Generalization of Quantum Mechanics

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The possibility of generalizing quantum mechanics in such a way as to retain its predictive results, while comprehending additional solutions, is examined. It is found that this can be done through a perfected formal correspondence with Hamilton-Jacobi mechanics, by which one is led to consider generalizations of the Heisenberg postulate of the form $p_kq_j-q_jp_k=S\delta_{jk}$, where S is a quantum analog of Hamilton's principal function. The formalism is shown to be equivalent to a simple change in Hamiltonian, with transformed momentum operators satisfying conventional commutation relations, and with an additional relationship involving formal analogs of the classical "initial constants" adjoined. A particular choice of S $(=\hbar/i)$ leads to a theory identical with wave mechanics a part from a constant (unobservable) phase factor on the wave function. The fact that S may possess other, nonconstant values, demonstrated by a specific example, suggests the ability of the mechanical equations to describe a broader class of physical states than has hitherto been investigated.

In the absence of a satisfactory theory of nuclear forces, the possibility of departures from the postulates of quantum mechanics cannot be excluded. Because of the extremely close agreement of quantum mechanics with, e.g., the observed spectrum of the H atom, no significant postulational modification can be tolerated in the description of atomic states. To be of physical interest, any generalization of the theory must therefore contain the familiar quantum-mechanical states as exact solutions, but may possess additional solutions not in conformity with the postulates of the more-specialized discipline. The present paper establishes the existence of one such generalization.

1. CORRESPONDENCE: NONRELATIVISTIC CASE

Any mechanics must exhibit some relationship to the classical canonical theory. The most direct relationship is an exact formal correspondence. The nonrelativistic equations of motion of an n-particle system are postulated to be

$$H(x_k,p_k,t)\Psi_f=-\frac{\partial}{\partial t}S\Psi_f,\quad k=1,2,\cdots,3n, \tag{1a}$$

$$p_k\Psi_f=\frac{\partial}{\partial x_k}S\Psi_f, \tag{1b}$$

$$-P_k\Psi_f=\frac{\partial}{\partial X_k}S\Psi_f, \tag{1c}$$

by correspondence with Hamilton-Jacobi mechanics.[1] As in that theory, the X_k and P_k are constants; S is some scalar function,

$$S=S(x_k,X_k,t),\quad k=1,2,\cdots,3n, \tag{2}$$

the x_k are rectangular coordinates in a Galilean frame of reference; and $\Psi_f=\Psi_f(x_k,X_k,P_k,t)$ is a formal operand or wave function. The present paper is concerned with the consequences of this postulate (in Sec. 1) and of its relativistic counterpart (in Sec. 2).

Equation (1) may be regarded as a set of simultaneous partial differential equations for the unknown functions S and Ψ_f. It possesses three classes of solutions, designated as

Class I: $\Psi_f=$ constant.

Class II: $S=$ constant.

Class III: $\Psi_f\neq$ constant, $S\neq$ constant.

These will be examined in turn.

Class I

If Ψ_f is an arbitrary, nonzero c number, it cancels from Eq. (1), reducing the latter to the Hamilton-Jacobi equations of motion. Thus the classical "states" are included among the exact solutions of Eq. (1).

Class II

If S is assigned a particular constant value, chosen to agree with experiment, viz.,

$$S=\hbar/i, \tag{3}$$

Eq. (1) reduces to quantum mechanics. Equation (1), with (3), becomes

$$H(x_k,p_k,t)\Psi_f=-\frac{\hbar}{i}\frac{\partial}{\partial t}\Psi_f, \tag{4a}$$

$$p_k\Psi_f=\frac{\hbar}{i}\frac{\partial}{\partial x_k}\Psi_f, \tag{4b}$$

$$-P_k\Psi_f=\frac{\hbar}{i}\frac{\partial}{\partial X_k}\Psi_f. \tag{4c}$$

Equation (4c) is seen by inspection to be satisfied by

$$\Psi_f=\exp[-(i/\hbar)\textstyle\sum_k P_kX_k]\Phi(x_k,t). \tag{5}$$

Equations (4a) and (4b) reduce, after cancellation of

* Operations Evaluation Group, Washington, D. C.
[1] H. Goldstein, *Classical Mechanics* (Addison-Wesley Press, Cambridge, Massachusetts, 1950).

the constant exponential factor, to

$$H(x_k,p_k,t)\Phi(x_k,t)=-\frac{\hbar}{i}\frac{\partial}{\partial t}\Phi(x_k,t), \tag{6a}$$

$$p_k\Phi(x_k,t)=\frac{\hbar}{i}\frac{\partial}{\partial x_k}\Phi(x_k,t), \tag{6b}$$

the Schrödinger equations.

By Eq. (5), the solutions Ψ_f of Eq. (1) for which Eq. (3) is satisfied differ from the Schrödinger wave functions Φ only by an arbitrary constant phase factor, which would in general be absorbed into the wave-function normalization factor. Equation (1), supplemented by Eq. (3), is therefore substantiated by experiments that confirm ordinary quantum mechanics.

Class III

If both S and Ψ_f are nonconstant, we have to deal with solutions of Eq. (1) of the most general type, for which some of the classical-analog operators become non-Hermitean, and for which the Heisenberg postulate (but none of the other postulates of ordinary quantum mechanics) is violated. From Eqs. (1b) and (2) we see that the Heisenberg postulate is generalized to

$$[p_kq_j-q_jp_k]\Psi_f=\left[\left(\frac{\partial}{\partial q_k}\right)Sq_j-q_j\left(\frac{\partial}{\partial q_k}\right)S\right]\Psi_f =S\delta_{jk}\Psi_f, \tag{7}$$

where δ_{jk} is the Kronecker symbol. It is convenient for further development to introduce, by means of

$$S\equiv(\hbar/i)s, \tag{8}$$

a scalar function s that is the generalized quantum analog of the classical Poisson bracket of conjugate position and momentum variables ($s=1$, the classical value of this P.B., yields the Class-II states discussed above, corresponding to ordinary quantum mechanics). Equation (1a), with (8), becomes

$$H\Psi_f=-\frac{\hbar}{i}\frac{\partial}{\partial t}s\Psi_f. \tag{9}$$

The assumption that s possesses an inverse permits us to introduce, by means of the transformation

$$\mathcal{H}\equiv Hs^{-1}, \tag{10a}$$

$$\Psi\equiv s\Psi_f, \tag{10b}$$

an operator $\mathcal{H}$ which, being conjugate to time, has the physical significance of an energy operator,

$$\mathcal{H}\Psi=-\frac{\hbar}{i}\frac{\partial}{\partial t}\Psi; \tag{11}$$

provided the Hermiticity of $\mathcal{H}$ can be established for classical-analog Hamiltonians H of physical interest.

The latter requirement is satisfied if and only if the function s is real. Thus, the nonrelativistic one-body energy operator,

$$\begin{aligned}\mathcal{H}=Hs^{-1}&=[(1/2m)\mathbf{p}\cdot\mathbf{p}+V]s^{-1}\\ &=-(\hbar^2/2m)\nabla s\cdot\nabla+Vs^{-1},\end{aligned} \tag{12}$$

is seen to be Hermitean if s is real; and the same is true in the many-body and general external-field cases. Here, use has been made of Eq. (1b), written as an operator equation,

$$\mathbf{p}=(\hbar/i)\nabla s. \tag{13}$$

(The differential operator, of course, acts on everything to its right, not merely on s.)

The transition to the Heisenberg picture is facilitated by defining a Hermitean momentum, $\mathfrak{P}\equiv\mathbf{p}s^{-1}$. Equation (13) indicates that $\mathfrak{P}$ obeys the commutation relations or ordinary quantum-mechanical momentum. From this fact the commutation relations of other quantities involved in the theory [see Eq. (7)] are readily deduced, as well as the fact that the classical-analog quantities $\mathbf{p}$ and H are *not* Hermitean if s is real (e.g., $\mathbf{p}=\mathfrak{P}s$ is the product of two noncommuting Hermitean operators). By Eq. (11), $\mathcal{H}$ is the generator of infinitessimal time displacements of the system. Therefore, the equation of motion of a Heisenberg variable v is

$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+\frac{1}{i\hbar}(v\mathcal{H}-\mathcal{H}v).$$

The remainder of the Heisenberg form of the theory is readily deduced.

The investigation of a generalized form of the Heisenberg postulate [Eq. (7)] has led us in this section, via a simple transformation [Eq. (10)], to a formulation mathematically identical with ordinary quantum mechanics, but differing in the explicit form of the energy operator. There is a further difference, in that the Hamilton-Jacobi form of the theory contains an extra relationship, Eq. (1c), to which no allusion has been made here. Discussion of this important equation, and of its use in determining s, is of greater interest in the context of a relativistic formulation; hence, it is deferred to Sec. 2. The present formal considerations impose upon s, in reference to Class III solutions, only the requirement that it be a real scalar function of the arguments indicated in Eq. (2), nonzero at all, or almost all, points of its argument range (in order that it may possess an inverse).

2. RELATIVISTIC THEORY

Attention will be confined to the one-body problem. Our procedure will be to postulate a relativistic equation of motion, demonstrate its Lorentz invariance, show that it agrees with Eq. (1) in the nonrelativistic limit, and examine the three classes of solutions, as before.

The equation of motion of a single particle of charge

e and mass m in an external field of vector potential $\mathbf{A}=(A_1,A_2,A_3)$ and scalar potential $\varphi=A_0$ is postulated to be

$$\{\alpha_\mu[p_\mu-(e/c)A_\mu]+\alpha_m mc\}\Psi_f=0, \tag{14a}$$

$$p_\mu\Psi_f=\frac{\partial}{\partial x_\mu}S\Psi_f, \tag{14b}$$

$$-P_\mu\Psi_f=\frac{\partial}{\partial X_\mu}S\Psi_f, \quad \mu=0,1,2,3, \tag{14c}$$

where (x_1,x_2,x_3,t) are space-time coordinates in a Lorentz frame; $x_0=-ct$; (p_1,p_2,p_3) are classical-analog momenta; $p_0=H/c$, where H is the linearized classical-analog Hamiltonian,[2]

$$H=-c\boldsymbol{\alpha}\cdot[\mathbf{p}-(e/c)\mathbf{A}]-\alpha_m mc^2+e\varphi; \tag{15}$$

the X_μ and P_μ are constants ($X_0=-ct_0$, where t_0 is the initial-time constant); $\alpha_0=1$, and the remaining α's are anticommuting unit elements, as given by Dirac[3]; $\Psi_f=(x_\mu,X_\mu,P_\mu)$; $S=S(x_\mu,X_\mu)$ is a scalar (spin-independent) function; summation on repeated indices is implied; and distinctions between contravariance and covariance are ignored.

If p_μ, A_μ transform as 4-vectors,

$$p_\mu=a_{\mu\nu}p_\nu^*, \quad A_\mu=a_{\mu\nu}A_\nu^*, \tag{16}$$

while the wave function transforms according to

$$\Psi_f^*=\gamma\Psi_f, \tag{17}$$

where

$$\bar{\gamma}\alpha_\nu\gamma=\alpha_\mu a_{\mu\nu}, \quad \bar{\gamma}\alpha_m\gamma=\alpha_m, \tag{18}$$

(bar denotes Hermitean conjugate), Dirac[4] has shown Eq. (14a) to be Lorentz-invariant. It remains to establish the convariance of Eqs. (14b) and (14c). Let $d_\mu\equiv\partial/\partial x_\mu$, $D_\mu\equiv\partial/\partial X_\mu$, $\mu=0,1,2,3$. Because x_μ and X_μ are of the nature of space-time coordinates (the X_k, $k=1,2,3$, are "generalized coordinates" in the classical transformation theory[1]) they transform as 4-vectors; consequently, d_μ and D_μ are 4-vectors,

$$d_\mu=a_{\mu\nu}d_\nu^*, \quad D_\mu=a_{\mu\nu}D_\nu^*. \tag{19}$$

Since the α's, and hence γ, commute with all operators appearing in Eqs. (14b) and (14c), the former may be multiplied from the left by $a_{\mu\nu'}\gamma$ to yield, with the help of Eq. (17) and the orthogonality relation $(a_{\mu\nu'}a_{\mu\nu}=\delta_{\nu'\nu})$,

$$a_{\mu\nu'}p_\mu\gamma\Psi_f=p_{\nu'}^*\Psi_f^*=a_{\mu\nu'}d_\mu S\gamma\Psi_f=d_{\nu'}^*S\Psi_f^*,$$

or, with ν' replaced by μ,

$$p_\mu^*\Psi_f^*=\frac{\partial}{\partial x_\mu^*}S\Psi_f^*, \quad \mu=0,1,2,3. \tag{20}$$

Similarly, reference to Eq. (14c) indicates that if P_μ transforms as a 4-vector $(P_\mu=a_{\mu\nu}P_\nu^*)$, D_μ and $(-P_\mu)$ may be substituted for d_μ and p_μ, respectively, in the foregoing derivation to establish that

$$-P_\mu^*\Psi_f^*=\frac{\partial}{\partial X_\mu^*}S\Psi_f^*, \quad \mu=0,1,2,3. \tag{21}$$

Comparison of Eqs. (20), (21) with Eqs. (14b), (14c) shows the latter relations to be form-invariant under a Lorentz transformation between starred and unstarred coordinate systems if and only if

$$S(x_\mu,X_\mu)=S^*(x_\mu^*,X_\mu^*). \tag{22}$$

It is therefore proven that Eq. (14) is Lorentz-invariant (and that x_μ, p_μ, X_μ, P_μ transform as 4-vectors) if and only if S transforms as a Lorentz scalar function [Eq. (22)].

The consistency of Eq. (14) with Eq. (1) in the nonrelativistic limit is easily established. For $\mu=1,2,3$, Eqs. (14b) and (14c) are identical with Eqs. (1b) and (1c), respectively. For $\mu=0$, Eq. (14b) may be written as

$$cp_0\Psi_f=c\frac{\partial}{\partial x_0}S\Psi_f=-\frac{\partial}{\partial t}S\Psi_f,$$

which is consistent with Eq. (1a), in view of the fact that $cp_0=H$, where H, given by Eq. (15), is known[2] in the field-free case to be essentially equivalent (except for possession of negative- as well as positive-energy solutions) to the classical relativistic Hamiltonian, $[m^2c^4+c^2(p_1^2+p_2^2+p_3^2)]^{\frac{1}{2}}$, which in turn has the one-body Hamiltonian of Eq. (1a) as its nonrelativistic limiting form. If an electromagnetic field is present, the nonrelativistic limiting form of Eq. (14a) (except in the case of Class I solutions, discussed below) differs from the classical-analog Hamiltonian of Eq. (1a) by α-dependent terms, often added as empirical (Pauli spin) corrections to the classical-analog Hamiltonian in ordinary quantum mechanics. These terms evidence the inadequacy of a nonrelativistic formulation.

It remains to confirm that Eq. (14c) for $\mu=0$, viz.,

$$-P_0\Psi_f=\frac{\partial}{\partial X_0}S\Psi_f, \tag{23}$$

is compatible with Eq. (1). Since two new parameters, X_0 and P_0, not explicitly contained in Eq. (1), enter here, it will suffice to show that particular values of these parameters can be chosen to render Eq. (23) nugatory in any particular Lorentz frame [for in this case Eq. (14) will possess solutions that can be correlated with those of Eq. (1) in the equivalent Galilean frame]. This is seen by inspection of Eq. (23) to be trivially the case, e.g., for the class of solutions of Eq. (14) for which $P_0=0$ and $S\Psi_f$ is not explicitly dependent on X_0; or for the class of solutions for which $P_0X_0=0$, S is independent of X_0, and Ψ_f depends on X_0 only through a phase factor $\exp(-S^{-1}P_0X_0)$.

[2] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), 3rd ed., p. 264.
[3] P. A. M. Dirac, reference 2, pp. 255–256.
[4] P. A. M. Dirac, reference 2, pp. 257–259.

Such correlations between Eqs. (14) and (1) apply only in a chosen Lorentz frame, since Eq. (1) is not Lorentz invariant.

We now examine the three classes of solutions of Eq. (14), defining these as in Sec. 1.

Class I

If Ψ_f is a matrix of constants,

$$\Psi_f=\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix},$$

Eq. (14a) reduces to a (linearized) form of relativistic classical mechanics first discussed by Pauli.[5] Multiplying Eq. (14a) from the left by

$$\{[p_0-(e/c)A_0]-\boldsymbol{\alpha}\cdot[\mathbf{p}-(e/c)\mathbf{A}]-\alpha_m mc\}$$

and also by a constant 4-entry row matrix (such that its product with Ψ_f is nonvanishing) to eliminate Ψ_f, we obtain

$$[p_0-(e/c)A_0]^2-[\mathbf{p}-(e/c)\mathbf{A}]^2-m^2c^2=0, \quad (24a)$$

the classical relationship. Thus, except for questions of algebraic sign of the energy, the linearized Hamiltonian of Eqs. (14a), (15) is equivalent to the classical nonlinear form embodied in Eq. (24a). Multiplication of Eqs. (14b), (14c) from the left by a constant 4-entry row-matrix eliminates Ψ_f, as before, so that these equations become

$$p_\mu=(\partial S/\partial x_\mu), \quad (24b)$$

$$-P_\mu=(\partial S/\partial X_\mu), \quad \mu=0,1,2,3. \quad (24c)$$

Pauli omits explicit mention of these supplementary relations, although Eq. (24b) is implicit in his discussion and Eq. (24c) is evidently needed to impart 4-vector significance to the generalized momentum P_μ. The classical "states" of special-relativistic motion are thus included among the exact solutions of Eq. (14).

Class II

If S is assigned the constant value $\hbar/i$ [see Eq. (3)] a theory results that is identical with the Dirac electron theory, except for a phase-factor distinction similar to that discussed for Class II solutions in Sec. 1. Equation (14) becomes

$$\{\alpha_\mu[p_\mu-(e/c)A_\mu]+\alpha_m mc\}\Psi_f=0, \quad (25a)$$

$$p_\mu\Psi_f=\frac{\hbar}{i}\frac{\partial}{\partial x_\mu}\Psi_f, \quad (25b)$$

$$-P_\mu\Psi_f=\frac{\hbar}{i}\frac{\partial}{\partial X_\mu}\Psi_f, \quad \mu=0,1,2,3. \quad (25c)$$

Equation (25) is satisfied by

$$\Psi_f=\exp[-(i/\hbar)P_\mu X_\mu]\Phi(x_{\mu'}), \quad (26)$$

where Φ is the Dirac wave function.

Class III

The formalism developed for Class III solutions in Sec. 1 applies also in the relativistic case. We have only to verify the Hermiticity of the energy operator,

$$\mathcal{H}\equiv Hs^{-1}=-c\boldsymbol{\alpha}\cdot[(\hbar/i)\nabla-(e/c)\mathbf{A}s^{-1}] -\alpha_m mc^2s^{-1}+e\varphi s^{-1} \quad (27)$$

[see Eqs. (8), (10a), (13), and (15)], which follows at once from the realness of s. The function s is assumed to satisfy the restrictions mentioned at the end of Sec. 1, and also to transform as a Lorentz scalar function; see Eqs. (8), (22). If the transformed wave function Ψ, defined by Eq. (10b) (a Lorentz-invariant relationship), is time-independent in a particular Lorentz frame of interest, the substitution

$$\Psi=\psi\exp[-(i/\hbar)E't]$$

reduces Eq. (11) to the eigenvalue equation

$$\mathcal{H}\psi=E'\psi, \quad (28)$$

where $\mathcal{H}$ is given by Eq. (27).

Both Eq. (28) [embodying the information in Eqs. (14a), (14b)] and Eq. (14c) must come into play if the correspondence-based relations are to prove both necessary and sufficient for determining the two unknowns, s and ψ (or S and Ψ_f). To establish that the present theory provides a nontrivial generalization of quantum mechanics, it is mandatory to exhibit an example of such a Class III solution. This can be done only in the context of a specific problem, since this type of solution by its nature lacks dynamical generality. The fact that s and ψ are determined by simultaneous partial differential equations implies that both are state-dependent; in contrast to the case of Class I or II solutions, in which ψ or s, respectively, is postulated *a priori* with complete (state-independent) dynamical generality.

Example

To exemplify Class III solutions a relativistic one-body central force problem will be considered. A light particle (electron or positron) of charge e and mass m is assumed bound in a stationary state to an infinitely-massive point center of nuclear and Coulombic force, of charge Ze, which is at rest at the origin of coordinates in the laboratory system. The light particle is described by the Hamiltonian of Eq. (27), or, more simply for present purposes, by the 2×2 component operator,[6]

$$\mathcal{H}=i\hbar c\epsilon\left(\frac{d}{dr}+\frac{1}{r}\right)-\frac{i\hbar cj}{r}\epsilon\rho_3-mc^2s^{-1}\rho_3+\frac{Ze^2}{r}s^{-1}, \quad (29)$$

[5] W. Pauli, *Handbuch der Physik* edited by S. Flügge (Verlag Julius Springer, Berlin, 1933), 2nd ed., Vol. XXIV, p. 241, Eq. (105_0).

[6] P. A. M. Dirac, reference 2, pp. 266–268.

on a 2-component radial wave function $\psi(r)$. Spherical symmetry permits s to be treated in its coordinate dependence as a scalar function of r alone. All customary considerations relating to angular momentum therefore remain valid (s commutes with j), and j has the same significance and, with the exception noted below, the same eigenvalue spectrum in the present theory as in Dirac's.[6] Choosing the matrix representation of the quantities ϵ and ρ_3 used by Dirac,[6] we find that the eigenvalue equation, (28), leads to the simultaneous equations

$$\psi_2'+\frac{(j+1)}{r}\psi_2+\left[\frac{s^{-1}}{\hbar c}\left(-mc^2+\frac{Ze^2}{r}\right)-\frac{E'}{\hbar c}\right]\psi_1=0, \quad (30a)$$

$$-\psi_1'+\frac{(j-1)}{r}\psi_1+\left[\frac{s^{-1}}{\hbar c}\left(mc^2+\frac{Ze^2}{r}\right)-\frac{E'}{\hbar c}\right]\psi_2=0, \quad (30b)$$

for the two components of ψ.

Equation (30) embodies the information about the radial part of the wave function contained in Eqs. (14a), (14b). It remains to determine s by means of Eq. (14c). The zeroth-component relationship may be rendered nugatory by one of the devices mentioned in the discussion of Eq. (23). (Since time is excluded from the problem by consideration of a stationary state, s and ψ can be assumed independent of $X_0=-ct_0$.) Equation (14c) then yields

$$-P_k s^{-1}\psi=\frac{\hbar}{i}\frac{\partial}{\partial X_k}\psi, \quad k=1,2,3. \quad (31)$$

Introducing polar coordinates, $R=(X_1^2+X_2^2+X_3^2)^{\frac{1}{2}}$, $X_1=R\sin\theta\cos\phi$, $X_2=R\sin\theta\sin\phi$, $X_3=R\cos\theta$, and assuming that the wave function is not an explicit function of θ, ϕ, so that

$$\frac{\partial}{\partial X_k}=\left(\frac{\partial R}{\partial X_k}\right)\frac{\partial}{\partial R}=\xi_k(\theta,\phi)\frac{\partial}{\partial R}, \quad k=1,2,3$$

(the ξ_k are direction cosines), we consider an arbitrary fixed direction, specified by θ, ϕ, such that $P_k\equiv P\xi_k$, where P is some constant. The three equations (31) then reduce to the single relation

$$Ps^{-1}\psi=-\frac{\hbar}{i}\frac{\partial}{\partial R}\psi. \quad (32)$$

In order to interpret the action of the operator $(\partial/\partial R)$ on the wave function, we must at least temporatily consider ψ, as well as s, to depend on $\mathbf{R}=(X_1,X_2,X_3)$. Our procedure will be to test the equations for a solution such that $\mathbf{r}$ and $\mathbf{R}$ occur only in the combination $|\mathbf{r}-\mathbf{R}|$. If this is the case, $\partial/\partial R\equiv-\partial/\partial r$, and Eq. (32) becomes

$$Ps^{-1}\psi=\frac{\hbar}{i}\frac{\partial}{\partial r}\psi. \quad (33)$$

To certify the originally-assumed spherical symmetry with respect to the origin of coordinates, it is necessary to impose a particular "initial condition," viz.,

$$\mathbf{R}=(X_1,X_2,X_3)=(0,0,0). \quad (34)$$

Only by this special choice of the initial condition can we assure that ψ and s dependence on r is equivalent to dependence on $|\mathbf{r}-\mathbf{R}|$, as is necessary for self-consistency of the above treatment. There is no doubt about the relativistic invariance of the initial condition that $\mathbf{R}$ shall coincide with the position of the heavy particle, since the latter, being infinitely massive, possesses an unambiguous world line. It will be noted that (a) any other initial condition than Eq. (34) would entail relativistic difficulties, (b) as in classical mechanics, after we have applied the initial condition, thereby specializing the dynamical description to that of a particular event, S (or ψ) no longer exhibits explicit dependence on the X_k. After imposing condition (34), we may replace partial by total differentiation in Eq. (33),

$$s^{-1}\psi=\frac{\hbar}{iP}\frac{d}{dr}\psi. \quad (35)$$

The operand here is the same radial wave function whose components appear in Eq. (30).

Separate equations for the two components of ψ are implied by Eq. (35). The scalar (spin-independent) nature of s establishes the equality of the logarithmic derivatives of these components; hence

$$\psi_2=C\psi_1, \quad C=\text{constant}. \quad (36)$$

Making use of Eqs. (35), (36) to eliminate s^{-1} and ψ_2 from Eq. (30), we obtain from the two parts of that equation

$$\psi_1'\left[C+\frac{imc}{P}-\frac{iZe^2}{cPr}\right]+\psi_1\left[-\frac{E'}{\hbar c}+\frac{(j+1)C}{r}\right]=0, \quad (37a)$$

$$\psi_1'\left[-1-\frac{imcC}{P}-\frac{iZe^2C}{cPr}\right]+\psi_1\left[-\frac{E'C}{\hbar c}+\frac{(j-1)}{r}\right]=0. \quad (37b)$$

Compatibility is secured by proper choice of the constants C and P, viz.,

$$C^2=\frac{j-1}{j+1}, \quad \frac{imcC}{P}=\frac{-j}{j+1}, \quad (38)$$

as may be seen by inspection, following multiplication of Eq. (37a) by C.

In order to specify these constants more definitely it is necessary to establish something about the behavior of the wave function, or of s, at infinity. From "insufficient reason" one may argue for asymptotic constancy of s for very large r, since there is in that region nothing physical to cause departures from constancy. We shall here suppose the constant value to agree with the classical P. B. value of unity, viz.,

$$\lim_{r\to\infty} s(r)=1, \tag{39}$$

since in this way we shall be assured that the Heisenberg postulate ($s=1$) is almost satisfied almost everywhere. By allowing only a local departure from that postulate we shall obtain a "gentlest possible" modification.

Considering Eq. (35) in the limit of large r, and making use of the condition (39), we see that ψ_1 and ψ_2 behave asymptotically like $\exp(iPr/\hbar)$. In view of the bound-state requirement that the wave function vanish at infinity, P must have a positive imaginary part. From Eq. (38), for j real, $|j|>1$, P must be pure imaginary,

$$P=iK^2,\quad K \text{ real}. \tag{40}$$

Equation (38) then yields

$$mc/K^2=|j|/(j^2-1)^{\frac{1}{2}}, \tag{41a}$$

$$C=-[(j-1)/(j+1)]^{\frac{1}{2}}. \tag{41b}$$

From Eq. (38) or (41a) it is apparent that the values $j=\pm1$, allowed in the Dirac theory,[6] must be excluded in the present one, because they do not correspond to bound states. All other nonzero integral values of j are allowed.

Substituting Eqs. (40), (37a) into (35), we find

$$s^{-1}=-\frac{\hbar}{K^2}\left(\frac{\psi_1'}{\psi_1}\right)$$
$$=\frac{\hbar}{K^2}\left[-\frac{E'}{\hbar c}+\frac{(j+1)C}{r}\right]\Big/\left[C+\frac{mc}{K^2}-\frac{Ze^2}{cK^2r}\right]. \tag{42}$$

The allowed energy values of the system are determined by again applying condition (39), viz.,

$$\lim_{r\to\infty} s(r)=1=\frac{C+(mc/K^2)}{-E'/K^2c}.$$

They are

$$E_j'/mc^2=-1/j,\quad j=\pm2,\pm3,\cdots. \tag{43}$$

The corresponding eigenfunctions are obtained by integrating Eq. (42), viz.,

$$\psi_{1j}=A_je^{-\alpha r}(r-\beta)^\gamma, \tag{44}$$

where

$$\alpha\equiv\frac{mc}{\hbar}\frac{(j^2-1)^{\frac{1}{2}}}{|j|},\quad \beta\equiv\frac{jZe^2}{mc^2},$$
$$\gamma\equiv j^2-1-\frac{Ze^2}{\hbar c}(j^2-1)^{\frac{1}{2}}\left(\frac{j}{|j|}\right),$$

with the second component, ψ_{2j}, given by Eqs. (36), (41b). A_j is a normalization constant. The final form of s,

$$s_j(r)=\left(1-\frac{Zj}{r}\frac{e^2}{mc^2}\right)\Big/\left(1-\frac{|j|(j^2-1)^{\frac{1}{2}}}{r}\frac{\hbar}{mc}\right), \tag{45}$$

is state-dependent, as indicated by our preliminary discussion.

The eigenvalue spectrum, Eq. (43), is a highly unusual one. It lies entirely within what is referred to by Pauli[7] as the Zwischengebiet, i.e., the region of imaginary particle momentum (real energy between mc^2 and $-mc^2$). The particle is able to enter this energy region only because the Heisenberg postulate is locally violated. Preliminary investigation, presented in an auxiliary report,[8] suggests the possibility of using such solutions in the description of nuclei. Our present objective, to exemplify a Class III solution of Eq. (14), and thereby to establish that the proposed generalization of quantum mechanics is not devoid of new results, has been accomplished by Eqs. (44), (45).

3. DISCUSSION

An example has been given of a type of mechanics that is in predictive agreement with both classical and ordinary quantum mechanics, yet is a true generalization, in the sense of comprehending additional solutions. The proposed theory rests on a perfected formal correspondence with classical mechanics. Heuristic justification for demanding such an exact correspondence may derive from an hypothecated "relativity of physical size," viz., the form of the equations of motion shall impart no meaning to the absolute largeness or smallness of a mechanical system.

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[7] W. Pauli, reference 5, p. 243.

[8] This auxiliary report has been deposited as Document No. 6257 with the ADI Auxiliary Publications Project, Photoduplication Service, Library of Congress, Washington 25, D. C. A copy may be secured by citing the Document number and by remitting \$1.25 for photoprints or \$1.25 for 35-mm microfilm. Advance payment is required. Make checks or money orders payable to: Chief, Photoduplication Service, Library of Congress.