Microwave Conductivity of a Plasma in a Magnetic Field*†

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The Boltzmann equation for electrons in a uniform isothermal plasma is solved by expressing the distribution function as a series of orthogonal polynomials in velocity space with time dependent expansion coefficients. The microwave conductivity is simply related to certain coefficients. Particular attention is devoted to the case in which the plasma is subject to a constant magnetic field and a microwave electric field. By introducing an "effective" electron temperature, convergence is attained for strong as well as weak electric fields. The formulation is particularly suited for problems involving partially ionized gases which contain several species of ions and neutrals. The conductivity of a completely ionized gas is calculated with and without consideration of electron-electron collisions, and the ratio (γ_E) of the two results is graphically illustrated as a function of microwave frequency.

INTRODUCTION

CURRENT microwave studies of electrical discharges demand a more detailed description of a plasma than a hydrodynamic treatment can supply. Such detail is contained in the electron distribution function, $f(\mathbf{r},\mathbf{v},t)$ which satisfies Boltzmann's equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_s f+\mathbf{a}\cdot\nabla_v f=\sum_j C(fF_j). \tag{1}$$

The quantity $f(\mathbf{r},\mathbf{v},t)d\mathbf{r}d\mathbf{v}$ denotes the probable number of electrons in the volume element $d\mathbf{r}$ about $\mathbf{r}$ having velocities in the range $d\mathbf{v}$ about $\mathbf{v}$ at time t. The configuration space gradient is ∇_s, ∇_v is the velocity space gradient, and $\mathbf{a}(t)$ is the acceleration due to external forces. $C(fF_j)$ denotes the rate of change in $f(\mathbf{r},\mathbf{v},t)$ as a result of elastic collisions with particles of type j having a distribution function F_j. The sum $\sum_j C(fF_j)$ might be written

$$\sum_j C(fF_j)=C(ff)+\sum_{\text{ions}} C(fF_i)+\sum_{\text{neutrals}} C(fF_n),$$

to exhibit the fact that in general there may be several types of heavy ions and neutral atoms and molecules present in the plasma. If only binary collisions are of importance, the explicit form of $C(fF_j)$ is

$$C(fF_j)=\int_{\mathbf{V}}\int_{\Omega} w\sigma_j(w,\psi)d\Omega(\chi,\psi)\times\{f(\mathbf{v}')F_j(\mathbf{V}')-f(\mathbf{v})F_j(\mathbf{V})\}d\mathbf{V}. \tag{2}$$

The primes denote velocities after scattering and

$$w=|\mathbf{v}-\mathbf{V}|=w' \tag{3}$$

is the relative speed, which is unchanged by an elastic collision. $\sigma_j(w,\psi)$ is the differential cross section for elastic scattering of an electron through an angle ψ in the center-of-mass system.

Our treatment is restricted from the beginning by the assumption that $\mathbf{a}$ arises from the Lorentz force of a constant magnetic field $\mathbf{H}$ and a microwave electric field $\mathbf{E}\cos\omega t$, i.e.,

$$\mathbf{a}(t)=(e/m)[\mathbf{E}\cos\omega t+(\mathbf{v}/c)\times\mathbf{H}]. \tag{4}$$

The rf magnetic field is taken to be negligible in comparison with $\mathbf{H}$, while $\mathbf{E}$ and $\mathbf{H}$ are constant in space. The geometry is chosen such that $\mathbf{H}$ lies along the z axis and $\mathbf{E}$ in the x-z plane making an angle β with $\mathbf{H}$.

It is further assumed that all particle concentrations are spatially uniform and that no temperature (here identified with mean total energy) gradients exist. If in addition we are allowed to ignore the effect of $\mathbf{a}$ on the heavy ions, it can be safely assumed that they maintain a Maxwellian velocity distribution. A future paper will deal with the more complicated problem of space-dependent distributions. Uniform temperature and concentration means: $\nabla_s f=0$ and $f(\mathbf{r},\mathbf{v},t)$ becomes a function of $\mathbf{v}$ and t alone.

II. EXPANSION OF $f(\mathbf{v},t)$

Our approach to the problem of determining $f(\mathbf{v},t)$ follows Landshoff[1] and Grad.[2] By developing $f(\mathbf{v},t)$ as a series of orthogonal polynomials in velocity space one obtains a set of coupled differential equations for the expansion coefficients. The choice of orthogonal functions is guided by the fact that the angle factors of the eigenfunctions of the collision operator $C(fF_j)$ are spherical harmonics in velocity space, $Y_l^m(\theta,\varphi)$. The appropriate radial functions prove to be generalized Laguerre polynomials,[3] $L_n^{l+\frac{1}{2}}(mv^2/2kT)$, apparently first used in kinetic theory by Burnett.[4] These polynomials are eigenfunctions of $C(fF_j)$ when $w\sigma_j(w,\psi)$ is independent of w.

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1 R. Landshoff, Phys. Rev. **76**, 907 (1949); **82**, 442 (1951).
2 H. Grad, Commun. Pure and Appl. Math. **2**, 331 (1949).
3 H. Buchholz, *Die Konfluente Hypergeometrische Funktion* (Springer-Verlag, Berlin, 1953).
4 D. Burnett, Proc. London Math. Soc. **39**, 385 (1935); **40**, 382 (1935).

We are thus led to write

$$f(\mathbf{u},t)=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}a_{nl}{}^{m}(t)\times\exp(-u^2)L_n{}^{l+\frac{1}{2}}(u^2)u^lY_l{}^m(\theta,\varphi),\quad(5)$$

where $\mathbf{u}$ is the dimensionless velocity variable

$$\mathbf{u}=(m/2kT)^{\frac{1}{2}}\mathbf{v}.\quad(6)$$

The electron temperature, T, is defined by the relation

$$T\equiv(1+\epsilon)T_g,\quad(7)$$

where T_g is the gas temperature, assumed common to all heavy particles. The parameter ϵ is a measure of the increase in electron energy due to the external fields. It is defined by the relation

$$\epsilon/2(1+\epsilon)\equiv-(a_{10}{}^0/a_{00}{}^0)_{\text{time average}}.\quad(8)$$

The time average of the mean electron energy is

$$\tfrac{3}{2}kT(1-a_{10}{}^0/a_{00}{}^0)_{\text{time av}}=\tfrac{3}{2}kT[1+\epsilon/2(1+\epsilon)],$$

so that T is not the temperature in the usual sense. This definition of T greatly improves the convergence of the expansion for $f(\mathbf{u},t)$. For example, suppose we wish to describe the velocity distribution of electrons in a microwave discharge. The mean energy of the heavy particles is $\frac{3}{2}kT_g$. On the other hand the electrons absorb microwave power and may attain a mean energy many times that of the heavy particles. If we take $T=T_g$, the expansion, Eq. (5), would require a prohibitive number of terms to approximate the distribution function. By choosing

$$kT\approx\text{mean energy of electrons}$$

we "center" the distribution function in the most densely populated region of velocity space.

The generalized Laguerre polynomials are defined as[3]

$$L_n{}^{l+\frac{1}{2}}(u^2)=\sum_{j=0}^{n}\binom{n+l+\frac{1}{2}}{n-j}\frac{(-u^2)^j}{j!}.\quad(9)$$

A more useful relation is

$$\sum_{n=0}^{\infty}s^nL_n{}^{l+\frac{1}{2}}(u^2)=(1-s)^{-(l+\frac{3}{2})}\exp\left(-\frac{s}{1-s}u^2\right).\quad(10)$$

The orthogonality condition, determined at once from Eq. (10), is

$$\int_0^\infty\exp(-u^2)L_n{}^{l+\frac{1}{2}}(u^2)L_p{}^{l+\frac{1}{2}}(u^2)u^{2l+1}d(u^2)=\frac{\Gamma(p+l+\frac{3}{2})}{p!}\delta_{p,n}.\quad(11)$$

The familiar spherical harmonics are defined in terms of Legendre polynomials by[5]

$$Y_l{}^m(\theta,\varphi)=(-1)^m\left(\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right)^{\frac{1}{2}}\sin^m\theta\times\frac{d^m}{d\cos\theta^m}P_l(\cos\theta)e^{im\varphi},\quad(12)$$

$$Y_l{}^{-m}(\theta,\varphi)=(-1)^mY_l{}^{m*}(\theta,\varphi).$$

They are orthonormal over the unit sphere

$$\int_\Omega Y_l{}^{m*}(\theta,\varphi)Y_{l'}{}^{m'}(\theta,\varphi)d\Omega(\theta,\varphi)=\delta_{l,l'}\delta_{m,m'}.$$

Since $f(\mathbf{u},t)$ is real we must have

$$a_{nl}{}^{-m}(t)=(-1)^ma_{nl}{}^{m*}(t).\quad(13)$$

The constant $a_{00}{}^0$ is fixed by the normalization condition

$$N_e=\int f(\mathbf{u},t)d\mathbf{u}(2kT/m)^{\frac{3}{2}};\quad N_e=\text{number of electrons per cc.}$$

Substituting for $f(\mathbf{u},t)$ from Eq. (5) and using Eq. (11) we find [remembering that $L_0{}^{\frac{1}{2}}(u^2)=1$]

$$a_{00}{}^0=(2N_e/\pi)(m/2kT)^{\frac{3}{2}}.\quad(14)$$

The distribution of heavy particles is

$$F_j(\mathbf{U})=A_{00}{}^0\exp(-U^2)L_0{}^{\frac{1}{2}}(U^2)Y_0{}^0(\Theta,\Phi),\quad(15)$$

where again

$$A_{00}{}^0=(2N_j/\pi)(M_j/2kT_g)^{\frac{3}{2}},\quad(16)$$

and

$$\mathbf{U}=(M_j/2kT_g)^{\frac{1}{2}}\mathbf{V},$$

N_j is the concentration of type j particles and M_j their mass.

III. CONDUCTIVITY

The conductivity tensor $\boldsymbol{\sigma}$ is a phenomenological constant relating the mean current density $\mathbf{J}$ and the electric field. We write

$$\mathbf{E}\cos\omega t=\mathrm{Re}(\mathbf{E}e^{i\omega t}).$$

The complex conductivity is then related to $\mathbf{E}$ by

$$\mathbf{J}=\mathrm{Re}(\boldsymbol{\sigma}\cdot\mathbf{E}e^{i\omega t})=\boldsymbol{\sigma}_R\cdot\mathbf{E}\cos\omega t+\boldsymbol{\sigma}_I\cdot\mathbf{E}\sin\omega t,\quad(17)$$

provided we take

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}_R-i\boldsymbol{\sigma}_I.\quad(18)$$

For the geometry chosen $\boldsymbol{\sigma}$ has the matrix form

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{xx}&\sigma_{xy}&0\\\sigma_{yx}&\sigma_{yy}&0\\0&0&\sigma_{zz}\end{bmatrix};\quad\sigma_{xx}=\sigma_{yy};\quad\sigma_{xy}=-\sigma_{yx}.\quad(19)$$

[5] M. E. Rose, *The Elementary Theory of Angular Momentum* (John Wiley and Sons, Inc., New York, 1957).

The kinetic expression for $\mathbf{J}$ is

$$\mathbf{J}=e\int \mathbf{u}f(\mathbf{u},t)d\mathbf{u}(2kT/m)^2, \quad (20)$$

where $d\mathbf{u}$ is an element of volume in $\mathbf{u}$ space.

Substituting for $f(\mathbf{u},t)$ and using the orthogonality relation Eq. (11) one finds

$$\mathbf{J}=\frac{3\pi}{8}(\tfrac{2}{3})^{\frac{1}{2}}e\left(\frac{2kT}{m}\right)^2[\mathbf{i}(a_{01}{}^{-1}-a_{01}{}^{+1}) +\mathbf{j}(-ia_{01}{}^{-1}-ia_{01}{}^{+1})+\mathbf{k}\sqrt{2}a_{01}{}^{0}]. \quad (21)$$

Here, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the x, y, and z directions. Thus, $\boldsymbol{\sigma}$ depends explicitly only on the $a_{01}{}^{m}$. However, we shall find these to be coupled to the $a_{n0}{}^{0}$.

IV. COUPLED EQUATIONS

Here, we only outline the method of obtaining the equations for the $a_{nl}{}^{m}(t)$; details are given elsewhere.[6] By substituting the expansion [Eq. (5)] for $f(\mathbf{u},t)$ into the Boltzmann equation and exploiting the orthogonality of the spherical harmonics and Laguerre polynomials one obtains the following equation coupling the $a_{nl}{}^{m}(t)$.

$$\begin{aligned}\left(\frac{\partial a_{pl}{}^{m}}{\partial t}-im\omega_H a_{pl}{}^{m}+\sum_{n=0}^{\infty}\langle\nu_l\rangle_{pn}a_{nl}{}^{m}\right) &-2\gamma(t)\cos\beta\left(\frac{l^2-m^2}{(2l+1)(2l-1)}\right)^{\frac{1}{2}}a_{pl-1}{}^{m}\\ &-\gamma(t)\sin\beta\left[-\left(\frac{(l+m-1)(l+m)}{(2l+1)(2l-1)}\right)^{\frac{1}{2}}a_{pl-1}{}^{m-1} +\left(\frac{(l-m-1)(l-m)}{(2l+1)(2l-1)}\right)^{\frac{1}{2}}a_{pl-1}{}^{m+1}\right]\\ &+\gamma(t)p\sin\beta\left[\left(\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}\right)^{\frac{1}{2}}a_{p-1\,l+1}{}^{m-1} -\left(\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}\right)^{\frac{1}{2}}a_{p-1\,l+1}{}^{m+1}\right]\\ &+2\gamma(t)p\cos\beta\left(\frac{(l+1)^2-m^2}{(2l+1)(2l+3)}\right)^{\frac{1}{2}}a_{p-1\,l+1}{}^{m}=0, \quad (22)\end{aligned}$$

where the various symbols are defined as

$$\begin{aligned}&\beta=\text{angle between } \mathbf{E} \text{ and } \mathbf{H},\\ &\gamma(t)=(eE/m)\cos\omega t(m/2kT)^{\frac{1}{2}},\end{aligned} \quad (23)$$

$$\omega_H=eH/mc. \quad (24)$$

[6] D. Kelly, Ph.D. thesis, Yale University, 1959 (unpublished).

The $\langle\nu_l\rangle_{pn}$ are matrix elements for the collision integral.

$$\sum_{n=0}^{\infty}a_{nl}{}^{m}\langle\nu_l\rangle_{pn}\frac{\Gamma(p+l+\frac{3}{2})}{2(p!)} =-\int_0^{\infty}u^{l+2}L_p{}^{l+\frac{1}{2}}(u^2)du\sum_j C(fF_j). \quad (25)$$

Inspection of Eq. (22) shows that the expansion coefficients for a given value of l are coupled with those for $l-1$ and $l+1$. In practice one is forced to place a finite limit on the summations over n and l, so that when $l=l_{\max}$ the coefficients involving $l+1$ must be dropped. The convergence of the $a_{nl}{}^{m}$ with n and l then depend on the parameter ϵ introduced in Eq. (7). For $\epsilon\ll1$ (weak electric field) we find

$$\left.\begin{aligned}&a_{nl}{}^{m}\propto\epsilon^{l/2}a_{00}{}^{0};\quad l\neq0\\ &a_{n0}{}^{0}\propto\epsilon a_{00}{}^{0};\quad n\neq0\end{aligned}\right\}\ \epsilon\ll1. \quad (26)$$

In this case the isotropic part of $f(\mathbf{u},t)$ deviates from the Maxwellian form by terms of order ϵ. For $\epsilon\gg1$ (strong electric field)

$$a_{nl}{}^{m}\propto(m/M)^{l/2}a_{00}{}^{0};\quad \epsilon\gg1. \quad (27)$$

Convergence of the $a_{n0}{}^{0}$ with n depends on the microwave frequency, ω, the characteristic frequency for momentum transfer, $\langle\nu_1\rangle_{00}$, and the concentration of electrons. Distortion of the isotropic part of $f(\mathbf{u},t)$ is slight in all cases. To determine the conductivity it is sufficient to consider Eq. (22) with $l=0$ and $l=1$. On examining Eqs. (22), (26), and (27) we see that for $l\neq0$ we are justified in dropping the terms involving $a_{p-1\,l+1}{}^{m,m\pm1}$ while retaining those with $a_{pl-1}{}^{m,m\pm1}$. For $l=0$, the coefficients of $a_{pl-1}{}^{m,m\pm1}$ vanish. This procedure leads to the following set of coupled equations;

$\mathbf{l=m=0}$

$$\frac{\partial a_{p0}{}^{0}}{\partial t}+\sum_{n=0}^{P}\langle\nu_0\rangle_{pn}a_{n0}{}^{0}+2(\tfrac{1}{3})^{\frac{1}{2}}\gamma(t)\cos\beta\, p a_{p-1\,1}{}^{0} +(\tfrac{2}{3})^{\frac{1}{2}}\gamma(t)\sin\beta\, p(a_{p-1\,1}{}^{-1}-a_{p-1\,1}{}^{+1})=0, \quad (22a)$$

$\mathbf{l=1; m=0}$

$$\frac{\partial a_{p1}{}^{0}}{\partial t}+\sum_{n=0}^{P}\langle\nu_1\rangle_{pn}a_{n1}{}^{0}=2(\tfrac{1}{3})^{\frac{1}{2}}\gamma(t)\cos\beta a_{p0}{}^{0}, \quad (22b)$$

$\mathbf{l=m=1}$

$$\frac{\partial a_{p}{}^{+1}}{\partial t}-i\omega_H a_{p1}{}^{+1}+\sum_{n=0}^{P}\langle\nu_1\rangle_{pn}a_{n1}{}^{+1} =-(\tfrac{2}{3})^{\frac{1}{2}}\gamma(t)\sin\beta a_{p0}{}^{0}. \quad (22c)$$

$a_{p1}{}^{-1}$ can be found from $a_{p1}{}^{+1}$ with the aid of Eq. (13). We have restricted the sum over n to a finite number

$(P+1)$ of terms. By comparing results for $P=0,1,2,\cdots$ one can test the convergence of this scheme. The solution of these equations for the $a_{01}{}^m$ needed to determine the conductivity is presented in the next section.

Calculation of the conductivity requires explicit forms for the collision-integral matrix elements. We may denote the expansion, Eq. (5), as

$$f(\mathbf{u},t)=f_0(u,t)+\varphi(\mathbf{u},t), \tag{5a}$$

where $f_0(u,t)$ is the isotropic part of $f(\mathbf{u},t)$ and $\varphi(\mathbf{u},t)$ is the small [as we have seen from Eqs. (26) and (27)] portion which describes how the external forces "warp" the distribution of velocities.

The portion of the collision integral describing *e-e* encounters, $C(ff)$, contains the terms

$$\{f_0(u')f_0(U')-f_0(u)f_0(U)\}+\{f_0(u')\varphi(\mathbf{U}')-f_0(u)\varphi(\mathbf{U})\}.$$

They yield what is referred to as the *linear* contribution of the *e-e* collisions. The remaining portion involves

$$\{\varphi(\mathbf{u}')\varphi(\mathbf{U}')-\varphi(\mathbf{u})\varphi(\mathbf{U})\},$$

which gives rise to what we call the *quadratic e-e* contribution.

Consideration of the convergence with l [Eqs. (26) and (27)] allows us to expand $\sum_j C(fF_j)$ and discard higher order terms. The details of this expansion are given elsewhere[6] and methods for calculating the matrix elements are given by Chapman and Cowling.[7] One finds that the $\langle\nu_l\rangle_{pn}$ may be written as the sum of three terms.

$$\langle\nu_l\rangle_{pn}=[2(p!)/\Gamma(p+l+\tfrac{3}{2})]\times[\langle X_l\rangle_{pn}+\langle Y_l\rangle_{pn}+\langle Z_l\rangle_{pn}]. \tag{28}$$

The $\langle X_l\rangle_{pn}$ contain the linear contribution of electron-electron (*e-e*) collisions

$$\sum_{n,p=0}^{\infty} s^n t^p\langle X_0\rangle_{pn}=\frac{\lambda}{2}(st)^2\frac{[1-\frac{1}{2}(s+t)]^{-\frac{3}{2}}}{1-st}, \tag{29}$$

$$\sum_{n,p=0}^{\infty} s^n t^p\langle X_1\rangle_{pn}=\lambda st\frac{[1-\frac{1}{2}(s+t)]^{-\frac{5}{2}}}{(1-st)^2}\{(1-\tfrac{1}{2}st)\times[1-\tfrac{1}{2}(s+t)]+\tfrac{3}{8}st(1-st)\}. \tag{30}$$

Here λ is a frequency characteristic of *e-e* collisions[8]

$$\lambda=N_e\frac{\pi}{2}\left(\frac{e^2}{kT}\right)^2\left(\frac{kT}{m}\right)^{\frac{1}{2}}\ln\left[1+\left(\frac{3kTD}{2e^2}\right)^2\right]. \tag{31}$$

D is the Debye length;

$$D^2=\frac{kT_g(1+\epsilon)}{4\pi N_e e^2[1+\mathfrak{z}(1+\epsilon)]}, \tag{32}$$

and $\mathfrak{z}$ is the mean ion charge for the plasma

$$\mathfrak{z}\equiv\sum_{\text{ions}}\mathfrak{z}_i{}^2N_i/N_e. \tag{33}$$

For very strong magnetic fields Tannenwald[9] has shown that D in the logarithmic term of Eq. (31) is replaced by a length proportional to the Larmor radius.

The $\langle Y_l\rangle_{pn}$ account for the quadratic electron-electron interaction and do not enter the equation for $l=1$. One may write

$$\langle Y_1\rangle_{pn}=0, \tag{33}$$

$$a_{n0}{}^0\langle Y_0\rangle_{pn}=3\sum_{q=0}^{\infty}\sum_{m=-1}^{+1}a_{q1}{}^m a_{n1}{}^{-m}\langle Y_0\rangle_{qpn}\times C(1,1,0;-m,m)C(1,1,0;0,0), \tag{34}$$

where

$$\sum_{q,p,n=0}^{\infty}r^q s^n t^p\langle Y_0\rangle_{qpn}=\frac{\lambda}{12a_{00}{}^0}t(1-t)(s-r)[1-\tfrac{1}{2}(s+t+r-rst)]^{-7/2}\times\{\tfrac{3}{2}\eta-\eta^2[\Lambda(r,s,t)-2\mu]\}, \tag{35}$$

provided

$$\eta\equiv\frac{1-\frac{1}{2}(r+s+t-rst)}{1-st},$$

$$\Lambda\equiv\frac{\frac{1}{2}(1-r)(1-s)(1-t)\left[\dfrac{s}{1-s}+\dfrac{t}{1-t}-\dfrac{r}{1-r}\right]^2}{1-\frac{1}{2}(r+s+t-rst)}, \tag{36}$$

$$\mu\equiv\frac{1-\frac{1}{2}(r+s+t-rst)}{(1-r)(1-s)(1-t)}.$$

The $C(1,1,0;m-m_2,m)$ in Eq. (34) are Clebsch-Gordan coefficients.[5] Finally, $\langle Z_l\rangle_{pn}$ is a sum of terms accounting for the electron-"heavy" [ions (*e-i*) and neutrals (*e-n*)] collisions. If we assume that the scattering cross section for these collisions is of the form of a power law

$$w\sigma_j(w,\psi)=S_{h,j}(\psi)w^h, \tag{37}$$

and define

$$\nu_l(h,j)=2\pi N_j\left(\frac{2kT}{m}\right)^{h/2}\times\int_\psi S_{h,j}(\psi)[1-P_l(\cos\psi)]\sin\psi d\psi, \tag{38}$$

[7] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1939).

[8] Collisions with an impact parameter greater than D are ignored.

[9] L. M. Tannenwald, Phys. Rev. **113**, 1396 (1959).

$P_l(\cos\psi)$ being a Legendre polynomial, then we obtain

$$\sum_{n,p=0}^{\infty} s^n t^p \langle Z_0\rangle_{pn}$$
$$=\sum_j \nu_1(h,j)\Gamma\left(\frac{h+5}{2}\right)\frac{m}{M_j}(1-s)^{h/2}(1-t)^{h/2}$$
$$\times(1-st)^{-[(h+5)/2]}\left[st-\frac{\epsilon}{1+\epsilon}t\right], \quad (39)$$

$$\sum_{n,p=0}^{\infty} s^n t^p \langle Z_1\rangle_{pn}$$
$$=\sum_j \nu_1(h,j)\Gamma\left(\frac{h+5}{2}\right)$$
$$\cdot\tfrac{1}{2}(1-s)^{h/2}(1-t)^{h/2}(1-st)^{-[(h+5)/2]}. \quad (40)$$

Although we have presented results only for power law cross sections, the theory readily adapts itself to situations involving more complicated forms of $\sigma_j(w,\psi)$. Of particular interest is the case $h=-3$ which describes *e-i* scattering. If all ions have the same mass, M_i, and the mean ion charge is $\mathfrak{z}$ we find

$$[Z_0]=\frac{2m}{M_i}\mathfrak{z}\lambda\sqrt{2}\begin{vmatrix} 0 & 0 & 0 & \cdots \\ -\delta & 1-\frac{3}{2}\delta & \frac{3}{2}-(15/8)\delta & \cdots \\ -\frac{3}{2}\delta & \frac{3}{2}-(15/8)\delta & 13/4-(69/16)\delta & \cdots \\ \vdots & & \vdots & \end{vmatrix};$$

$$\delta\equiv\frac{\epsilon}{1+\epsilon},$$

$$[Z_1]=\mathfrak{z}\lambda\sqrt{2}\begin{vmatrix} 1 & 3/2 & 15/8 & \cdots \\ 3/2 & 13/4 & 69/16 & \cdots \\ 15/8 & 69/16 & 433/64 & \cdots \\ \vdots & & \vdots & \end{vmatrix}.$$

V. SOLUTION OF COUPLED EQUATIONS

In seeking the solutions of Eqs. (22a)–(22c) we employ a substitution analysis and write

$$a_{n1}{}^m = A_n{}^m \sin\omega t + B_n{}^m \cos\omega t, \quad (43)$$

$$a_{n0}{}^0 = C_n + D_n \sin 2\omega t + F_n \cos 2\omega t. \quad (44)$$

We also assume

$$D_n,\ F_n \ll C_n.$$

It is found, *a posteriori*, that

$$D_n,\ F_n \sim (m/M_i)C_n;\quad D_0=F_0=0.$$

On substituting Eqs. (43) and (44) into Eq. (22b) we obtain the results

$$A_n{}^0 = 2\left(\frac{m}{6kT}\right)^{\frac{1}{2}}\frac{eE}{m}\cos\beta\sum_{q=0}^{P}\omega C_q\frac{L^{qn}}{|L|}, \quad (45)$$

$$B_n{}^0 = 2\left(\frac{m}{6kT}\right)^{\frac{1}{2}}\frac{eE}{m}\cos\beta\sum_{r,q=0}^{P}\langle\nu_1\rangle_{qr}C_r\frac{L^{qn}}{|L|}, \quad (46)$$

where

$$|L| = \det[L_{qn}], \quad (47)$$

$$L_{qn} = \omega^2\delta_{q,n} + \langle\nu_1{}^2\rangle_{qn}, \quad (48)$$

$$\langle\nu_1{}^2\rangle_{qn} = \sum_{r=0}^{P}\langle\nu_1\rangle_{qr}\langle\nu_1\rangle_{rn}, \quad (49)$$

and L^{qn} is the cofactor[10] of L_{qn}. In similar fashion it is found from Eq. (22c) that

$$A_n{}^{+1} = -\left(\frac{m}{3kT}\right)^{\frac{1}{2}}\frac{eE}{m}\sin\beta\sum_{q=0}^{P}\omega C_q\frac{D^{qn}}{|D|}, \quad (50)$$

$$B_n{}^{+1} = -\left(\frac{m}{3kT}\right)^{\frac{1}{2}}\frac{eE}{m}\sin\beta$$
$$\times\sum_{q=0}^{P}\sum_{r=0}^{P}\{\langle\nu_1\rangle_{qr} - i\omega_H\delta_{q,r}\}C_r\frac{D^{qn}}{|D|}, \quad (51)$$

with

$$D_{qn} = (\omega^2-\omega_H{}^2)\delta_{q,n} + \langle\nu_1{}^2\rangle_{qn} - 2i\omega_H\langle\nu_1\rangle_{qn}. \quad (52)$$

Unfortunately it is not possible to solve Eq. (22a) by using Eqs. (44) and (45)–(51) because the matrix elements $\langle\nu_0\rangle_{pn}$ contain a contribution from the quadratic *e-e* scattering. Explicitly,

$$\sum_{n=0}^{P} a_{n0}{}^0\langle\nu_0\rangle_{pn}\frac{\Gamma(p+\frac{3}{2})}{2\cdot p!}$$
$$=\sum_{n=0}^{P} a_{n0}{}^0[\langle X_0\rangle_{pn}+\langle Z_0\rangle_{pn}] + 3\sum_{n=0}^{P}\sum_{q=0}^{P}\sum_{m=-1}^{+1} a_{q1}{}^m a_{n1}{}^{-m}$$
$$\times\langle Y_0\rangle_{qpn}C(1,1,0;m_1-m)C(1,1,0;0,0).$$

The last term on the right introduces products of the C_n so that Eq. (22a) gives a set of nonlinear equations of the form

$$\sum_{n=0}^{P}\mu_{pn}C_n + \sum_{m,n=0}^{P}\eta_{pmn}C_mC_n = \kappa_p.$$

We must therefore determine under what conditions it is permissible to ignore the quadratic term. This term will be of little consequence if the electron concentration is so low that

$$\lambda/\langle\nu_1\rangle_{00} \ll 1,$$

i.e., if *e-e* and *e-i* collisions are infrequent in comparison with *e-n* encounters. This will usually be the case for a degree of ionization below 10^{-5}.[11,12] In case we are content with the approximation afforded by taking $P=1$ the quadratic term vanishes—a result of the fact that *e-e* collisions cannot change the concentration or mean energy. However the collisions do affect the results indirectly through their contributions to $\langle\nu_1\rangle$ and $\langle\nu_1{}^2\rangle$.

[10] H. Margenau and G. Murphy, *The Mathematics of Physics and Chemistry* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1955).
[11] J. H. Cahn, Phys. Rev. **75**, 293 (1949); **75**, 838 (1949).
[12] J. M. Anderson and L. Goldstein, Phys. Rev. **100**, 1037 (1955).

(a) Quadratic Term Negligible

We consider first the case where the quadratic term is negligible, by virtue of a low electron concentration or because we limit the analysis to $P=1$. The C_n are then determined by the equations

$$\sum_{n=1}^{P} \mu_{pn} C_n = -\mu_{p0} C_0, \quad p=1, 2, \cdots, P, \tag{53}$$

where

$$\mu_{pn} = \langle \nu_0 \rangle_{pn} + \tfrac{2}{3}\Omega^2 p \Bigg[\sum_{q=0}^{P} \sin^2\beta \cdot \mathrm{Re}\left(\{\langle \nu_1 \rangle_{qn} - i\omega_H \delta_{q,n}\} \cdot \frac{D^{qp-1}}{|D|} \right) + \cos^2\beta \langle \nu_1 \rangle_{qn} \frac{L^{qp-1}}{|L|} \Bigg], \tag{54}$$

$$\Omega^2 \equiv (m/2kT)(eE/m)^2. \tag{55}$$

In the $P=1$ approximation we have

$$-C_1/C_0 = \mu_{10}/\mu_{11}. \tag{56}$$

In view of Eq. (54) this becomes

$$-\frac{C_1}{C_0} = \frac{\langle \nu_0 \rangle_{10}}{\langle \nu_0 \rangle_{11}} + \frac{2\Omega^2}{3\langle \nu_0 \rangle_{11}} \Bigg[\sum_{q=0,1} \cos^2\beta \langle \nu_1 \rangle_{q0} \frac{L^{q0}}{|L|} + \sin^2\beta \, \mathrm{Re}\left(\{\langle \nu_1 \rangle_{q0} - i\omega_H \delta_{q,0}\} \frac{D^{q0}}{|D|} \right) \Bigg]. \tag{56a}$$

We now consider a special case; it is assumed that the electron and ion concentrations are low enough to permit ignoring *e-e* and *e-i* collisions. We further assume that only one species of neutral scatterers is present and that it is of the $h=0$ type. This is referred to as the constant collision frequency case. One finds

$$\langle \nu_1 \rangle_{pn} = \nu_1(0,j)\delta_{p,n} \equiv \nu \delta_{p,n}, \tag{57}$$

$$\langle \nu_0 \rangle_{pn} = \frac{2m}{M_j} \nu \left[\delta_{p,n} - \frac{\epsilon}{1+\epsilon} \delta_{p,n+1} \right] (1-\delta_{p,0}). \tag{58}$$

In this instance the Laguerre polynomials are eigenfunctions of the collision integral. In general one finds that the rate of convergence depends on the degree to which the scattering departs from the $h=0$ form. Noting that

$$-C_1/C_0 = (a_{10}^0/a_{00}^0)_{\text{time av}} \equiv \epsilon/2(1+\epsilon),$$

one finds

$$\epsilon = \frac{M_j}{3kT_j} \left(\frac{eE}{m} \right)^2 \left\{ \frac{\cos^2\beta}{\nu^2+\omega^2} + \frac{\sin^2\beta}{2} \times \left[\frac{1}{\nu^2+(\omega-\omega_H)^2} + \frac{1}{\nu^2+(\omega+\omega_H)^2} \right] \right\}. \tag{59}$$

Although Eq. (59) is derived for constant collision frequencies it provides a reasonable estimate of ϵ for other situations with ν replaced by $\langle \nu_1 \rangle_{00}$.[13]

Returning to the conductivity problem we use Eq. (56) in Eqs. (45), (46), (50), and (51) to determine the A_0^m and B_0^m.

$$\sqrt{2} A_0^0 = 2 \left(\frac{m}{3kT} \right)^{\frac{1}{2}} \frac{eE}{m} \cos\beta \times \sum_{q=0}^{P} \left\{ \delta_{q,0} - \sum_{p=1}^{P} \frac{\mu_{p0} \mu^{pq}}{\mu^{00}} (1-\delta_{q,0}) \right\} \omega C_0 \frac{L^{q0}}{|L|}, \tag{45a}$$

$$\sqrt{2} B_0^0 = 2 \left(\frac{m}{3kT} \right)^{\frac{1}{2}} \frac{eE}{m} \cos\beta \sum_{r,q=0}^{P} \langle \nu_1 \rangle_{qr} \times \left\{ \delta_{r,0} - \sum_{p=1}^{P} \frac{\mu_{p0} \mu^{pr}}{\mu^{00}} (1-\delta_{r,0}) \right\} C_0 \frac{L^{q0}}{|L|}, \tag{46a}$$

$$A_0^{\div 1} = -\left(\frac{m}{3kT} \right)^{\frac{1}{2}} \frac{eE}{m} \sin\beta \times \sum_{q=0}^{P} \left\{ \delta_{q,0} - \sum_{p=1}^{P} \frac{\mu_{p0} \mu^{pq}}{\mu^{00}} (1-\delta_{q,0}) \right\} \omega C_0 \frac{D^{q0}}{|D|}, \tag{50a}$$

$$B_0^{\div 1} = -\left(\frac{m}{3kT} \right)^{\frac{1}{2}} \frac{eE}{m} \sin\beta \sum_{q,r=0}^{P} \{\langle \nu_1 \rangle_{qr} - i\omega_H \delta_{q,r}\} \times \left\{ \delta_{q,0} - \sum_{p=1}^{P} \frac{\mu_{p0} \mu^{pr}}{\mu^{00}} (1-\delta_{r,0}) C_0 \frac{D^{q0}}{|D|} \right\}. \tag{51a}$$

Substituting into Eq. (21) we can find $\mathbf{J}$ and thus $\boldsymbol{\sigma}$. The results are valid for arbitrary field strength in cases where *e-e* collisions are unimportant, and for arbitrary electron concentration with $P=1$. For the constant collision frequency case we obtain the well known result[14–16]

$$\sigma_{xx} = \sigma_{yy} = \frac{N_e e^2}{2m} \left[\frac{1}{\nu^2+(\omega-\omega_H)^2} + \frac{1}{\nu^2+(\omega+\omega_H)^2} \right] \times \left[\nu - i\omega \frac{\nu^2+\omega^2-\omega_H{}^2}{\nu^2+\omega^2+\omega_H{}^2} \right], \tag{60}$$

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[13] Note that for all but the $h=0$ case, $\langle \nu_1 \rangle_{00}$ depends on T and thus ϵ.

[14] H. Margenau, Phys. Rev. **69**, 508 (1946).

[15] W. P. Allis, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956).

[16] L. Goldstein, M. Gilden, University of Illinois Electrical Engineering Research Laboratory Technical Report No. 9, 1956 (unpublished).

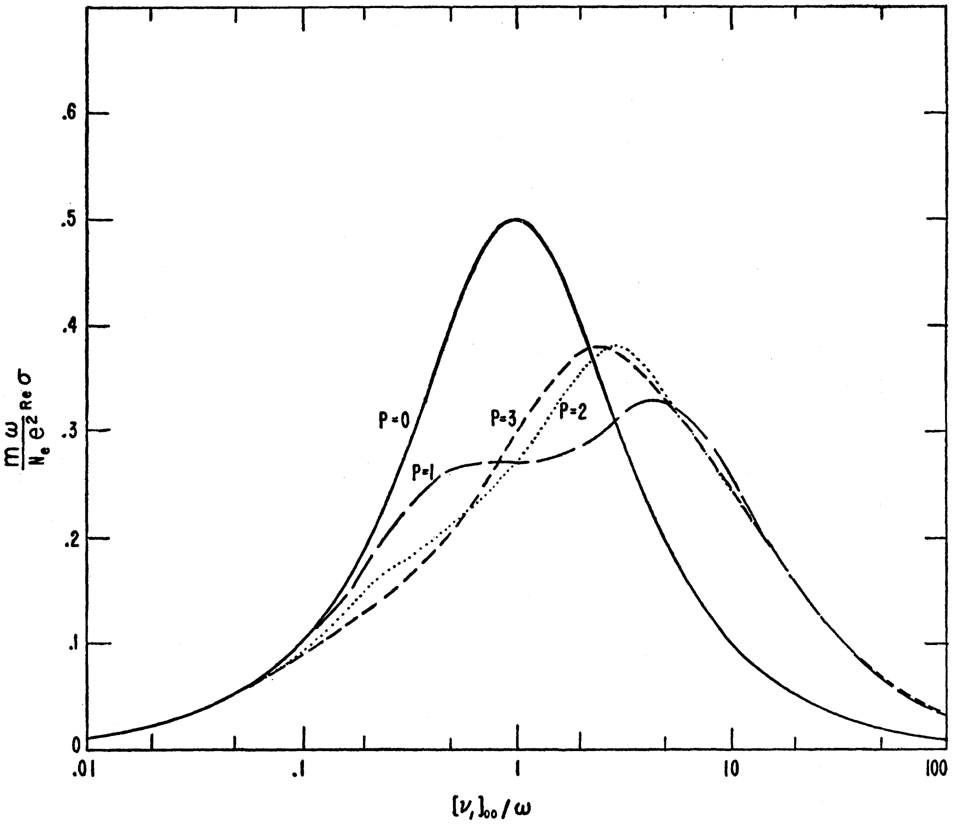

FIG. 1. $(m\omega/N_e e^2)$ Reσ vs $\langle\nu_1\rangle_{00}/\omega$ for a completely ionized gas, ignoring e-e collisions, in the $P=0$, 1, 2, 3 approximations.

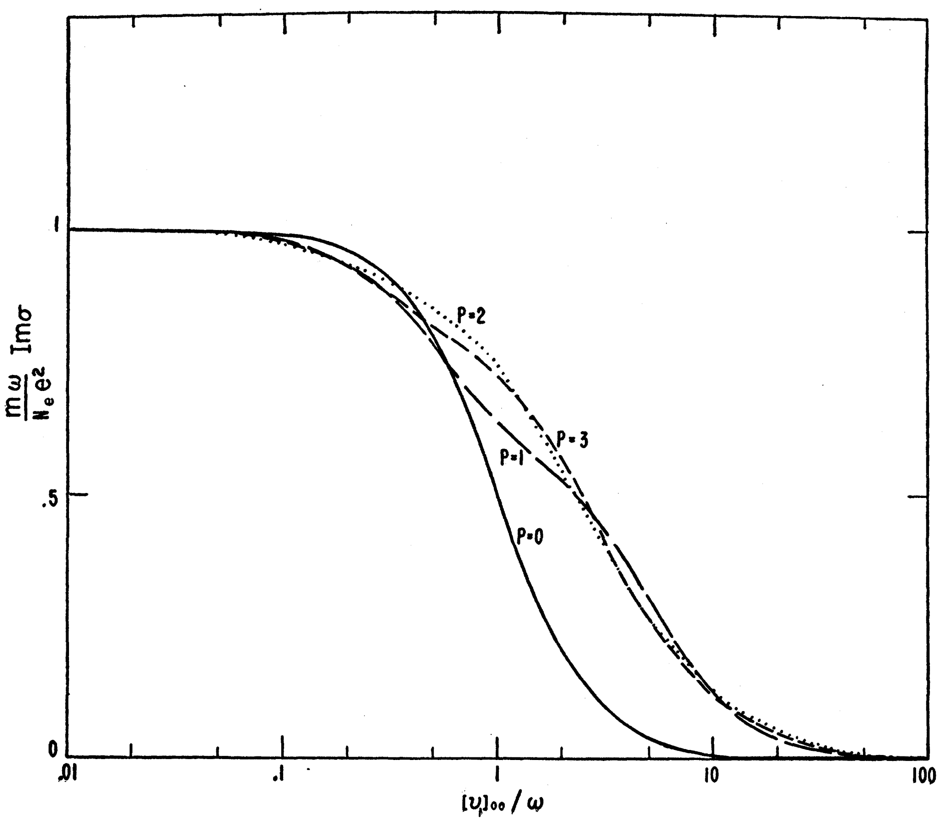

FIG. 2. $(m\omega/N_e e^2)$ Imσ vs $\langle\nu_1\rangle_{00}/\omega$ for a completely ionized gas, ignoring e-e collisions, in the $P=0$, 1, 2, 3 approximations.

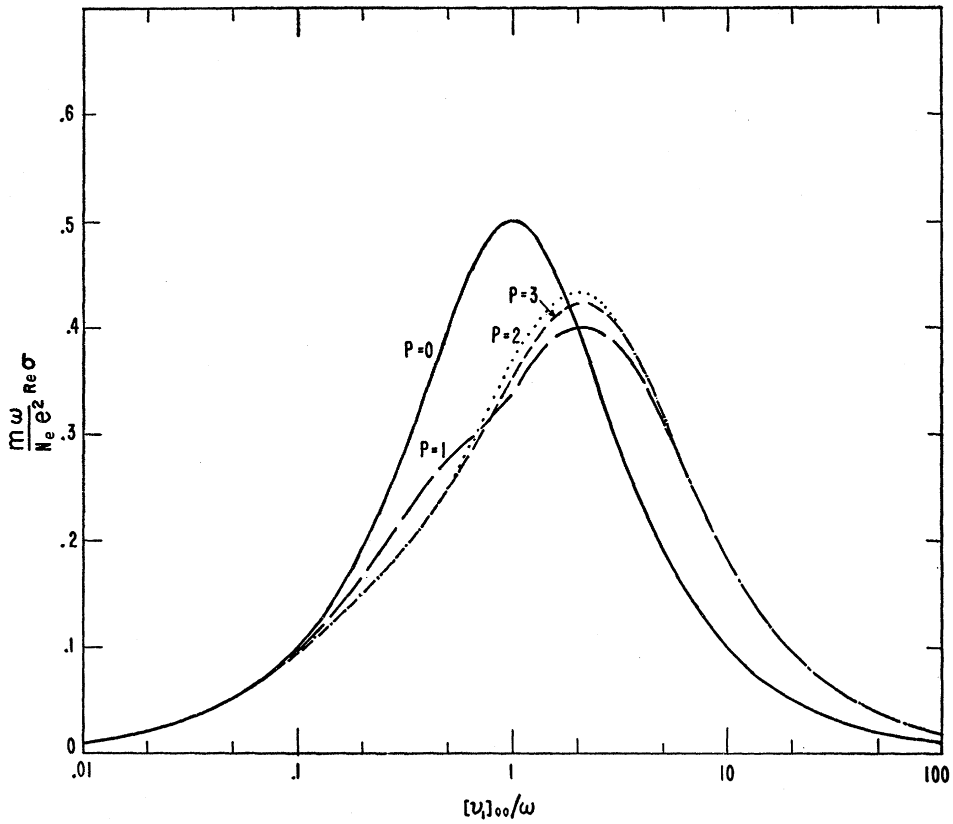

FIG. 3. $(m\omega/N_e e^2)$ Reσ vs $\langle\nu_1\rangle_{00}/\omega$ for a completely ionized gas, including e-e collisions, in the $P=0$, 1, 2, 3 approximations.

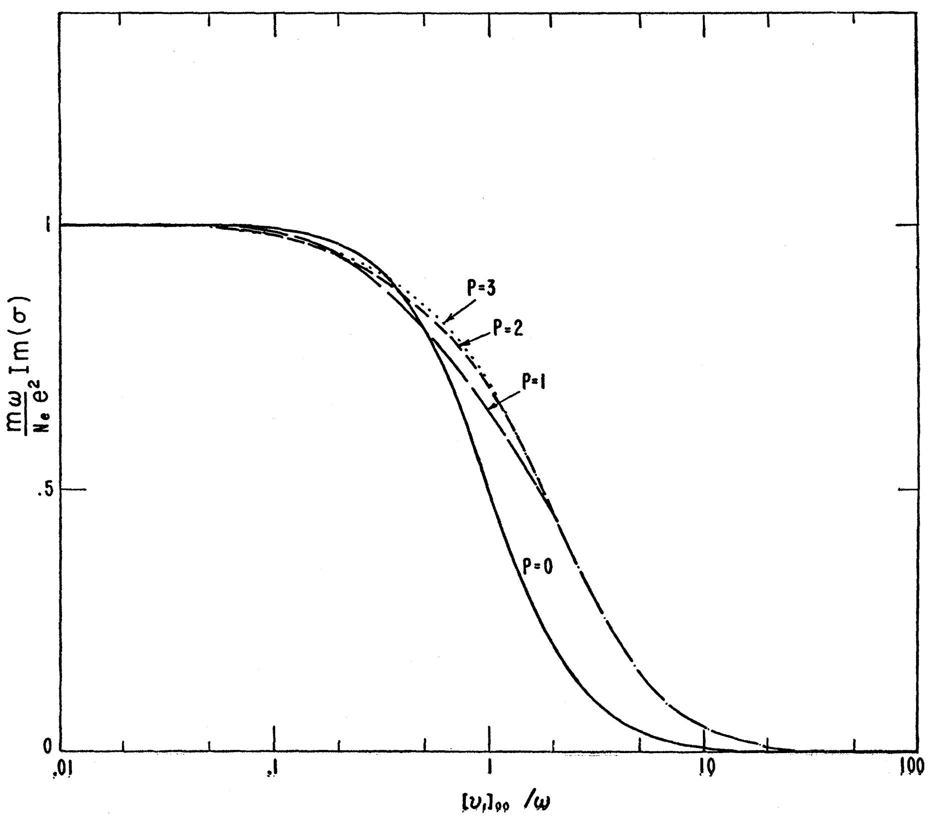

FIG. 4. $(m\omega/N_e e^2)$ Imσ vs $\langle\nu_1\rangle_{00}/\omega$ for a completely ionized gas, including e-e collisions, in the $P=0$, 1, 2, 3 approximations.

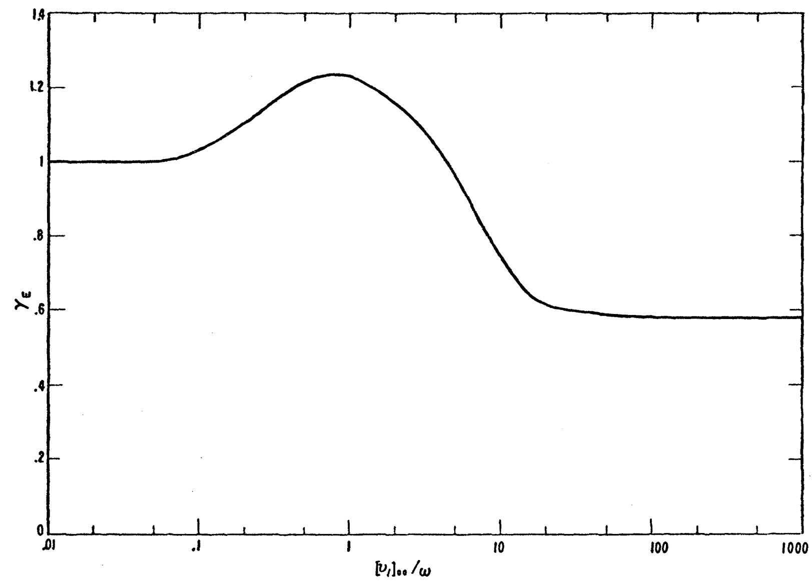

FIG. 5. The ratio γ_E defined by Eq. (66) as a function of $\langle\nu_1\rangle_{00}/\omega$.

$$\sigma_{yx}=-\sigma_{xy}=\frac{N_e e^2}{2m}\left[\frac{\omega_H}{\nu^2+(\omega-\omega_H)^2}+\frac{\omega_H}{\nu^2+(\omega+\omega_H)^2}\right]\times\left[\frac{\omega^2-\omega_H^2-\nu^2+2i\nu\omega}{\omega^2+\omega_H^2+\nu^2}\right], \quad (61)$$

$$\sigma_{zz}=\frac{N_e e^2}{m}\left[\frac{\nu-i\omega}{\nu^2+\omega^2}\right]. \quad (62)$$

(b) Weak Electric Field, $\varepsilon\ll1$; Quadratic Term Included

The quadratic e-e term in Eq. (22a) changes the results for C_n, $n\neq0$, but does not affect[17] C_0. When $\epsilon\ll1$, $C_n\sim\epsilon C_0$ and the conductivity depends only on $C_0=a_{00}{}^0$. Thus, providing $\epsilon\ll1$, it is not necessary to solve Eq. (22a) with its troublesome quadratic term. One uses Eqs. (45)–(51) with $C_r=C_0\delta_{r,0}$ and substitutes in Eq. (21) to obtain the result

$$\sigma_{xx}=\sigma_{yy}=\frac{N_e e^2}{m}\left[\operatorname{Re}\sum_{q=0}^{P}\left(\langle\nu_1\rangle_{q0}-i\omega_H\delta_{q,0}\right)\times\frac{D^{q0}}{|D|}-i\omega\operatorname{Re}\frac{D^{00}}{|D|}\right], \quad (63)$$

$$\sigma_{yx}=-\sigma_{xy}=\frac{N_e e^2}{m}\left[\operatorname{Im}\sum_{q=0}^{P}\left(\langle\nu_1\rangle_{q0}-i\omega_H\delta_{q,0}\right)\times\frac{D^{q0}}{|D|}-i\omega\operatorname{Im}\frac{D^{00}}{|D|}\right], \quad (64)$$

$$\sigma_{zz}=\frac{N_e e^2}{m}\left[\sum_{q=0}^{P}\frac{\langle\nu_1\rangle_{q0}L^{q0}-i\omega L^{00}}{|L|}\right]. \quad (65)$$

[17] $C_0=a_{00}{}^0\propto T^{-\frac{3}{2}}\simeq T_g{}^{-\frac{3}{2}}$ when $\epsilon\ll1$.

For the constant collision frequency case, Eqs. (63)–(65) give results identical with Eqs. (60)–(62).

When there is no magnetic field present $\boldsymbol{\sigma}$ becomes diagonal

$$\sigma_{xx}=\sigma_{yy}=\sigma_{zz}\equiv\sigma;\quad H=0.$$

Figures 1–4 illustrate the rapid convergence of the method. Figures 1 and 2 are plots of $(m\omega/N_e e^2)\sigma$ vs $\langle\nu_1\rangle_{00}/\omega$ for a completely ionized gas, $\vartheta=1$, ignoring e-e collisions while Figs. 3 and 4 show the same quantity, but include both e-e and e-i collisions. $\langle\nu_1\rangle_{00}$ is a frequency characteristic of the rate of momentum transfer between electrons and ions:

$$\langle\nu_1\rangle_{00}=\tfrac{4}{3}(2/\pi)^{\frac{1}{2}}\vartheta\lambda.$$

Spitzer and Härm[18] and Landshoff[1] have calculated σ for the d-c ($\omega=0$) case. Spitzer introduced a quantity γ_E to relate σ(e-i only) and σ(e-e and e-i).

$$\sigma_{e\text{-}e\text{ and }e\text{-}i}=\gamma_E\sigma_{e\text{-}i}, \quad (66)$$

and found $\gamma_E=0.582$. Using the $P=3$ approximation γ_E was determined as a function of $\langle\nu_1\rangle_{00}/\omega$. The result is shown in Fig. 5. Figures 6–8 compare $\operatorname{Re}\sigma_{xx}$ for a completely ionized gas with and without e-e collisions for three values of $\langle\nu_1\rangle_{00}/\omega$, in the $P=1$ approximation. Figures 9–11 show $\operatorname{Re}\sigma_{yx}$ for the same values of $\langle\nu_1\rangle_{00}/\omega$.

VI. EXPERIMENTS; APPLICATIONS OF THE THEORY

Very few experiments have been performed with electron concentrations high enough to show the effects of e-e and e-i collisions. Of particular interest is the work of Anderson and Goldstein[12] which shows that there is a critical degree of ionization above which the conductivity is determined by e-i collisions. They measured the conductivity of a plasma formed by the afterglow of a

[18] L. Spitzer and R. Härm, Phys. Rev. 89, 977 (1953).

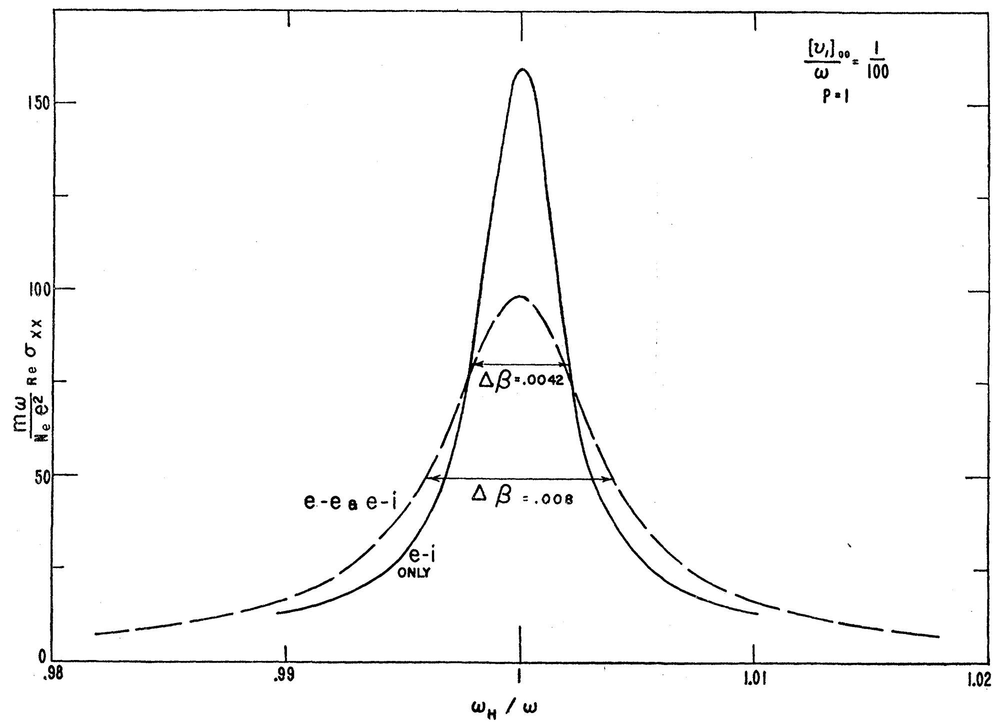

FIG. 6. $(m\omega/N_e e^2)$ Reσ_{xx} vs ω_H/ω for $\langle\nu_1\rangle_{00}/\omega=1/100$, for $P=1$, showing the effect of excluding e-e collisions.

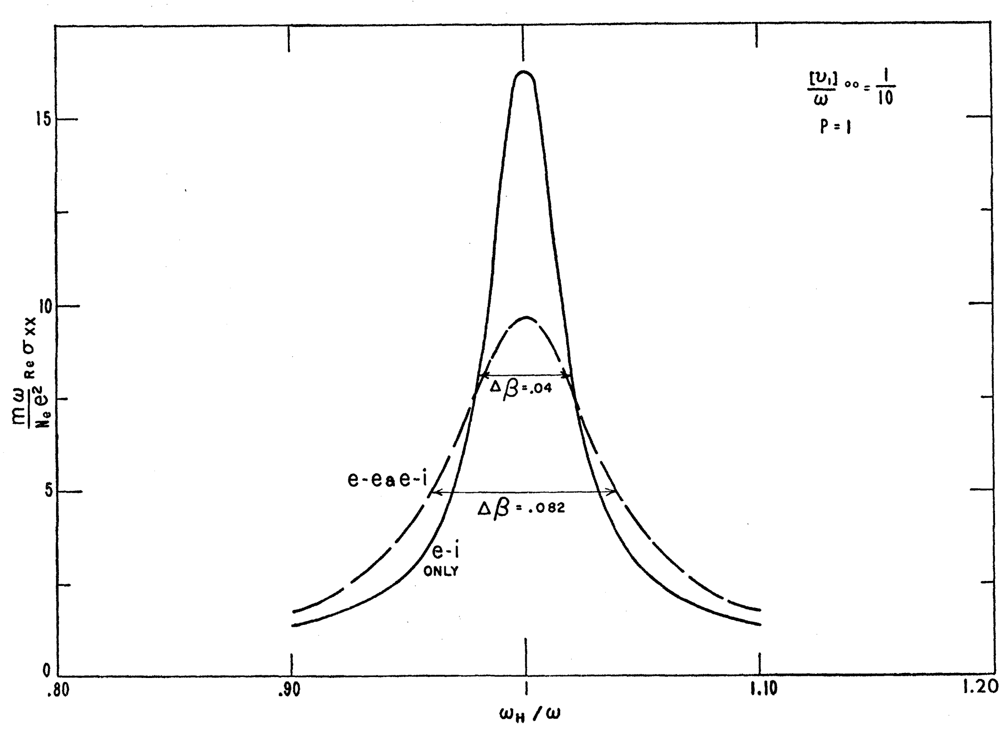

FIG. 7. $(m\omega/N_e e^2)$ Reσ_{xx} vs ω_H/ω for $\langle\nu_1\rangle_{00}/\omega=1/10$, for $P=1$, showing the effect of excluding e-e collisions.

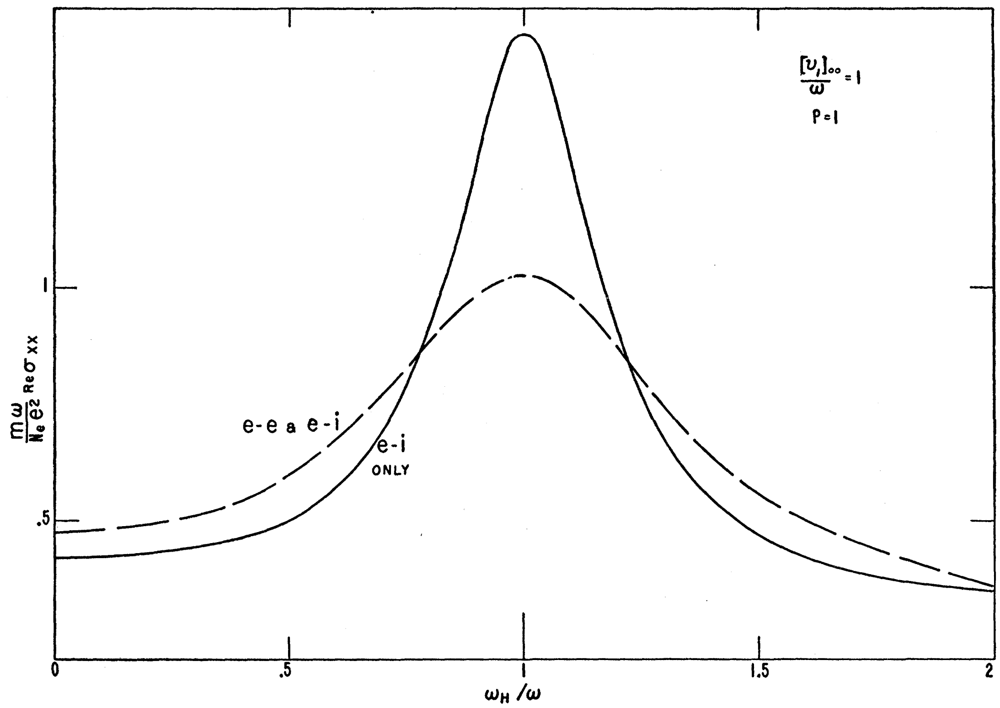

FIG. 8. $(m\omega/N_e e^2)$ Reσ_{xx} vs ω_H/ω for $\langle\nu_1\rangle_{00}/\omega=1$, for $P=1$, showing the effect of excluding *e-e* collisions.

helium discharge, concluding that for pressures above about 5 mm Hg the conductivity was determined by *e-n* collisions. The "effective" *e-n* collision frequency was found to be

$$\nu_{e\text{-}n}=3.12\times10^8 P(T/300)^{\frac{1}{2}},\quad p=\text{pressure mm Hg.} \tag{67}$$

The "effective" collision frequency as it is used here refers to the particular average value of the velocity dependent collision frequency which allows one to write the conductivity in the form

$$\sigma=\frac{N_e e^2}{m}\frac{\nu_{e\text{-}j}-i\omega}{\nu_{e\text{-}j}{}^2+\omega^2},$$

when only type j scatterers are of importance. The conductivity appears in this form in the present theory only in the $P=0$ approximation, where it reads

$$\sigma=\frac{N_e e^2}{m}\frac{\langle\nu_1(j)\rangle_{00}-i\omega}{\langle\nu_1(j)\rangle_{00}{}^2+\omega^2}.$$

At lower pressures *e-i* collisions determine the conductivity, the effective collision frequency being determined as

$$(\nu_{e\text{-}i})_{\text{exp}}=3.6N_iT^{-\frac{3}{2}}\log(3.7\times10^3T^{\frac{3}{2}}N_i^{-\frac{1}{3}}). \tag{68}$$

This should be compared with the corresponding quantity in our calculations

$$(\nu_{e\text{-}i})_{\text{theory}}=\langle\nu_1(\text{ion})\rangle_{00}=\tfrac{4}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lambda$$
$$=3.6N_iT^{-\frac{3}{2}}\log(4.39\times10^3\cdot T^{\frac{3}{2}}N_i^{-\frac{1}{3}}), \tag{69}$$

a result which is almost identical with that of Ginsburg[19] who found

$$(\nu_{e\text{-}i})_{\text{theory}}=3.59N_iT^{-\frac{3}{2}}\log(3.22\times10^3T^{\frac{3}{2}}N_i^{-\frac{1}{3}}).$$

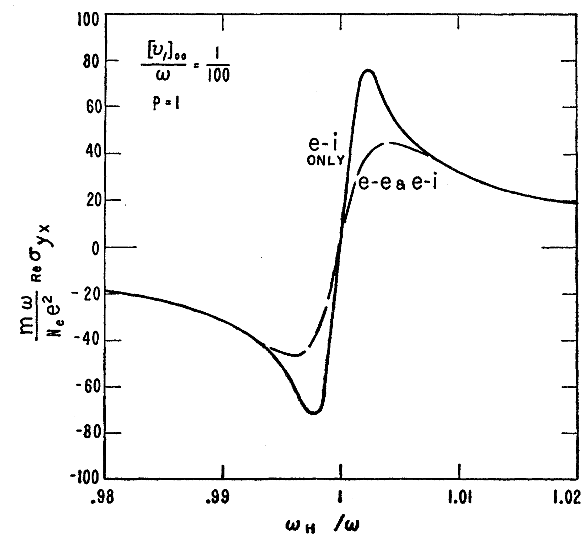

FIG. 9. $(m\omega/N_e e^2)$ Reσ_{yx} vs ω_H/ω for $\langle\nu_1\rangle_{00}/\omega=1/100$, for $P=1$, showing the effect of excluding *e-e* collisions.

[19] V. L. Ginsburg, J. Phys. U.S.S.R. 8, 253 (1944).

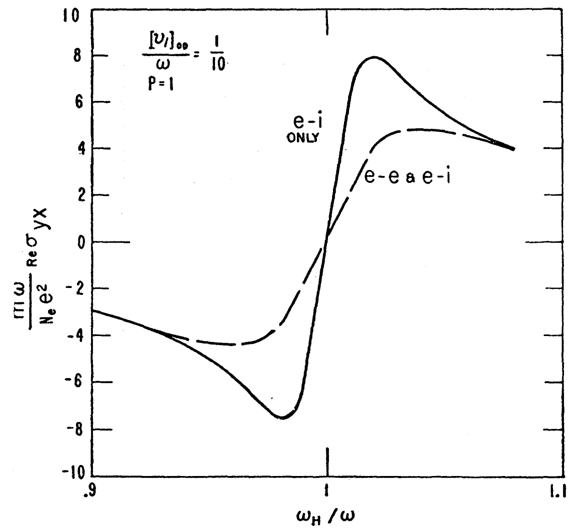

FIG. 10. $(m\omega/N_e e^2)$ Reσ_{yx} vs ω_H/ω for $\langle\nu_1\rangle_{00}/\omega=1/10$, for $P=1$, showing the effect of excluding *e-e* collisions.

The critical degree of ionization may be defined by

$$\nu_{e\text{-}n}=\nu_{e\text{-}i}.$$

Using (67) and (69) one finds that the critical degree of ionization for helium is 2.7×10^{-6} at $T=300$°K.

For hydrogen, Allis and Brown[20] found

$$\nu_{e\text{-}n}=5.9\times10^9 p;\ \text{hydrogen}$$

for electron energies above 2 volts. For air, the work of Rose and Brown[21] indicates that for electron energies of several volts

$$\nu_{e\text{-}n}=4.3\times10^9 p;\ \text{air}.$$

Because of the $T^{-\frac{3}{2}}$ factor in $(\nu_{e\text{-}i})$ the critical degree of ionization is strongly temperature dependent.

Unfortunately the concentration of charged particles in the experiment of Anderson and Goldstein was not high enough to allow verification of the *e-e* contributions to the conductivity calculated in this paper.

Lin, Resler, and Kantrowitz[22] measured the *d-c* conductivity ($\omega=0$) of a high temperature (15 000°) shock-produced plasma. They attained degrees of ionization in excess of 10^{-3}. Their results show favorable agreement with the theory of Spitzer and Härm[18] for the *d-c* conductivity. Our results for the *d-c* case ($\epsilon\ll1$) are of course identical with those of Landshoff, the calculations agreeing to better than one percent with those of Spitzer and Härm in the $P=3$ approximation.

The theory developed here is also useful in interpreting cyclotron resonance experiments.[23,24] The microwave power absorbed by a plasma subjected to a constant magnetic field and microwave electric field

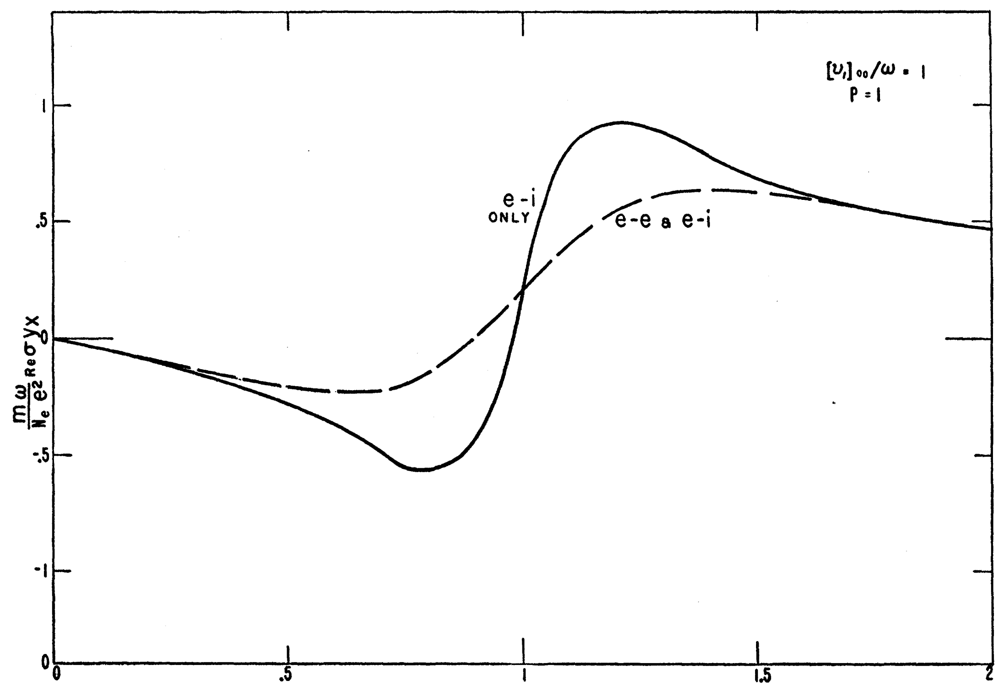

FIG. 11. $(m\omega/N_e e^2)$ Reσ_{yx} vs ω_H/ω for $\langle\nu_1\rangle_{00}/\omega=1$, for $P=1$, showing the effect of excluding *e-e* collisions.

[20] W. P. Allis and S. C. Brown, Phys. Rev. **87**, 419 (1952).
[21] D. J. Rose and S. C. Brown, J. Appl. Phys. **28**, 561 (1957).
[22] S.-C. Lin, E. L. Resler, and A. Kantrowitz, J. Appl. Phys. **26**, 95 (1955).
[23] D. Kelly, H. Margenau, and S. C. Brown, Phys. Rev. **108**, 1367 (1957).
[24] R. M. Hill, Sylvania Microwaves Physics Laboratory, Technical Report MPL-13, 1958 (unpublished).

exhibits a strong resonance peak at $\omega=\omega_H$, provided $\omega\gg\langle\nu_1\rangle_{00}$. For the constant collision frequency case the absorbed power density is

$$P_{\text{abs}}=\frac{N_e e^2 E^2}{4m}\left[\frac{\nu}{\nu^2+(\omega-\omega_H)^2}+\frac{\nu}{\nu^2+(\omega+\omega_H)^2}\right].$$

The width of the resonance curve at half its maximum value is 2ν. For cases where the collision frequency is velocity dependent one generally assumes a power law behavior for the cross section, i.e., one takes

$$w\sigma_j(w,\psi)=S_{h,j}(\psi)w^h. \tag{37}$$

The half-width of the absorption curve allows the determination of h. In this respect, calculations based on the theory developed here[6] are in good agreement with results obtained earlier by the present author[23] and also by Hill[24] who considered the cases $h=0,\ \pm1,\ \pm2$.

There is one notable example where the theory developed here has proved inferior to other approaches. It involves a method developed by Hirshfield and Brown[25] to measure collision frequencies of slow electrons. From Eq. (63) one sees that the imaginary part of σ_{xx} vanishes when

$$\mathrm{Re}D^{00}/|D|=0. \tag{70}$$

At this point the plasma becomes purely resistive and the resonant frequency of a plasma filled cavity is the same as that for an empty cavity. Hirshfield and Brown show that (provided $\omega\gg$mean collision frequency) the condition for zero frequency shift is

$$0=\int_0^\infty\left(1-\frac{\omega_H^2-\omega^2}{\nu^2}\right)v^3\frac{\partial f_0}{\partial v}dv$$
$$=-\int_0^\infty\left\{3v^2-(\omega_H^2-\omega^2)\frac{d}{dv}\left(\frac{v^3}{\nu^2}\right)\right\}f_0dv,$$

[25] J. L. Hirshfield and S. C. Brown, J. Appl. Phys. **29**, 1749–1752 (1958).

with $f_0(v)$ denoting the isotropic part of the distribution function. Taking $\nu=cv^h$ and assuming $f_0(v)$ is Maxwellian they obtain the result

$$\omega_H^2-\omega^2=c^2\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}-h)}\left(\frac{2kT}{m}\right)^h.$$

The theory developed here does not give satisfactory agreement with this result because

$$\int_0^\infty\frac{d}{dv}\left(\frac{v^3}{\nu^2}\right)f_0dv=\int_0^\infty\left[\frac{3}{\nu^2}-\frac{2v}{\nu^3}\frac{d\nu}{dv}\right]f_0v^2dv$$
$$\equiv3\left(\frac{1}{\nu^2}\right)_{\text{av}}-2\left(\frac{v}{\nu^3}\frac{d\nu}{dv}\right)_{\text{av}}$$

is approximated by

$$3\frac{1}{(\nu)_{\text{av}}^2}-2\frac{1}{(\nu)_{\text{av}}^3}\left(v\frac{d\nu}{dv}\right)_{\text{av}}$$

in the $P=0$ case. Taking $P=1$ does little to improve the agreement. More ambitious calculations with larger values of P might prove Eq. (70) useful in situations where the mean collision frequency is comparable with ω.

ACKNOWLEDGMENTS

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