L^3}{(2\pi)^3}4\pi\int_0^\infty k^4 dk\frac{d}{dE_k}\left(\frac{1}{1+\epsilon^{\beta E_k}}\right)\frac{1}{\beta}$$
$$=3L^3(1/\beta)\rho_n, \tag{A.12}$$

which was obtained by Bardeen[34] in his discussion of a two-fluid model for superconductivity. Substituting (A.12) into (A.11) and using (41), there results finally,[35]

$$\mathcal{I}(T)=\tfrac{1}{6}N_n ML^2, \tag{A.13}$$

and hence,

$$\mathcal{I}(T)=(N_n/N)\mathcal{I}_{\text{rigid}}. \tag{A.14}$$

[33] Note that in the zero temperature limit, $f_{\mathbf{k}}, f_{\mathbf{k}+\mathbf{r}}\to 0$, one recovers the familiar expression for the superconducting moment,

$$\mathcal{I}_x(0)=2\sum_{\mathbf{k}r}\left(\frac{k_y}{r}\right)^2\left\{\frac{1}{2}\frac{1}{E_{\mathbf{k}}+E_{\mathbf{k}+\mathbf{r}}}\left(1-\frac{\epsilon_{\mathbf{k}}\epsilon_{\mathbf{k}+\mathbf{r}}+I^2}{E_{\mathbf{k}}E_{\mathbf{k}+\mathbf{r}}}\right)\right\}.$$

[34] J. Bardeen, Phys. Rev. Letters **1**, 399 (1958). ρ_n is the density of the normal component (of one spin).

[35] This result has also been obtained by K. Sawada (private communication) using the method of A. A. Abrikosov, L. P. Gor'kov, and I. M. Khalatnikov [J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 265 (1958) [translation: Soviet Phys.-JETP **8**, 182 (1959)].