out that a preliminary value of m of about 8 is too optimistic; based on his values of abundance and cross sections he estimated the value of k to be a small fraction. In order that no neutrons are lost by spontaneous decay it is necessary that the reaction (capture and fission) lifetime of neutron be much shorter than its spontaneous decay lifetime. As the expansion of the supernova shell goes on, its density decreases and the time required for a neutron to react with heavy nuclides increases. Fowler[9] pointed out that after several hundred days the density will be so small that spontaneous decay will take place before any reaction. This situation is not improved even assuming the exploding material to be confined in a shell.[10] It appears that the necessary conditions are not realized in supernova type I and thus other explanations perhaps are necessary.[11] In view of the plausibility of the fission chain reaction and its possible occurrence in other circumstances it was decided to publish these results and point out the difficulties regarding the supernova type I.

[9] W. A. Fowler (private communication).

[10] J. Greenstein pointed out some difficulties of such an assumption in a private communication although a calculation shows that a shell of a thickness of the sun-radius is stable against thermal diffusion and leak-proof for the neutrons.

[11] E. Anders has now published a paper in Astrophys. J. **129**, 327 (1959) considering Fe^{59}, instead of Cf^{254}, to be responsible for the exponential decline of the supernova light curve.

Another source of fission contributing to the chain reaction may be mentioned here. The odd A, very heavy nuclides ($A>254$), such as E^{255}, have long half-life against spontaneous fission. After capturing a neutron and converting itself to an even-even nuclide (by β decay if necessary), such a nuclide will become one with extremely short spontaneous fission half-life and therefore will undergo fission "immediately." Thus the neutron capture processes of such very heavy odd-A nuclides also contribute to the fission chain reaction just as does the induced fission. However, the amount due to this source is likely to be small.

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Two-Nucleon Stripping Process*

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A new expression is derived for the differential cross section of processes in which two nucleons are captured from an incident alpha particle or similar projectiles. The formula derived is compared with a similar one previously obtained together with some experimental data on the $O^{16}(d,\alpha)N^{14}$ reaction. Fairly good agreement is observed.

It is well known that the theories of the stripping reactions[1] of deuterons were successful in describing the main features of many such reactions, especially the experimentally observed forward peaking. On the other hand, some deuteron stripping reactions showed a backward peaking of the stripped particles distribution, which was explained in terms of the heavy-particle stripping process, as developed by Owen and Madansky.[2]

Quite recently, however, experimental results have accumulated on reactions in which two nucleons are stripped from the projectile and captured by the target nucleus. This is the case for (α,d) and (He^3,p) reactions[3] and data are also available for their inverse. These cases show also the forward and backward peaking characterizing the deuteron stripping process. This may suggest that in the two-nucleon stripping reaction, a mechanism similar to that responsible for deuteron stripping may take place. An expression for the differential cross section of this process, based on the first

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† On leave of absence from the Faculty of Science, Cairo University, Egypt, U.A.R.

[1] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951); A. B. Bhatia, K. Huang, R. Huby, and H. C. Newns, Phil. Mag. **43**, 485 (1952); R. Huby, Proc. Roy. Soc. (London) **A215**, 385 (1952); F. L. Friedman and W. Tobocman, Phys. Rev. **92**, 93 (1953); W. Tobocman, Phys. Rev. **94**, 1655 (1954); R. G. Thomas, Phys. Rev. **100**, 25 (1955).

[2] L. Madansky and G. E. Owen, Phys. Rev. **99**, 1608 (1955); G. E. Owen and L. Madansky, Phys. Rev. **105**, 1766 (1957).

[3] T. S. Green and R. Middleton, Proc. Phys. Soc. (London) **A69**, 28 (1956); D. A. Bromley, E. Almzvist, H. E. Gove, A. E. Litherland, E. B. Paul, and A. J. Ferguson, Phys. Rev. **105**, 957 (1957); R. L. Johnston, H. D. Holmgren, E. A. Wolicki, and E. G. Illsley, Phys. Rev. **109**, 884 (1958); J. B. Marion and G. Weber, Phys. Rev. **102**, 1355 (1956).

Born approximation was derived, predicting the observed forward and backward peakings.[4]

In the simple theory of deuteron stripping, such as the (d,p) reaction, the interaction $V(|\mathbf{r}_p-\mathbf{r}_n|)$ outside the target nucleus appears in Butler's treatment as the quantity responsible for the stripping process. Here $\mathbf{r}_p$ and $\mathbf{r}_n$ refer to the coordinates of the constituents of the deuteron. On the other hand, in the first-Born-approximation method (Bhatia, Huby et al.) the interaction potential $V(|\mathbf{r}_n-\boldsymbol{\xi}|)$ is taken to be responsible for the stripping process, with $\boldsymbol{\xi}$ referring to the collective coordinates of the target nucleus. The particles in the core, forming the target nucleus, are not treated separately, some of the aspects of the actual many-body problem being thus left out of account.

In his analysis of the deuteron stripping theory, Gerjuoy[5] has shown that in the first Born approximation method for the (d,p) stripping reaction, the potentials $V(|\mathbf{r}_n-\mathbf{r}_p|)$ and $V(|\mathbf{r}_n-\boldsymbol{\xi}|)$ are equivalent in calculations of the cross section, in one of the several developments based on different sets of assumptions. In the theory of the two-nucleon stripping process mentioned above,[4] the interactions $V(|\mathbf{r}_n-\boldsymbol{\xi}|)+V(|\mathbf{r}_p-\boldsymbol{\xi}|)$ are used, with $\mathbf{r}_p$ and $\mathbf{r}_n$ standing for the coordinates of the neutron and proton captured from the projectile, and $\boldsymbol{\xi}$ referring to the collective coordinates of the target nucleus.

By considering the (α,d) stripping reactions on similar lines to those given by Gerjuoy[5] for the (d,p), it can be shown that in the first Born approximation, the matrix elements of the potentials

$$V(|\mathbf{r}_n-\boldsymbol{\xi}|)+V(|\mathbf{r}_p-\boldsymbol{\xi}|)$$

and

$$V\equiv V(|\mathbf{r}_d-\mathbf{r}_n|)+V(|\mathbf{r}_d-\mathbf{r}_p|)+V(|\mathbf{r}_n-\mathbf{r}_p|)$$

are equivalent in calculations of cross sections for the (α,d) stripping reactions. Here $\mathbf{r}_d$ refers to the coordinate of the outgoing deuteron (see Fig. 1). The precise meaning of the equivalence of the two potentials is presented in the Appendix to the present note.

Below, the latter form of the potential, referred to as V, will be used in the first-Born-approximation matrix element for the (α,d) reaction, together with the delta-function representation for the three interparticle potentials. This representation has been successfully employed in deuteron stripping theories.[6] It will be seen that the differential cross section σ may be represented in a compact form suitable for comparison with experimental data.

In the center-of-mass system the cross section may be written

$$d\sigma=(1/4\pi^2\hbar^4)M_d^*M_\alpha^*(k_d/k_\alpha)\sum|I|^2. \qquad (1)$$

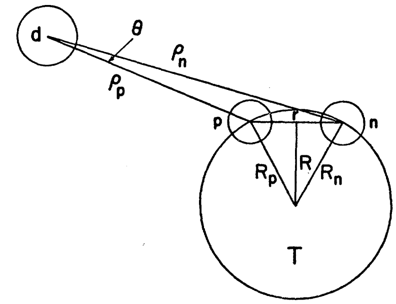

FIG. 1. The two-nucleon stripping process. The proton p and neutron n are captured by the target nucleus T from an incident α particle. The outgoing deuteron is shown as d.

Here the summation sign implies the sum over final states and average over the initial state, and

$$I=\int d\mathbf{r}_p d\mathbf{r}_n d\mathbf{r}_d d\boldsymbol{\xi}\, \exp(-i\mathbf{k}_d\cdot\mathbf{r}_d')\Psi_{I_f}{}^{M_f*}(\mathbf{r}_p,\mathbf{r}_n,\boldsymbol{\xi})V \times\exp(i\mathbf{k}_\alpha\cdot\mathbf{r}_\alpha)\varphi_\alpha(\mathbf{r}_p,\mathbf{r}_n,\mathbf{r}_d)\psi_{I_i}{}^{M_i}(\boldsymbol{\xi}), \qquad (2)$$

while $\mathbf{r}_p$, $\mathbf{r}_n$, $\mathbf{r}_d$ are the vector displacements of the proton, neutron and deuteron. The quantity

$$\mathbf{r}_d'=\mathbf{r}_d-[(M_n+M_p)/M_f]\mathbf{R},$$

where $\mathbf{R}=\frac{1}{2}(\mathbf{r}_p+\mathbf{r}_n)$, is the vector displacement of the deuteron with respect to the center of the final nucleus. The notation $M_d^*=M_dM_f/(M_d+M_f)$, $M_\alpha^*=M_\alpha M_i/(M_\alpha+M_i)$, with M_i and M_f standing for the masses of the initial and final nuclei will be used and the customary symbols $k_d=(2M_d^*E_d)^{\frac{1}{2}}/\hbar$ and $k_\alpha=(2M_\alpha^*E_\alpha)^{\frac{1}{2}}/\hbar$ with E_d and E_α standing for the energies of the deuteron and α particle in the center-of-mass system will be employed.

The internal wave function for the α particle will be denoted by $\varphi_\alpha(\mathbf{r}_p,\mathbf{r}_n,\mathbf{r}_d)$. This will be approximated by the Gaussian form $\exp(-\gamma^2\sum_{i<j}r_{ij}^2)$. Here γ is a constant, r_{ij} is the distance between the two particles i and j in the α particle, and summation extends over i and j from 1 to 4 with the condition that $i<j$.

The total exponent of e in the matrix element (2) is given by

$$\begin{aligned}X&=-i\mathbf{k}_d\cdot\mathbf{r}_d'+i\mathbf{k}_\alpha\cdot\mathbf{r}_\alpha\\ &=-i\mathbf{k}_d\cdot\{\mathbf{r}_d-[(M_n+M_p)/M_f]\mathbf{R}\}\\ &\qquad+i\mathbf{k}_\alpha\cdot\tfrac{1}{4}(\mathbf{r}_p+\mathbf{r}_n+2\mathbf{r}_d)\\ &=i\mathbf{Q}\cdot\mathbf{R}+\tfrac{1}{2}i\mathbf{K}\cdot\boldsymbol{\rho}_p+\tfrac{1}{2}i\mathbf{K}\cdot\boldsymbol{\rho}_n,\end{aligned} \qquad (3)$$

where

$$\mathbf{r}_d=\mathbf{r}_p+\boldsymbol{\rho}_p=\mathbf{r}_n+\boldsymbol{\rho}_n=\mathbf{R}+\boldsymbol{\rho},$$

$$\mathbf{Q}=k_\alpha-(M_i/M_f)\mathbf{k}_d,\quad \mathbf{K}=\tfrac{1}{2}\mathbf{k}_\alpha-\mathbf{k}_d.$$

Similarly one obtains for the exponent of the internal wave function of the α particle

$$-\gamma^2\sum_{i<j}^{4}r_{ij}^2=-\gamma^2(2a^2+3\rho_p^2+3\rho_n^2-2\rho_p\rho_n\cos\theta), \qquad (4)$$

where a is the distance between the proton and neutron which will constitute the outgoing deuteron after their detachment and θ is the angle between $\boldsymbol{\rho}_p$ and $\boldsymbol{\rho}_n$.

[4] M. el Nadi, Proc. Phys. Soc. (London) **A70**, 62 (1957); M. el Nadi and M. el Khishin, Proc. Phys. Soc. (London) **73**, 705 (1959).

[5] E. Gerjuoy, Phys. Rev. **91**, 645 (1953).

[6] G. Breit, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41, Part 1, p. 327.

The wave function of the residual nucleus $\psi_{I_f}{}^{M_f}(\xi,\mathbf{r}_p,\mathbf{r}_n)$ will now be expanded in terms of the wave function of the target nucleus $\Psi_{I_i}{}^{M_i}(\xi)$ and that of the captured particles as follows:

$$\Psi_{I_f}{}^{M_f}(\xi,\mathbf{r}_p,\mathbf{r}_n)=\sum_{I_i,L} A_{I_i,L}\Psi_{I_i}{}^{M_i}(\xi)f_{l_pl_n}(\mathbf{r}_p\mathbf{r}_n)$$
$$\times\sum_M\begin{pmatrix} I_i & L & I_f \\ M_i & M & M_f\end{pmatrix}\Psi_{I_i}{}^{M_i}(\omega_\xi)\mathcal{Y}_L{}^M(\Omega), \quad (5)$$

with

$$\mathcal{Y}_L{}^M(\Omega)=\sum_{m_pm_n}\begin{pmatrix} l_p & l_n & L \\ m_p & m_n & M\end{pmatrix}Y_{l_p}{}^{m_p}(\omega_p)Y_{l_n}{}^{m_n}(\omega_n), \quad (6)$$

where the two bracket symbols denote the Wigner vector-addition coefficients, $f_{l_pl_n}(r_pr_n)$ is the product of the radial parts of the wave functions of the bound proton and neutron, ω_ξ, ω_p, and ω_n denote in each case collectively the polar angles defining the directions of the vectors ξ, $\mathbf{r}_p$, and $\mathbf{r}_n$, respectively.

The coefficient $A_{I_i,L}$ in (5) gives the probability amplitude that the final nucleus is composed of a core with angular momentum I_i and a proton-neutron system with total orbital angular momentum L. It is proportional to the reduced width amplitude. The integral over ξ in (2) gives

$$\int\Psi_{I_f}{}^{M_f*}(\xi,\mathbf{r}_p,\mathbf{r}_n)\Psi_{I_i}{}^{M_i}(\xi)d\xi$$
$$=\sum_{LM}A_{I_i,L}f_{l_pl_n}(r_pr_n)\begin{pmatrix} I_i & L & I_f \\ M_i & M & M_f\end{pmatrix}\mathcal{Y}_L{}^{M*}(\Omega). \quad (7)$$

Using Eqs. (3), (4), and (7) one finds for the matrix element (2)

$$I=\sum_{LM}A_{I_i,L}f_{l_pl_n}(R_0)\begin{pmatrix} I_i & L & I_f \\ M_i & M & M_f\end{pmatrix}$$
$$\times\int\exp\{i\mathbf{Q}\cdot\mathbf{R}+\tfrac{1}{2}i\mathbf{K}\cdot(\boldsymbol{\varrho}_p+\boldsymbol{\varrho}_n)-\gamma^2(2a^2+3\rho_p{}^2$$
$$+3\rho_n{}^2-2\rho_p\rho_n\cos\theta)\}\mathcal{Y}_L{}^{M*}(\Omega)d\boldsymbol{\varrho}_pd\boldsymbol{\varrho}_nd\Omega. \quad (8)$$

In the matrix element (8), R_0 is substituted for $|\mathbf{r}_p|$ and $|\mathbf{r}_n|$, where R_0 is the nuclear radius. This follows (Bhatia, Huby et al.) from the assumption that stripping takes place at the nuclear surface. The integration over Ω is readily performed by means of

$$\int\exp(i\mathbf{Q}\cdot\mathbf{R})\mathcal{Y}_L{}^{M*}(\Omega)d\Omega$$
$$=i^L[4\pi(2L+1)]^{\frac{1}{2}}j_L(QR_0)\delta_{M,0}, \quad (9)$$

where j_L is the spherical Bessel function of order L.

The evaluation of the integral in (8) over $\boldsymbol{\varrho}_p$ and $\boldsymbol{\varrho}_n$, is greatly facilitated by the assumption of the δ-function interaction potential

$$-V_0[\delta(|\boldsymbol{\varrho}_n|)+\delta(|\boldsymbol{\varrho}_p|)+\delta(|\boldsymbol{\varrho}_p-\boldsymbol{\varrho}_n|)], \quad (10)$$

where V_0 is a constant.[7]

Now, since the integrand in (8) is symmetrical with respect to $\boldsymbol{\varrho}_p$ and $\boldsymbol{\varrho}_n$, the first two terms on the right-hand side of (1) give equal contributions. Using the term $-V_0\delta(|\boldsymbol{\varrho}_n|)$ in the integral (8), one obtains for the integral over $\boldsymbol{\varrho}_p$

$$-V_0\int_0^\infty\exp(i\cdot\mathbf{K}\cdot\boldsymbol{\varrho}_p/2-3\gamma^2\rho_p{}^2)d\boldsymbol{\varrho}_p$$
$$=-(\pi^{\frac{3}{2}}V_0/3\sqrt{3}\gamma^3)\exp(-K^2/48\gamma^2). \quad (11)$$

Similarly, using the third term on the right-hand side of (1), i.e., $-V_0\delta(|\boldsymbol{\varrho}_p-\boldsymbol{\varrho}_n|)=-V_0\delta(|\mathbf{r}|)$, in the integral over ρ, one gets

$$-V_0\int_0^\infty\exp(i\mathbf{K}\cdot\boldsymbol{\varrho}_p-4\gamma^2\rho_p{}^2)d\boldsymbol{\varrho}_p$$
$$=-(\pi^{\frac{3}{2}}V_0/8\gamma^3)\exp(-K^2/16\gamma^2). \quad (12)$$

Thus the complete expression for the matrix element (8) becomes

$$I\simeq\sum_L A_{I_i,L}f_{l_pl_n}(R_0)\begin{pmatrix} I_i & L & I_f \\ M_i & 0 & M_f\end{pmatrix}$$
$$\times i^L(2L+1)^{\frac{1}{2}}[3.07\exp(-K^2/48\gamma^2)$$
$$+\exp(-K^2/16\gamma^2)]j_L(QR_0). \quad (13)$$

Squaring (13), substituting in (1), summing over the final and averaging over the initial states, one gets for the differential cross section in the center-of-mass system

$$d\sigma\cong[3.07\exp(-K^2/48\gamma^2)+\exp(-K^2/16\gamma^2)]^2$$
$$\times f_{l_pl_n}{}^2(R_0)\sum_L|A_{I_i,L}|^2j_L{}^2(QR_0). \quad (14)$$

The angular dependence enters through both Q and K, as

$$Q^2=[k_\alpha-(M_i/M_f)k_d]^2+4(M_i/M_f)k_\alpha k_d\sin^2(\tfrac{1}{2}\theta),$$
$$K^2=(\tfrac{1}{2}k_\alpha-k_d)^2+2k_\alpha k_d\sin^2(\tfrac{1}{2}\theta).$$

[7] For simplicity of calculation the coefficients of the three δ functions have been assumed to be the same. There is no compelling reason for believing that the best representation of actual conditions is obtained in this manner. If the relative spin orientations of the two protons and the two neutrons are taken to be the same as in the ground state of the alpha particle and if one employs the Majorana-Heisenberg mixture $(1-g)P^M+gP^H$ for the interaction potential, then one obtains in fact a somewhat smaller effective interaction between p and n than between p and d, the ratio of the latter to the former being $\sim$1.5. It is not certain, however, th atthis method of estimating the ratio is sufficiently complete and it is also not clear that the difference between using different coefficients of the δ functions in Eq. (10) cannot be compensated for in its effect on the angular distribution by a suitable adjustment of the value of the nuclear radius R. The simplifying assumption of equal coefficients has been used, therefore, in the present note. This problem does not, of course, arise in the stripping of a three-body projectile; e.g., in the (He^3,p) reaction. The writer is indebted to Professor G. Breit for drawing his attention to the possible importance of the effect discussed in this footnote.

The quantity inside the first parentheses of (14) gives the form factor for the (α,d) reaction.

Equation (14) can also be used for (He^3,p), (T,p) or (T,n) reactions with the following modifications: The form factor in (14) is replaced by $[2\exp(-K^2/32\gamma^2)+\exp(-K^2/8\gamma^2)]^2$.

$$Q^2=[k_h-(M_i/M_f)k_p]^2+4(M_i/M_f)k_hk_p\sin^2(\tfrac{1}{2}\theta),$$
$$K^2=(\tfrac{1}{3}k_h-k_p)^2+\tfrac{4}{3}k_hk_p\sin^2(\tfrac{1}{2}\theta).$$

In the derivation of Eq. (14), the spins of the particles were neglected. If this is taken into account, then the following selection rules, associated with the matrix element for the general case, apply

$$\mathbf{I}_f=\mathbf{I}_i+\mathbf{L}+\tfrac{1}{2}+\tfrac{1}{2},$$

or

$$|\mathbf{I}_f-\mathbf{I}_i|-1\leq l_n+l_p\leq J_f+J_i+\tfrac{1}{2}+\tfrac{1}{2},$$

where

$$\mathbf{L}=\mathbf{l}_p+\mathbf{l}_n,$$

and I_i and I_f are the spins of the initial and final nuclei, respectively. Also the conservation of parity implies that the total orbital angular momentum L must be even or odd according as the parities of the initial and final nuclei are the same or different.

In Eq. (14), there is no interference between different values of L, so that fitting with experimental data can be accomplished by one or more values of L.

Comparing the expression (14) with the previously derived expression,[4] one observes the simplicity of the present formula as regards to its fitting to experimental data. Although the two formulas are basically the same, the present form combines the different summations, expressed in the earlier formula, in one or more simple terms depending on the possible values of L.

To compare the formula[8] (14) with the experimental data, the observations of Green and Middleton[3] on the reaction $\mathrm{O}^{16}(d,\alpha)\mathrm{N}^{14}$, for the ground level transition were considered. The ground level of N^{14} is $I=1^+$ and that of O^{16} is $I=0^+$, so that there is no change of parity in this transition. Hence L is even. Assuming the particles to be picked up in the $p_{\frac{1}{2}}$ state, the values of L are limited to 0 and 2 only.

The value of the constant γ defining the Gaussian wave function of the α particle is given by $(1/\gamma)=4.5\times10^{-13}$ cm as this is found to give reasonable values for the binding energies of the α particle and the triton.[9]

Figure 2 shows the theoretical curve calculated, according to formulas (14) above, and to the corresponding formula in reference 4, both for the same value of the radius 6.0×10^{-13} cm. In fitting formula (14) to the experimental data, the following ratio for the coefficients A_2 and A_0 was obtained:

$$|A_2/A_0|^2=10/3.$$

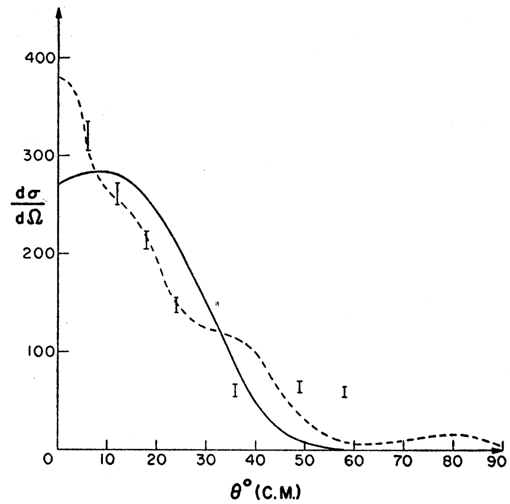

FIG. 2. Comparison of theory and experiment for $\mathrm{O}^{16}(d,\alpha)\mathrm{N}^{14}$. The dashed curve is calculated by means of Eq. (14) of the text while the solid curve is obtained by the corresponding formula of reference 4. The experimental points are from Green and Middleton.

The difference between the two theoretical formulas, as shown in Fig. 2, may be attributed to the fact that in the earlier work, the differential cross section was expressed as an infinite series and that only the first few terms of this were calculated. Moreover, in the present calculations, a different value for the constant γ of the α-particle wave function is used.

Satchler and Sawicki[10] have recently shown, that when deuterons are stripped by spheroidal nuclei, some additional selection rules are implied, which can prevent the mixing of different angular momentum quantum numbers L in the cross section for some particular nuclei.

As in the formula (14) above for the (α,d) reaction, summation is implied on all the possible values of L, it may be of interest to consider if this mixing is limited when the α or the He^3 particles are stripped by spheroidal nuclei. The cross section is still given by Eqs. (1) and (2) with the following expressions for the initial and final nuclei:

$$\Psi_i=(2I_i+1/16\pi^2)^{\frac{1}{2}}[\chi(\Omega_i;i')\mathfrak{D}_{M_iK_i}{}^{I_i}(\theta_i)+(-)^{I_i-l_i}\chi(-\Omega_i;i')\mathfrak{D}_{M_i,-K_i}{}^{I_i}(\theta_i)]\varphi_0(\xi),\quad(15)$$

and

$$\Psi_f=(2I_f+1/16\pi^2)^{\frac{1}{2}}[\chi(\Omega_f;i'p'n')\mathfrak{D}_{M_fK_f}{}^{I_f}(\theta_i)+(-)^{I_f-l_f}\chi(-\Omega_f;i'p'n')\mathfrak{D}_{M_f,-K_f}{}^{I_f}(\theta_i)]\varphi_0(\xi),\quad(16)$$

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[8] In a recent publication [S. Hinds and R. Middleton, Proc. Phys. Soc. (London) **74**, 196 (1959)] of further experimental data on $\mathrm{Be}^9(\mathrm{He}^3,p)\mathrm{B}^{11}$, mention was made of some work, under publication, by H. C. Newns, in which the differential cross section for the two-nucleon stripping process is given by

$$d\sigma(\theta)\cong\Sigma_L|A(L)j_L(QR_0)|^2.$$

Apart from the form factor given in (14), the formulas are of the same form.

[9] T. Muto and T. Sebe, Progr. Theoret. Phys. (Kyoto) **18**, 621 (1957).

[10] J. Sawicki, Nuclear Phys. **6**, 575 (1958); S. Yoshida, Progr. Theoret. Phys. (Kyoto) **12**, 141 (1954); G. R. Satchler, Phys. Rev. **97**, 1416 (1955); G. R. Satchler, Ann. Phys. **3**, 275 (1958); D. A. Bromley, H. E. Gove, and A. E. Litherland, Can. J. Phys. **35**, 1057 (1957).

where $\varphi_0(\xi)$ is the spinless core wave function of the vibrational ground state, $\chi(\Omega_i;i)$ the wave function of the particles outside the core, i' denotes the coordinates of these particles in the intrinsic system, $\mathfrak{D}_{MK}{}^I(\theta)$ are the usual rotational matrices depending on the Eulerian angles θ_i, and K represents the component of the total angular momentum I along the nuclear symmetry axis. The quantity K is assumed to be nonnegative.

The wave function $\chi(\Omega_f;i'p'n')$ may now be written down as

$$\chi(\Omega_f;i'p'n')=\chi(\Omega_i;i')\sum_{\Omega_n\Omega_p}\begin{pmatrix} l_p & l_n & l \\ \Omega_p & \Omega_n & \Omega\end{pmatrix}\times\chi^{l_p}(\Omega_p;p')\chi^{l_n}(\Omega_n;n'). \quad (17)$$

Using (15), (16), and (17) the overlap integral becomes

$$\int d\xi d\mathbf{i}'\,\Psi_f{}^*\Psi_i =[(2I_i+1)(2I_f+1)/(16\pi^2)^2]^{\frac{1}{2}}\sum_{\Omega_p\Omega_n}\begin{pmatrix} l_p & l_n & l \\ \Omega_p & \Omega_n & \Omega\end{pmatrix}$$
$$\times\{\chi^*(\Omega_n;n')\chi^*(\Omega_p;p')\mathfrak{D}_{M_fK_f}{}^{I_f*}(\theta_i)\mathfrak{D}_{M_iK_i}{}^{I_i}(\theta_i)$$
$$+(-)^{I_f-I_i-l}\chi^*(-\Omega_n;n')\chi^*(-\Omega_p;p')$$
$$\times\mathfrak{D}_{M_f,-K_f}{}^{I_f*}(\theta_i)\mathfrak{D}_{M_i,-K_i}{}^{I_i}(\theta_i)\}. \quad (18)$$

The wave functions of the captured nucleons $\chi(\Omega_n;n')$ and $\chi(\Omega_p;p')$ will now be transformed from the intrinsic nuclear system to the fixed system by means of

$$\chi(\Omega_n;n')=R(r_n)\textstyle\sum_{m_n}\mathfrak{D}_{m_n\Omega_n}{}^{l_n}(\theta_i)Y_{l_n}{}^{m_n}(\omega_n),$$
$$\chi(\Omega_p;p')=R(r_p)\textstyle\sum_{m_p}\mathfrak{D}_{m_p\Omega_p}{}^{l_p}(\theta_i)Y_{l_p}{}^{m_p}(\omega_p). \quad (19)$$

Inserting (18) and (19) in the reaction amplitude I, integrating over θ_i, and summing over Ω_p and Ω_n, it is found that

$$I=\frac{1}{4\pi^2}[(2I_i+1)/(2I_f+1)]^{\frac{1}{2}}\sum_{m_n,m_p,L}(-)^{m_n+m_p+K_f-K_i}$$
$$\times\begin{pmatrix} l_n & l_p & L \\ m_n & m_p & \end{pmatrix}\begin{pmatrix} L_i & L & I_f \\ K_i & \Omega & K_f\end{pmatrix}\begin{pmatrix} I_i & L & I_f \\ M_f & - & M_i\end{pmatrix}$$
$$\times\int d\mathbf{r}_p d\mathbf{r}_n d\mathbf{r}_d R(r_n)R(r_p)Y_{l_n}{}^{m_n*}(\omega_n)Y_{l_p}{}^{m_p*}(\omega_p)$$
$$\times\exp[i\mathbf{Q}\cdot\mathbf{R}+i\mathbf{K}\cdot(\boldsymbol{\varrho}_p+\boldsymbol{\varrho}_n)/2 -\gamma^2(2a^2+3\rho_n{}^2+3\rho_p{}^2-2\rho_p\rho_n\cos\theta)].$$

Using the formula

$$Y_{l_n}{}^{m_n}(\omega_n)Y_{l_p}{}^{m_p}(\omega_p)=\sum_{L'M'}\begin{pmatrix} l_n & l_p & L' \\ m_n & m_p & M'\end{pmatrix}\mathcal{Y}_{L'}{}^{M'}(\Omega),$$

in the above integral and proceeding in the same manner as before, the following expression is obtained:

$$I\simeq F_{l_nl_p}(R_0)[3.07\exp(-K^2/48\gamma^2)+\exp(-K^2/16\gamma^2)]$$
$$\times\textstyle\sum_L(-)^{K_f-K_i}\begin{pmatrix} I_i & L & I_f \\ K_i & \Omega & K_f\end{pmatrix}\begin{pmatrix} I_i & L & I_f \\ M_i & 0 & M_f\end{pmatrix}$$
$$\times(i)^L(2L+1)^{\frac{1}{2}}j_L(QR_0). \quad (20)$$

Squaring (20), summing over M_f and averaging over M_i, one gets for the differential cross section

$$d\sigma\simeq F_{l_pl_n}(R_0)[3.07\exp(-K^2/48\gamma^2)+\exp(-K^2/16\gamma^2)]^2$$
$$\times\textstyle\sum_L\left|\begin{pmatrix} I_i & L & I_f \\ K_i & \Omega & K_f\end{pmatrix}\right|^2 j_L{}^2(QR_0). \quad (21)$$

This is similar to the expression (14) derived above, except for the term

$$\left|\begin{pmatrix} I_i & L & I_f \\ K_i & \Omega & K_f\end{pmatrix}\right|^2,$$

which implies, when the spins of the captured particles are taken into account, the following selection rules:

$$|\mathbf{I}_f-\mathbf{I}_i|\leq|\mathbf{L}+\tfrac{1}{2}+\tfrac{1}{2}|\leq I_f+I_i,$$
$$K_f=K_i+\Omega',\quad|\mathbf{L}+\tfrac{1}{2}+\tfrac{1}{2}|\geq\Omega', \quad (22)$$

where Ω' is the projection of the vector $\mathbf{L}+\frac{1}{2}+\frac{1}{2}$ along the body axis. If either I_i or I_f is zero, as in the case of even nuclei in their ground states, then (22) together with the parity conservation rule will, generally, limit L to one value only.

In the integrals over $\mathbf{r}_p$ and $\mathbf{r}_n$ which lead to (20), the lower limit was taken as the nuclear radius R_0, instead of the corresponding values for spheroidal nuclei, as this was shown by Satchler and Sawicki[11] to have negligible effects on the scattering amplitude.

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APPENDIX. DERIVATION FOR THE MATRIX ELEMENT OF THE (α,d) REACTION

To derive an expression for the scattering amplitude for the (α,d) reaction, the method used by Gerjuoy[5] for the (d,p) stripping reactions will be followed and a similar notation will be used.

If the nucleus is pictured as a fixed center of force, then the solution Ψ at infinity must be of the form

$$\Psi=\psi_\alpha+\Phi, \quad (A1)$$

where ψ_α is the plane wave of α particles incident on

[11] J. Sawicki and G. R. Satchler, Nuclear Phys. 7, 296 (1958).

the initial nucleus, and Φ is everywhere outgoing. Allowing the energy to have an infinitesimal positive imaginary part, the outgoing character of Φ is secured provided it remains bounded as r_n, r_p, or r_d or all approach infinity. Let $\varphi(\mathbf{r}_n,\mathbf{r}_p)$ denote the wave functions of the final nucleus in a (α,d) reaction in which one neutron "n" and one proton "p" are captured into bound states. Then the scattering amplitude $A(\mathbf{n})$ in the direction $\mathbf{n}$ will be given by

$$A(\mathbf{n})\frac{e^{ikr_d}}{r_D}=\lim_{r_d\to\infty}\int d\mathbf{r}_p d\mathbf{r}_n \varphi^*\Psi, \tag{A2}$$

and r_d approaches infinity along $\mathbf{n}$.

A solution is sought with outgoing deuterons whose energy corresponds to leaving the neutron and proton bound to the center of force schematically representing the target nucleus. The total Hamiltonian is given by

$$H=T_d+T_p+T_n+V_d+V_p+V_n+V_{np}+V_{dp}+V_{dn}, \tag{A3}$$

where T represents the kinetic energy, V the potential energy, and the symbols d, p, n the deuteron, proton, and neutron, respectively. The solution satisfies

$$(H-E)\Psi=0, \tag{A4}$$

with Ψ having the form (A1), where

$$\psi_\alpha=\varphi(\alpha)\exp[i\mathbf{k}_\alpha\cdot(2\mathbf{r}_d+\mathbf{r}_p+\mathbf{r}_n)/4], \tag{A5}$$

so that $\psi_\alpha(\alpha)$ satisfies the wave equation

$$(T_d+T_n+T_p+V_{dn}+V_{dp}+V_{np}-E)\psi_\alpha=0. \tag{A6}$$

The solution of Eq. (A4) can now be written as

$$\Psi=\psi_0-G(V_d+V_{nd}+V_{pd}+V_{np})\Psi, \tag{A7}$$

where

$$(T_d+T_n+T_p+V_n+V_p-E)\psi_0=0, \tag{A8}$$

$$(T_d+T_n+T_p+V_n+V_p-E)G=1=\delta(\mathbf{r}_d-\mathbf{r}_d')\delta(\mathbf{r}_p-\mathbf{r}_p')\delta(\mathbf{r}_n-\mathbf{r}_n'). \tag{A9}$$

The solution of (A7) satisfies the boundary conditions at the nuclear radius, and satisfies the boundary condition at ∞ if G is the outgoing Green's function,

$$G(\mathbf{r}_d,\mathbf{r}_d';\mathbf{r}_n\mathbf{r}_p,\mathbf{r}_n'\mathbf{r}_p')=\sum\nolimits_\lambda g(E-\lambda)\varphi(\mathbf{r}_n\mathbf{r}_p,\lambda)\varphi^*(\mathbf{r}_n',\mathbf{r}_p',\lambda). \tag{A10}$$

$\varphi(\lambda)$ form the complete set of eigenfunctions of the neutrons and protons in the field of the initial nucleus so that

$$(T_n+T_p+V_n+V_p-\lambda)\varphi(\lambda)=0, \tag{A11}$$

while $g(E-\lambda)$ is the outgoing free space Green's function for the deuteron satisfying

$$(T_d-E+\lambda)g(E-\lambda)=\delta(\mathbf{r}_d-\mathbf{r}_d'), \tag{A12}$$

and has the form

$$g(\mathbf{r}_d-\mathbf{r}_d')=\frac{1}{4\pi}\frac{2M_d}{\hbar^2}\frac{\exp[i(E-\lambda)^{\frac{1}{2}}|\mathbf{r}_d-\mathbf{r}_d'|]}{|\mathbf{r}_d-\mathbf{r}_d'|}. \tag{A13}$$

From Eq. (A6) one gets

$$\psi_\alpha=\psi_0+G(V_n+V_p-V_{dn}-V_{dp}-V_{np})\psi_\alpha. \tag{A14}$$

Eliminating ψ_0 from (A7) and (A14) one finds

$$\Psi=\psi_\alpha-G(V_n+V_p-V_{dn}-V_{dp}-V_{np})\psi_\alpha-G(V_d+V_{nd}+V_{pd}+V_{np})\Psi. \tag{A15}$$

Substituting (A15) into Eq. (A2), and using Eq. (A5) it is seen that the contribution from the first term on the right-hand side of (A15) vanishes exponentially as $r_d\to\infty$, since both φ and $\varphi(\alpha)$ are bound states. The second and third term on the right-hand side of (A15) can be simplified by using Eqs. (A10), (A13) together with the orthonormality of the set $\varphi(\lambda)$. Letting $r_d\to\infty$, then one has

$$A(\mathbf{n})=A_1(\mathbf{n})+A_2(\mathbf{n}), \tag{A16}$$

where

$$A_1(\mathbf{n})=-\frac{1}{4\pi}\frac{2M_d}{\hbar^2}\int d\mathbf{r}_d d\mathbf{r}_p d\mathbf{r}_n e^{-i\mathbf{k}_d\cdot\mathbf{r}_d}\varphi^*(\mathbf{r}_n,\mathbf{r}_p,\lambda_f)\times(V_n+V_p-V_{dn}-V_{dp}-V_{np})\psi_\alpha, \tag{A17}$$

$$A_2(\mathbf{n})=-\frac{1}{4\pi}\frac{2M_d}{\hbar^2}\int d\mathbf{r}_d d\mathbf{r}_p d\mathbf{r}_n e^{-i\mathbf{k}_d\mathbf{r}_d}\varphi^*(\mathbf{r}_n,\mathbf{r}_p,\lambda_f)\times(V_d+V_{nd}+V_{pd}+V_{np})\Psi, \tag{A18}$$

where λ_f is the total energy of the bound proton and neutron in the final nucleus:

Eliminating the potentials in (A17), by using Eqs. (A6) and (A11), it can be shown that $A_1(\mathbf{n})=0$.

In the first Born approximation one may replace Ψ by ψ_α, so that the scattering amplitude will be given by (A18) with Ψ replaced by ψ_α.

Neglecting V_d, Eq. (A17) shows that in the first-Born-approximation amplitude either of the following interaction potentials may be used:

$$V_n+V_p \quad\text{or}\quad V_{dn}+V_{dp}+V_{nd}.$$

This proves the equivalence of the two methods for the derivation of the cross section.