To a given order r is considerably simpler than T. There are no singular integrals appearing in the expression for r; the singularities in $T^{(3)}$ being canceled by $D^{(2)}T^{(1)}+T^{(2)}D^{(1)}$ etc.

P-wave $T=1$:

$$T_{11}=-(1/16\pi^2)(q/\omega)[5(\lambda/4\pi)^2A_2^1+(5/4\pi)(\lambda/4\pi)^3(7A_3^1-4B_3^1)],$$

where

$$A_2^1(\omega)=-\left(\frac{\mu}{q}\right)^2\left[\left(\frac{q}{\mu}\right)^2-1+2\frac{\omega}{\mu}\left(\frac{\sinh^{-1}(q/\mu)}{q/\mu}\right)-\left[1+2\left(\frac{q}{\mu}\right)^2\right]\left(\frac{\sinh^{-1}(q/\mu)}{q/\mu}\right)^2\right],$$

$$A_3^1(\omega)=\int_{-1}^{1}\alpha d\alpha\left\{2\left[\left(1+\frac{2\mu^2}{q^2(1-\alpha)}\right)^{\frac{1}{2}}\right]\times\sinh^{-1}\left[\left(\frac{q^2(1-\alpha)}{2\mu^2}\right)^{\frac{1}{2}}\right]-2-c_0\right\}^2,$$

$$B_3^1(\omega)=-\frac{2}{q^2}\int_{\mu^2}^{\infty}d\omega'^2\frac{q'}{\omega'}A_2^0(\omega')\times\left\{\left(1+\frac{2\omega'^2}{q^2}\right)\ln\left(1+\frac{q^2}{\omega'^2}\right)-2\right\}.$$

For the P-wave, to order λ^3, $r=T$ since $T=0$ in first order in λ.

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Foldy Transformation in the Pion-Hyperon System*

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A unitary transformation, which plays the same role as the Foldy transformation in the pion-nucleon system, is constructed for the case where the pion interacts with both Σ and Λ hyperons through γ_5 couplings. The transformation function and the transformed Hamiltonian are very similar to those of the Foldy transformation, in spite of the complexity of our system in isotopic spin space. The application to practical problems is not considered in this paper.

WE consider a system where the pion interacts with both Σ and Λ hyperons through charge independent γ_5 couplings. The interaction Hamiltonian is

$$H_I=H_{\Sigma\Sigma\pi}+H_{\Sigma\Lambda\pi}, \tag{1}$$

with

$$H_{\Sigma\Sigma\pi}=if_\Sigma\int d\mathbf{x}\ \boldsymbol{\psi}_\Sigma^*\rho_2\times\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\varphi},$$

$$H_{\Sigma\Lambda\pi}=f_\Lambda\int d\mathbf{x}\ (\psi_\Lambda^*\rho_2\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\varphi}+\boldsymbol{\psi}_\Sigma^*\rho_2\psi_\Lambda\cdot\boldsymbol{\varphi}).$$

Here the bold faced letters represent vectors in isotopic spin space.

We wish to find a unitary transformation,

$$H'=e^{iS}(H_{\text{free}}+H_I)e^{-iS}, \tag{2}$$

which eliminates the ρ_2 components completely from the sum $H_I+H_{\text{h.m.}}$, where H_I represents the interaction and $H_{\text{h.m.}}$ the hyperon-mass terms in the free Hamiltonian H_{free}. According to Foldy[1] such a transformation corresponds to a certain rotation around the ρ_1 axis in ρ space. Then, we assume[2]

$$\begin{aligned}S&=S_1+S_2,\\ S_1&=-i\int d\mathbf{x}\ \boldsymbol{\psi}_\Sigma^*\rho_1\times\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\varphi}[\chi(\phi)/\phi],\\ S_2&=\int d\mathbf{x}\ \boldsymbol{\psi}_\Sigma^*\cdot\boldsymbol{\varphi}\rho_1\psi_\Lambda[\omega(\phi)/\phi]+\text{H.c.},\end{aligned} \tag{3}$$

with $\phi=(\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})^{\frac{1}{2}}$. Here χ and ω are odd functions of ϕ only and correspond to the angle of rotation in ρ space. Using formulas (A-6) in the Appendix, we see that

$$\begin{aligned}&e^{iS}[H_{\text{h.m.}}+H_I]e^{-iS}\\ &=-i\int d\mathbf{x}\ \boldsymbol{\psi}_\Sigma^*\rho_2\times\boldsymbol{\psi}_\Sigma(m_\Sigma\sin2\chi-f_\Sigma\phi\cos2\chi)\\ &\quad+\left\{\int d\mathbf{x}\ \boldsymbol{\psi}_\Sigma^*\cdot\boldsymbol{\varphi}\rho_2\phi^{-1}\times[\tfrac{1}{2}(m_\Sigma+m_\Lambda)\sin2\omega-f_\Lambda\phi\cos2\omega]\psi_\Lambda+\text{H.c.}\right\}\\ &\quad+\text{terms proportional to }\rho_3.\end{aligned}$$

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[1] L. L. Foldy, Phys. Rev. 84, 168 (1951). J. M. Berger, L. L. Foldy, and R. K. Osborn, Phys. Rev. 87, 1061 (1952).

[2] In (3) we may add one more independent term;

$$S_3=\int(\boldsymbol{\psi}_\Sigma^*\rho_1\boldsymbol{\varphi})(\boldsymbol{\varphi}\boldsymbol{\psi}_\Sigma)\chi'd\mathbf{x},$$

where χ' is an even function of ϕ. Since this term gives no change in the final results, we omit it for the simplicity.

Hence we get

$$\chi=-\tfrac{1}{2}\tan^{-1}(f_\Sigma\phi/m_\Sigma),\quad \omega=\tfrac{1}{2}\tan^{-1}(f_\Lambda\phi/m), \tag{4}$$

with $m=\frac{1}{2}(m_\Sigma+m_\Lambda)$.

The transformation of the total Hamiltonian (2) is now easily carried out with the help of several formulas in the Appendix. The transformed Hamiltonian is

$$H'=H_{\text{free}}+\sum_{i=1}^{8}H_i, \tag{5}$$

with

$$H_1=\int d\mathbf{x}\,\boldsymbol{\psi}_\Sigma^*\rho_3\boldsymbol{\psi}_\Sigma(m_\Sigma^*-m_\Sigma)+\int d\mathbf{x}\,\psi_\Lambda^*\rho_3\psi_\Lambda[(m_\Lambda^*-m_\Lambda)-\Delta m]+\int d\mathbf{x}\,\phi^{-2}\boldsymbol{\psi}_\Sigma^*\rho_3\cdot\boldsymbol{\varphi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi}_\Sigma(m_\Lambda^*-m_\Sigma^*+\Delta m),$$

$$H_2=\frac{if_\Sigma}{4}\int d\mathbf{x}\left[\frac{m_\Sigma}{(m_\Sigma^*)^2}\boldsymbol{\psi}_\Sigma^*\times\boldsymbol{\psi}_\Sigma\cdot(\boldsymbol{\sigma}\nabla\boldsymbol{\varphi}+\rho_1\boldsymbol{\pi})-\text{H.c.}\right],$$

$$H_3=-\frac{f_\Lambda}{4}\int d\mathbf{x}\left\{\frac{m}{(m_\Lambda^*)^2},\boldsymbol{\psi}_\Sigma^*(\boldsymbol{\sigma}\nabla\boldsymbol{\varphi}+\rho_1\boldsymbol{\pi})\psi_\Lambda\right\}_++\text{H.c.},$$

$$H_4=-\frac{i}{2}\int d\mathbf{x}\{A_\Sigma(\phi^2)\boldsymbol{\psi}_\Sigma^*\times\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\varphi}\times[\boldsymbol{\varphi}\times(\boldsymbol{\sigma}\nabla\boldsymbol{\varphi}+\rho_1\boldsymbol{\pi})]-\text{H.c.}\},$$

$$H_5=\frac{1}{2}\int d\mathbf{x}\{A_\Lambda(\phi^2),\,\boldsymbol{\psi}_\Sigma^*\cdot\boldsymbol{\varphi}\times[\boldsymbol{\varphi}\times(\boldsymbol{\sigma}\nabla\boldsymbol{\varphi}+\rho_1\boldsymbol{\pi})]\psi_\Lambda\}_++\text{H.c.},$$

$$H_6=\frac{i}{2}\int d\mathbf{x}[B(\phi^2)\boldsymbol{\psi}_\Sigma^*\times\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\varphi}\times(\boldsymbol{\alpha}\nabla\boldsymbol{\varphi}+\boldsymbol{\pi})-\text{H.c.}], \tag{6}$$

$$H_7=\frac{1}{2}\int d\mathbf{x}\{C(\phi^2),\,\boldsymbol{\psi}_\Sigma^*\cdot\boldsymbol{\varphi}\times(\boldsymbol{\alpha}\nabla\boldsymbol{\varphi}+\boldsymbol{\pi})\psi_\Lambda\}_++\text{H.c.},$$

$$H_8=\int d\mathbf{x}\sum\nolimits_s\left|\frac{if_\Sigma m_\Sigma}{2(m_\Sigma^*)^2}\boldsymbol{\psi}_\Sigma^*\rho_1\times\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\delta}^s-\frac{f_\Lambda m}{2(m_\Lambda^*)^2}(\boldsymbol{\psi}_\Sigma^*\rho_1\cdot\boldsymbol{\delta}^s\psi_\Lambda+\text{H.c.})-iA_\Sigma(\phi^2)\boldsymbol{\psi}_\Sigma^*\rho_1\times\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\varphi}\times(\boldsymbol{\varphi}\times\boldsymbol{\delta}^s)+[A_\Lambda(\phi^2)\boldsymbol{\psi}_\Sigma^*\cdot\boldsymbol{\varphi}\times(\boldsymbol{\varphi}\times\boldsymbol{\delta}^s)\rho_1\psi_\Lambda+\text{H.c.}]-iB(\phi^2)\boldsymbol{\psi}_\Sigma^*\times\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\delta}^s\times\boldsymbol{\varphi}+C(\phi^2)\times(\boldsymbol{\psi}_\Sigma^*\cdot\boldsymbol{\varphi}\times\boldsymbol{\delta}^s\psi_\Lambda+\text{H.c.})\right|^2,$$

where

$$m_\Sigma^*=m_\Sigma[1+(f_\Sigma\phi/m_\Sigma)^2]^{\frac{1}{2}},$$

$$m_\Lambda^*=m[1+(f_\Lambda\phi/m)^2]^{\frac{1}{2}},\quad \Delta m=\tfrac{1}{2}(m_\Sigma-m_\Lambda)$$

and

$$A_\Sigma(\phi^2)=\phi^{-3}\left(\cos\omega\,\sin\chi-\phi\frac{d\chi}{d\phi}\right)=\frac{f_\Sigma}{2(m_\Sigma^*)^2\phi^2}\times\left\{m_\Sigma^*\left[\left(1+\frac{m}{m_\Lambda^*}\right)^{\frac{1}{2}}\Big/\left(1+\frac{m_\Sigma}{m_\Sigma^*}\right)^{\frac{1}{2}}\right]-m_\Sigma\right\},$$

$$A_\Lambda(\phi^2)=\phi^{-3}\left(\cos\chi\,\sin\omega-\phi\frac{d\omega}{d\phi}\right)=\frac{f_\Lambda}{2(m_\Lambda^*)^2\phi^2}\times\left\{m_\Lambda^*\left[\left(1+\frac{m_\Sigma}{m_\Sigma^*}\right)^{\frac{1}{2}}\Big/\left(1+\frac{m}{m_\Lambda^*}\right)^{\frac{1}{2}}\right]-m\right\},$$

$$B(\phi^2)=\phi^{-2}(1-\cos\omega\,\cos\chi)=\phi^{-2}\{1-\tfrac{1}{2}[(1+m_\Sigma/m_\Sigma^*)(1+m/m_\Lambda^*)]^{\frac{1}{2}}\},$$

$$C(\phi^2)=\phi^{-2}\sin\omega\,\sin\chi=(2\phi^2)^{-1}[(1-m_\Sigma/m_\Sigma^*)(1-m/m_\Lambda^*)]^{\frac{1}{2}}.$$

In (6) we understand the expression like $\boldsymbol{\psi}_\Sigma^*\times\boldsymbol{\psi}_\Sigma(\boldsymbol{\sigma}\nabla\boldsymbol{\varphi}+\rho_1\boldsymbol{\pi})$ is the abbreviation of $\boldsymbol{\psi}_\Sigma^*\times\sigma_i\boldsymbol{\psi}_\Sigma\cdot\nabla_i\boldsymbol{\varphi}+\boldsymbol{\psi}_\Sigma^*\times\rho_1\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\pi}$. In H_8, the $\boldsymbol{\delta}^s$'s are the one-row matrices given in (A-1). In the weak coupling limit, m^*, A, B, and C are reduced to

$$m_\Sigma^*\to m_\Sigma+(f_\Sigma\phi)^2/2m_\Sigma,\quad m_\Lambda^*\to m+(f_\Lambda\phi)^2/2m,$$

$$A_\Sigma\to\frac{1}{16}\frac{f_\Sigma}{m_\Sigma}\left[5\left(\frac{f_\Sigma}{m_\Sigma}\right)^2-\left(\frac{f_\Lambda}{m}\right)^2\right],$$

$$A_\Lambda\to\frac{1}{16}\frac{f_\Lambda}{m}\left[5\left(\frac{f_\Lambda}{m}\right)^2-\left(\frac{f_\Sigma}{m_\Sigma}\right)^2\right],$$

$$B\to\frac{1}{8}\left[\left(\frac{f_\Sigma}{m_\Sigma}\right)^2+\left(\frac{f_\Lambda}{m}\right)^2\right],\quad C\to\frac{f_\Sigma f_\Lambda}{4m_\Sigma m}.$$

We now discuss briefly the various interaction terms in (5). In the static limit H_1 represents the coupling of the S-wave pions with hyperons. The first two terms of H_1 are the so-called spin-independent S-wave pair terms in the weak coupling limit, while the last term depends on the total isotopic spin of the system and has no counter terms in the pion-nucleon system. This term vanishes if the interaction has global symmetry in Gell-Mann's sense. H_2 and H_3 and the well known $\boldsymbol{\sigma}\nabla$ couplings of a P-wave pion. H_4 and H_5 represent rather complicated pv couplings. H_6 and H_7 are the terms corresponding to the $\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\times\boldsymbol{\pi}$ term in the pion-nucleon interaction. The last term in (5) is the so-called direct interaction term.

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APPENDIX

Here, we will briefly discuss the transformation properties of the field operators. First we introduce three matrices $\boldsymbol{\delta}^s$, T^s, and D^{st} in isotopic spin space through

$$\psi_\Sigma^* \cdot \boldsymbol{\varphi}\psi_\Lambda = \psi_{\Sigma i}^* \delta_i^s \psi_\Lambda \varphi_s, \quad \psi_\Sigma^* \times \psi_\Sigma \cdot \boldsymbol{\varphi} = i\psi_{\Sigma i}^* T_{ij}^s \psi_{\Sigma j}\varphi_s,$$
$$(\psi_\Sigma^* \cdot \boldsymbol{\varphi})(\boldsymbol{\varphi}\cdot\psi_\Sigma) = \psi_{\Sigma i}^* D_{ij}^{st}\psi_{\Sigma j}\varphi_s\varphi_t. \quad \text{(A-1)}$$

Further, we define the three operators

$$\mathbf{e} = \boldsymbol{\delta}^s\varphi_s/\phi, \quad R = T^s\varphi_s/\phi, \quad Z = D^{st}\varphi_s\varphi_t/\phi^2. \quad \text{(A-2)}$$

These operators satisfy several simple relations, which very much facilitate the ensuing calculation. The relations are

$$R^2 = 1-Z, \quad R^3 = R, \quad RZ = ZR = 0$$
$$Z = \mathbf{ee}^*, \quad \mathbf{e}^*\mathbf{e} = 1, \quad R\mathbf{e} = \mathbf{e}^*R = 0, \quad R^* = R, \quad Z^* = Z, \quad \text{(A-3)}$$

and

$$\nabla\mathbf{e}^*\cdot\mathbf{e} = \mathbf{e}^*\cdot\nabla\mathbf{e} = 0, \quad \nabla R = Z\cdot\nabla R + \nabla R\cdot Z,$$
$$Z\cdot\nabla Z - \nabla Z\cdot Z = e\cdot\nabla\mathbf{e}^* - \nabla\mathbf{e}\cdot\mathbf{e}^* = R\cdot\nabla R - \nabla R\cdot R. \quad \text{(A-4)}$$

Here the symbol * means the Hermitian conjugate of the relevant operator.

Now let us consider the transformation properties of the field quantities. The operator S is now written simply as

$$S = \int d\mathbf{x}\ \psi_\Sigma^*\rho_1 R\psi_\Sigma\chi + \left[\int d\mathbf{x}\ \psi_\Sigma^*\rho_1\mathbf{e}\psi_\Lambda\omega + \text{H.c.}\right].$$

According to Foldy,[1] we put

$$\psi_\Sigma(\xi) = e^{i\xi S}\psi_\Sigma e^{-i\xi S}, \quad \psi_\Lambda(\xi) = e^{i\xi S}\psi_\Lambda e^{-i\xi S}.$$

Then, these quantities satisfy the following coupled differential equations;

$$\begin{aligned} d\psi_\Sigma(\xi)/d\xi &= -i\rho_1 R\chi\psi_\Sigma(\xi) - i\rho_1\omega\mathbf{e}\psi_\Lambda(\xi), \\ d\psi_\Lambda(\xi)/d\xi &= -i\rho_1\omega\mathbf{e}^*\psi_\Sigma(\xi). \end{aligned} \quad \text{(A-5)}$$

The solution is easily found to be

$$\begin{aligned} \psi_\Sigma(\xi) &= \cos(Z\omega\xi)e^{-i\rho_1\chi R\xi}\psi_\Sigma - i\rho_1\mathbf{e}\sin(Z\omega\xi)\psi_\Lambda, \\ \psi_\Lambda(\xi) &= \cos(\omega\xi)\psi_\Lambda - i\rho_1\mathbf{e}^*\sin(Z\omega\xi)e^{-i\rho_1\chi R\xi}\psi_\Sigma. \end{aligned} \quad \text{(A-6)}$$

Hence we obtain on setting $\xi = 1$;

$$\begin{aligned} e^{iS}\psi_\Sigma e^{-iS} &= \{[Z\cos\omega + (1-Z)\cos\chi] \\ &\quad - i\rho_1 R\sin\chi\}\psi_\Sigma - i\rho_1\mathbf{e}\sin\omega\psi_\Lambda, \\ e^{iS}\psi_\Lambda e^{-iS} &= \cos\omega\psi_\Lambda - i\rho_1\mathbf{e}^*\sin\omega\psi_\Sigma. \end{aligned} \quad \text{(A-7)}$$

The transform of π_i is also obtainable by similar method. As before we put $\pi_i(\xi) = \exp(iS)\pi_i\exp(-iS)$, which satisfies the equation

$$\frac{d\pi_i(\xi)}{d\xi} = i[S,\pi_i(\xi)] = \frac{d}{d\xi}\left[\int d\mathbf{x}\ \psi_\Sigma^*(\xi)\pi_i\psi_\Sigma(\xi) + \int d\mathbf{x}\ \psi_\Lambda^*(\xi)\pi_i\psi_\Lambda(\xi)\right]. \quad \text{(A-8)}$$

The solution is

$$\pi_i(\xi) = \pi_i + \int d\mathbf{x}\,[\psi_\Sigma^*(\xi)\pi_i\psi_\Sigma(\xi) + \psi_\Lambda^*(\xi)\pi_i\psi_\Lambda(\xi)] - \int d\mathbf{x}[\psi_\Sigma^*\pi_i\psi_\Sigma + \psi_\Lambda^*\pi_i\psi_\Lambda]. \quad \text{(A-9)}$$

Finally we get, after some calculations,

$$\begin{aligned} e^{iS}\pi_i e^{-iS} &= \pi_i + \int d\mathbf{x}\ \psi_\Sigma^*\left[i\left(\frac{dR}{d\varphi_i}R - R\frac{dR}{d\varphi_i}\right)(1-\cos\omega\cos\chi)\right. \\ &\quad \left. -\rho_1\left(\frac{dR}{d\varphi_i}\cos\omega\sin\chi + R\frac{d\chi}{d\varphi_i}\right)\right]\psi_\Sigma \\ &\quad + \left\{\int d\mathbf{x}\psi_\Sigma^*\left[-\rho_1\left(\mathbf{e}\frac{d\omega}{d\varphi_i} + \frac{d\mathbf{e}}{d\varphi_i}\sin\omega\cos\chi\right)\right.\right. \\ &\quad \left.\left. -iR\frac{d\mathbf{e}}{d\varphi_i}\sin\omega\sin\chi\right]\psi_\Lambda + \text{H.c.}\right\}. \end{aligned} \quad \text{(A-10)}$$