angles. In the formulas (7, 8 of the preceding paper) for D and P_2', Palmieri's values of P_2 were used for these angles while those of this experiment were used at 20°, 25°, and 30°. The uncertainty in D and P_2' arising from the choice of P_2 is negligible compared to the error from counting statistics.

The angular resolution of the second scattering is 3° rms full width at $\theta_2=10°$ and 15°, and 4° at $\theta_2=20°$ 25°, and 30°. The multiple scattering in the energy degrader contributed the bulk of this, namely 3°.

The errors quoted include errors in e_{3n} and P_1P_3 due to counting statistics and misalignment. The only significant error is counting statistics of the e_{3n} measurement. The largest correction to D for misalignment is 0.04, at $\theta_2=30°$. The largest change in D from adjustment of background for correct energy conditions (see section on Backgrounds) is 0.02, at $\theta_2=10°$. The correction to D from the energy variation across the incident beam (see section on The Beam) is between 8% and 14% of the quoted error.

The values of D reported here lie below those measured at 140 Mev[1] by an amount consistent with the Gammel and Thaler potential,[7] which potential correctly predicts values of D at 140 Mev and 315 Mev.[3]

The values of P_2' agree with the P_2 measurements of Palmieri *et al.*,[8] as they should if *p-p* scattering is invariant under time reversal. The values of P_2 also agree, within statistics, with the measurements of Palmieri *et al.*[8]

The values of P_3, when plotted as a function of scattering energy, fall on the smooth curve suggested by the polarization measurements of Dickson and Salter[9] and of Hwang *et al.*[1]

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Decay of μ^- Mesons Bound in the K Shell of Light Nuclei

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For μ^- mesons bound in the K shell of light nuclei of atomic number Z, we calculate the decay electron spectrum accurately up to the first power in Z, both for point and extended nuclei. The decay rate is evaluated accurately up to the second power for point nuclei. Our results for the spectrum show the Doppler smearing of its upper end as obtained previously, and demonstrate the small effect of the nuclear extension. The decay rate is obtained as a monotonically decreasing function of Z, and we cannot explain recent experiments which show a maximum of the decay rate around $Z\sim 26$. We also find that the decay rate in second order decreases much more slowly with Z than what would be obtained from a phase space consideration alone.

I. INTRODUCTION

It has been known for a considerable time that the decay characteristics of a μ^- meson should be different from those of a μ^+ meson. This is due to the fact that μ^- mesons brought to rest in matter end up in a bound state of a mesonic atom, believed to be the ground state,[1] and decay from there. Decay electron spectra have been obtained theoretically,[2–5] essentially for the case of muons bound by light point nuclei only, and show predominantly a Doppler smearing of the upper end of a free muon decay spectrum caused by the orbital motion of the muon ("Primakoff effect"). Recent experiments,[6–11] however, have until now measured only the decay rate λ_- of bound μ^- mesons. Their results show significant deviations from the free μ^+ decay rate λ_+, such that $\lambda_->\lambda_+$ for $Z\sim 26$, $\lambda_-<\lambda_+$ for $Z\gtrsim 30$. A simple phase space argument[8] gives λ_-/λ_+ as a monotonically decreasing function of Z (the

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atomic number of the stopping material), and one is tempted to attribute the rise of λ_- beyond λ_+ at low Z to the attractive Coulomb interaction between the nucleus and the decay electron[7,10]; one might for example speculate that λ_-/λ_+ have an expansion of the form $1+aZ+bZ^2$, where $a>0$ comes from the Coulomb effect on the electron wave function and $b<0$ from the phase space reduction and from Doppler smearing and relativistic time dilatation effects on the muon wave function, with the third term outweighing the second for $Z\gtrsim 30$. It will be shown in Sec. IV that this is not so, i.e., that $a=0$ in the customary theory of muon decay, at least if the muon is assumed to decay from the lowest Bohr orbit of a mesonic atom. This is true whether the binding nucleus has a point charge or an extended one. Moreover, b differs considerably from its value given by the simple phase space argument. We have to conclude, therefore, that the experimental results for $Z\lesssim 30$ cannot be explained at present within the assumptions underlying this paper.

In the following sections, we shall set up the calculation as follows: Sec. II states the wave functions for a point nucleus in a form suitable for an expansion of the matrix element in powers of Z; Sec. III gives expressions for the muon decay probability, and in Sec. IV, the decay electron spectrum for a point nucleus is obtained, accurate up to the first power in Z. In Sec. V, the same is done for an extended nucleus, and in Sec. VI, the decay rate for a point nucleus up to the second power of Z is worked out; the results are discussed in Sec. VII. Three appendices deal with the evaluation of Born approximation integrals and the derivation of the ground-state wave function of a muon bound by an extended nucleus.

II. ELECTRON AND MUON WAVE FUNCTIONS IN THE FIELD OF A POINT NUCLEUS

We shall measure lengths in units of the electron Compton wavelength, and energies and momenta (the latter ones considered to be multiplied by c) in units of the electron rest mass. The ground-state Dirac wave function of a μ^- meson bound by a point nucleus can be written as

$$\psi_\mu(\mathbf{r})=N\begin{pmatrix}(1/r)f(r)\\(1/ir)g(r)\boldsymbol{\sigma}\cdot\hat{r}\end{pmatrix}\varphi_\mu,\tag{1a}$$

with

$$(1/r)f(r)=e^{-\mu\gamma r},\quad (1/r)g(r)=-\tfrac{1}{2}\gamma e^{-\mu\gamma r},\tag{1b}$$

$$N^2=\gamma^3\mu^3/\pi,\tag{1c}$$

with $\boldsymbol{\sigma}$ being the Pauli spin vector, φ_μ the corresponding Pauli spinor, $\hat{r}=\mathbf{r}/r$, and $\gamma=Ze^2/\hbar c=Z/137$. This wave function is accurate *in the zeroth and first power of Z*, or γ; (we shall show later that this is sufficient for obtaining the muon decay rate up to *second order in Z*—the spectrum only up to first order, though), and therefore, the usual factor r to the power $[(1-\gamma^2)^{\frac{1}{2}}-1]$ was left out.[12] For the same reason, we can in (1) take for μ not the muon rest energy μ_0, but the total energy of a muon in the ground state, accurate up to second order in γ:

$$\mu=\mu_0(1-\gamma^2/2).\tag{2}$$

The electron wave function cannot be taken as a plane wave,[4] as has been done frequently and incorrectly[2,3]; but since it is needed accurately only up to the second power in Z, it may be obtained[13] by an iteration of the Dirac equation

$$[i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+p-V(\mathbf{r})]\psi_e(\mathbf{r})=0,\quad \boldsymbol{\alpha}=\begin{pmatrix}0&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{pmatrix}.\tag{3}$$

We use a Yukawa potential

$$V(\mathbf{r})=-(1/r)\gamma_e e^{-\beta r},\tag{4}$$

which will later be made to coincide with the Coulomb potential which binds the muon, by letting $\beta\to 0$; the introduction of β is useful to circumvent certain well-known divergences[14,15] which would occur in individual second-order terms if a pure Coulomb potential were used, but which cancel out when all these terms are combined, as it must be physically. Again $\gamma_e=Z/137$; the index e serves just for identifying the origin of the terms later. Note that the approximation $p\gg 1$ has been made in (3), which is permissible as the electrons from muon decay appear predominantly with extreme relativistic momenta p. The solution satisfying (3) up to second order in γ_e is

$$\psi_e(\mathbf{r})=\frac{1}{\sqrt{2}}\{e^{i\mathbf{p}\cdot\mathbf{r}}+(i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}-p)[\Pi(\mathbf{r})+\Lambda(\mathbf{r})]\}\begin{pmatrix}1\\\boldsymbol{\sigma}\cdot\hat{p}\end{pmatrix}\varphi_e,\tag{5}$$

$$\Pi(\mathbf{r})=\int\frac{d^3s}{(2\pi)^3}\frac{e^{i\mathbf{s}\cdot(\mathbf{r}-\mathbf{r}')}}{s^2-p^2+i\eta}V(\mathbf{r}')e^{i\mathbf{p}\cdot\mathbf{r}'}d^3r',\tag{5a}$$

$$\Lambda(\mathbf{r})=\int\frac{d^3s}{(2\pi)^3}\frac{e^{i\mathbf{s}\cdot(\mathbf{r}-\mathbf{r}')}}{s^2-p^2+i\eta}V(\mathbf{r}')(i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}'-p)\times\Pi(\mathbf{r}')d^3r'$$
$$=i\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda}(\mathbf{r})-p\Omega(\mathbf{r});\tag{5b}$$

here $\eta\to +0$ ensures (5) to represent incoming scattered waves appropriate for the use of $\psi_e(\mathbf{r})$ as a final state,[16] although we will find that outgoing waves might just as well have been used. (This is a property of transition probabilities integrated over angles.[17]) Again, φ_e is a Pauli spinor, and $\hat{p}=\mathbf{p}/p$.

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[13] See, e. g., H. Olsen, Kgl. Norske Videnskab. Selskabs, Forh. **31**, No. 11 (1958).
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III. DERIVATION OF THE TRANSITION PROBABILITY

The interaction Hamiltonian responsible for the muon decay is taken as

$$H=\sum_{i=V,A} f_i(\psi_e{}^\dagger O_i\psi_\mu)(\psi_\nu{}^\dagger O_i\psi_\nu), \tag{6}$$

$O_V=\gamma_4\gamma_\mu$, $O_A=i\gamma_4\gamma_5\gamma_\mu$, with left-handed two-component plane-wave neutrinos. For the decay probability, we then obtain (neglecting all radiative corrections, which are of order 1/137):

$$w_-=(2\pi)^{-8}\int|\mathfrak{M}|^2\delta(\mu-p-p_1-p_2)d^3pd^3p_1d^3p_2, \tag{7}$$

containing the matrix element—or rather, for the sake of convenience, its conjugate complex—,

$$\mathfrak{M}^*=\sum_i f_i^*(u_2{}^\dagger O_iu_1)M_i, \tag{7a}$$

$$M_i=\int e^{i\mathbf{k}\cdot\mathbf{r}}(\psi_\mu{}^\dagger O_i\psi_e)d^3r, \tag{7b}$$

$$\mathbf{k}=\mathbf{q}-\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2.$$

Here, we introduced the vector $\mathbf{q}=\mathbf{p}+\mathbf{p}_1+\mathbf{p}_2$ which represents the momentum of the muon in the Bohr orbit. The u_k are the neutrino spinors, and $\mathbf{p}_1$, $\mathbf{p}_2$ the neutrino momenta. Assuming unpolarized muons and thus averaging over muon spins and also summing over electron spins, we get

$$|\mathfrak{M}|^2=\tfrac{1}{2}\sum_{i,k} f_if_k^*Z_{ik}\sum_{s_e,s_\mu} M_i^*M_k, \tag{8}$$

with the neutrino trace

$$Z_{ik}=\mathrm{Tr}P_2\tfrac{1}{2}(1+\gamma_5)O_i{}^\dagger P_1O_k\tfrac{1}{2}(1+\gamma_5), \tag{9}$$

using neutrino positive energy projection operators

$$P_n=p_{n\alpha}\gamma_\alpha\gamma_4/2p_{n4},\quad \alpha=1\cdots4,\quad p_{n4}=ip_n.$$

The muon-electron matrix element can be decomposed according to the number of interactions of the final electron with the nucleus:

$$M_i=2^{-\frac{1}{2}}N(\varphi_\mu{}^\dagger[T_i+U_i+V_i]\varphi_e); \tag{10}$$

here, T_i, U_i, and V_i correspond to the three Feynman diagrams (a), (b), and (c) of Fig. 1, respectively, and are thus of relative order 1, γ, and γ^2. They read as follows:

$$T_i=(F,\boldsymbol{\sigma}\cdot\hat{q}G)O_i\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\hat{p}\end{pmatrix}, \tag{11}$$

containing the integrals

$$F(q)=\int\frac{1}{r}f(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3r=\frac{4\pi}{q}\int_0^\infty f(r)\sin qr\,dr, \tag{11a}$$

$$\boldsymbol{\sigma}\cdot\hat{q}G(q)=i\int\frac{1}{r}g(r)\boldsymbol{\sigma}\cdot\hat{r}e^{i\mathbf{q}\cdot\mathbf{r}}d^3r$$
$$=-\boldsymbol{\sigma}\cdot\hat{q}\frac{4\pi}{q}\int_0^\infty g(r)\left(\frac{\sin qr}{qr}-\cos qr\right)dr. \tag{11b}$$

Furthermore,

$$U_i=\{(1,0)O_i[-\boldsymbol{\alpha}\cdot(\mathbf{k}\mathrm{I}+i\gamma\mu\mathbf{K})+p\mathrm{I}]-\tfrac{1}{2}i\gamma\mathbf{K}\cdot(0,\boldsymbol{\sigma})O_i(-\boldsymbol{\alpha}\cdot\mathbf{k}+p)\}\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\hat{p}\end{pmatrix}, \tag{12}$$

containing the integrals

$$\mathrm{I}(\mathbf{k},\alpha)=-\int\Pi(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}-\alpha r}d^3r=-\frac{2\gamma_e}{\pi}\frac{\partial}{\partial\alpha}\mathrm{II}(\mathbf{k},\alpha), \tag{12a}$$

$$\mathbf{K}(\mathbf{k},\alpha)=-\int\Pi(\mathbf{r})\hat{r}e^{i\mathbf{k}\cdot\mathbf{r}-\alpha r}d^3r=-\frac{2i\gamma_e}{\pi}\boldsymbol{\nabla}_{\mathbf{k}}\mathrm{II}(\mathbf{k},\alpha), \tag{12b}$$

and

$$\mathrm{II}(\mathbf{k},\alpha)=\int\frac{d^3s}{s^2-p^2+i\eta}\frac{1}{(\mathbf{s}-\mathbf{p})^2+\beta^2}\frac{1}{(\mathbf{s}+\mathbf{k})^2+\alpha^2}, \tag{12c}$$

with $\alpha=\mu\gamma$. In obtaining (12), a partial integration was performed, and terms of order higher than γ^2 were discarded. Finally,

$$V_i=(1,0)O_i(-\boldsymbol{\alpha}\cdot\mathbf{k}+p)(i\boldsymbol{\alpha}\cdot\mathbf{R}-pQ)\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\hat{p}\end{pmatrix}, \tag{13}$$

containing the integrals

$$Q(\mathbf{k},\alpha)=-\int\Omega(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}-\alpha r}d^3r=\frac{\gamma_e{}^2}{\pi^3}\frac{\partial}{\partial\alpha}\mathrm{III}(\mathbf{k},\alpha), \tag{13a}$$

$$\mathbf{R}(\mathbf{k},\alpha)=-\int\boldsymbol{\Lambda}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}-\alpha r}d^3r=\frac{i\gamma_e{}^2}{\pi^3}\frac{\partial}{\partial\alpha}\mathbf{III}(\mathbf{k},\alpha), \tag{13b}$$

and

$$\mathrm{III}(\mathbf{k},\alpha)=\int\frac{d^3s}{s^2-p^2+i\eta}\frac{1}{(\mathbf{s}+\mathbf{k})^2+\alpha^2}\int\frac{d^3t}{t^2-p^2+i\eta}\times\frac{1}{(\mathbf{t}-\mathbf{p})^2+\beta^2}\frac{1}{(\mathbf{t}-\mathbf{s})^2+\beta^2}, \tag{13c}$$

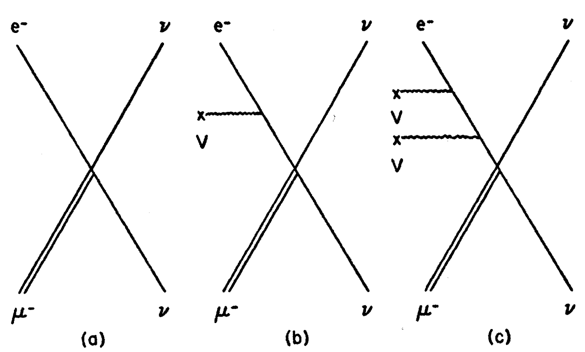

FIG. 1. Feynman diagrams for the decay of bound muons. The double line indicates a bound state, V the final electron-nucleus interaction.

$$\mathrm{III}(\mathbf{k},\alpha)=\int\frac{d^3s}{s^2-p^2+i\eta}\frac{1}{(\mathbf{s}+\mathbf{k})^2+\alpha^2}\int\frac{d^3t}{t^2-p^2+i\eta}\times\frac{\mathbf{t}}{(\mathbf{t}-\mathbf{p})^2+\beta^2}\frac{1}{(\mathbf{t}-\mathbf{s})^2+\beta^2}. \quad (13\text{d})$$

To obtain this, we used the integral representation of the potential,

$$V(\mathbf{r})=-\gamma_e\int_\beta^\infty e^{-\xi r}d\xi.$$

The matrix element (8) now becomes

$$|\mathfrak{M}|^2=\tfrac{1}{4}N^2\sum_{i,k}f_if_k^*Z_{ik}\times[\mathrm{Tr}T_i^\dagger T_k+\mathrm{Tr}(T_i^\dagger U_k+U_i^\dagger T_k)+\mathrm{Tr}U_i^\dagger U_k+\mathrm{Tr}(V_i^\dagger T_k+T_i^\dagger V_k)]; \quad (14)$$

inserting into (7) gives after laborious evaluation of the traces:

$$\begin{aligned}w_-={}&N^2(2\pi)^{-8}(|f_V|^2+|f_A|^2)\\&\times d^3p\int d^3qd^3p_1\,\delta(\mu-p-p_1-p_2)\\&\times\{(F^2+G^2)(1-(\hat{p}_1+\hat{p}_2)\cdot\hat{p}/2)\\&+FG[\hat{p}_1\cdot\hat{p}\hat{p}_2\cdot\hat{q}+\hat{p}_2\cdot\hat{p}\hat{p}_1\cdot\hat{q}-(\hat{p}_1+\hat{p}_2)\cdot\hat{q}]\\&+2F\hat{p}\cdot\mathrm{Re}\mathbf{J}-F(\hat{p}_1+\hat{p}_2)\cdot\mathrm{Re}\mathbf{J}+\hat{p}_1\cdot\boldsymbol{\kappa}\hat{p}_2\cdot\mathrm{Re}\mathbf{P}\\&+\hat{p}_2\cdot\boldsymbol{\kappa}\hat{p}_1\cdot\mathrm{Re}\mathbf{P}-\hat{p}\cdot\boldsymbol{\kappa}(\hat{p}_1+\hat{p}_2)\cdot\mathrm{Re}\mathbf{P}\\&+|\mathrm{I}|^2[\kappa^2(1+(\hat{p}_1+\hat{p}_2)\cdot\hat{p}/2)-\hat{p}\cdot\boldsymbol{\kappa}(\mathbf{p}_1+\mathbf{p}_2)\cdot\boldsymbol{\kappa}]\\&+F[-2\boldsymbol{\kappa}\cdot\mathrm{Re}(\mathbf{p}Q-i\mathbf{R})+(\hat{p}_1+\hat{p}_2)\\&\cdot(\boldsymbol{\kappa}p\,\mathrm{Re}Q-\hat{p}\,\mathrm{Re}i(\boldsymbol{\kappa}\cdot\mathbf{R})-\mathrm{Re}i\mathbf{R}(\hat{p}\cdot\mathbf{q})\\&+\mathbf{q}\,\mathrm{Re}i(\hat{p}\cdot\mathbf{R}))]\}, \quad (15)\end{aligned}$$

with the notation

$$\begin{aligned}&\mathbf{J}=\boldsymbol{\kappa}\mathrm{I}-i\gamma\mu\mathbf{K}, &&\mathbf{P}=-\tfrac{1}{2}i\gamma F\mathbf{K}+G\mathrm{I}\hat{q},\\&\boldsymbol{\kappa}=2\mathbf{p}-\mathbf{q}, &&\hat{p}_i=\mathbf{p}_i/p_i.\end{aligned} \quad (15\text{a})$$

Next, we shall perform the phase space integration over d^3p_1.[18] The δ function eliminates the dp_1 integration, expressing p_1 as a function of the polar angle, which we choose to be measured with respect to $\mathbf{k}$ as the polar axis. At this point, we shall also introduce a dimensionless notation for the momenta:

$$p=\tfrac{1}{2}\mu x,\quad q=\tfrac{1}{2}\mu y,\quad w=\cos(\mathbf{p},\mathbf{q}). \quad (16)$$

Then, we us enew expressions for the quantities (11)–(13). First, since the only vectors left are $\mathbf{p}$ and $\mathbf{q}$, we can express the vector quantities as follows:

$$\begin{aligned}-\tfrac{1}{2}\gamma\,\mathrm{Re}(i\mathbf{K})&=S_1\mathbf{q}+S_2\mathbf{p},\\-\mathrm{Re}(i\mathbf{R})&=H_1\mathbf{q}+H_2\mathbf{p}.\end{aligned} \quad (17)$$

For getting rid of some inconvenient factors, we now replace the integrals $F, G, \mathrm{I}, S_{12}, Q, H_{12}$ by the new quantities (all functions of x, y, and w):

$$\{\Phi;\Gamma\}=\tfrac{1}{2}(2\pi)^{-\frac{3}{2}}(N^2\pi\mu^3)^{\frac{1}{2}}\{F;G\}, \quad (18\text{a})$$

$$\Theta=\tfrac{1}{2}(2\pi)^{-\frac{3}{2}}(N^2\pi\mu^3)^{\frac{1}{2}}\tfrac{1}{2}\mu\mathrm{I}, \quad (18\text{b})$$

$$\{\Psi_{12};\Sigma;\Delta_{12}\}=\tfrac{1}{2}(2\pi)^{-\frac{3}{2}}(N^2\pi\mu^3)^{\frac{1}{2}}\times(\tfrac{1}{2}\mu)^2\{S_{12};\mathrm{Re}Q;H_{12}\}. \quad (18\text{c})$$

Thus, we finally end up with the expression for the decay probability:

$$\begin{aligned}w_-={}&w_+\epsilon^5(3-2x)^{-1}\int y^2dydw\\&\times\{(\Phi^2+\Gamma^2)W_0-2\Phi\Gamma W_1+2\Phi\,\mathrm{Re}\Theta W_{1e}+4\Phi\Psi_1W_{2e}\\&+4\Phi\Psi_2W_{2e}'-4\Gamma\,\mathrm{Re}\Theta W_{2e}''+4|\Theta|^2W_{2ee}\\&-4\Phi(\Sigma+\Delta_2)\bar{W}_{2ee}-4\Phi\Delta_1\bar{W}_{2ee}'\}, \quad (19)\end{aligned}$$

with the following notation:

$$w_+=(\mu_0^5/3.2^7\pi^3)(|f_V|^2+|f_A|^2)(3-2x)x^2dx \quad (19\text{a})$$

is the decay probability of the free muon; $\epsilon=1-\gamma_p^2/2$ is the ratio of bound and free muon total energy [see Eq. (2)], and ϵ^5 represents the "phase space factor," expressing a reduction of decay rate, as the bound muon has less final states available. For identification purposes, we again used γ_p instead of γ. Further:

$$\begin{aligned}W_0&=3-2x-(1-x)yw-y^2/4,\\W_1&=(1-2x)w+(1+xw^2)y,\\W_{1e}&=2x(3-2x)-(3-2x^2)yw,\\W_{2e}&=(6-5x+2x^2)yw,\\W_{2e}'&=(6-5x+2x^2)x,\\W_{2e}''&=(1-2x)xw,\\W_{2ee}&=\bar{W}_{2ee}=(3-2x)x^2,\\\bar{W}_{2ee}'&=(3-2x)xyw.\end{aligned} \quad (19\text{b})$$

The indices of the W's are chosen to indicate the order of magnitude of the various terms; index i means the term will furnish a contribution of order γ^i to w_-—unless there is a cancellation—plus possible higher order contributions. It should be pointed out that we have dropped from the W's some terms containing higher powers of y: it will be shown later that all the quantities Φ, Γ, Θ, etc., are rapidly decreasing functions of y, which are large only within a region $y\lesssim\gamma$. Therefore, each power of y means a corresponding power of γ in the result, and we can discard the higher powers. The further subscripts e and ee of the W's indicate the origin of the term: e means cross-term between diagrams (a)

[18] See the Appendix by R. R. Rau in J. Tiomno and J. A. Wheeler, Revs. Modern Phys. 21, 144 (1949).

and (b) of Fig. 1, *ee* is the square-term (b) (first Born approximation of the electron wave function), and $\bar{W}$ is the cross-term (a) and (c) (second Born approximation).

The limits of integration in (19) have to be found from the conservation laws:

$$\mu=p+p_1+p_2, \quad \mathbf{q}=\mathbf{p}+\mathbf{p}_1+\mathbf{p}_2. \tag{20}$$

In our dimensionless notation, they are found as follows:

(1) for $0\leqslant x\leqslant 1$:

$$\begin{aligned}&\text{(a)}\quad 0\leqslant y\leqslant 2|1-x|, \quad -1\leqslant w\leqslant 1,\\ &\text{(b)}\quad 2|1-x|\leqslant y\leqslant 2, \quad w_0\leqslant w\leqslant 1;\end{aligned} \tag{20a}$$

(2) for $1\leqslant x\leqslant 2$:

$$\begin{aligned}&\text{(a)}\quad 0\leqslant y\leqslant 2|1-x|, \quad \text{no contribution},\\ &\text{(b)}\quad 2|1-x|\leqslant y\leqslant 2, \quad w_0\leqslant w\leqslant 1,\end{aligned} \tag{20b}$$

where

$$w_0=\frac{1}{x}\frac{y}{2}-\frac{1-x}{x}\frac{2}{y}. \tag{20c}$$

Note that the electron spectrum for bound muons reaches in principle up to $x=2$, whereas for free muons, the conservation laws permit only $x\leqslant 1$.

Thus, the integral in (19) has to be understood to mean

$$\int y^2dydw=\vartheta(1-x)\int_0^{2(1-x)} y^2dy\int_{-1}^1 dw + \int_{2|1-x|}^2 y^2dy\int_{w_0}^1 dw, \tag{21}$$

with the step function $\vartheta(z)=1(z>0)$, $0(z<0)$.

We also observe that (19) contains only real parts and one squared imaginary part of the integrals II, III, and **III**. This means that the result is independent of the sign of η, and thus the same whether we used in- or outgoing scattered waves, as has been stated before.

IV. ELECTRON SPECTRUM: CASE OF POINT NUCLEUS

We wish to calculate the spectrum accurately up to the first power in γ. For this, only the terms with W_0, W_1, and W_{1e} of (19) are needed. We can then use the muon wave functions (1), and obtain from (11a), (11b), and (19a):

$$\Phi=\left(\frac{2\gamma}{\pi}\right)^{\frac{1}{2}}\frac{2^4\gamma^2}{Y^2}, \quad \Gamma=\left(\frac{2\gamma}{\pi}\right)^{\frac{1}{2}}\frac{2^2\gamma^2y}{Y^2}, \tag{22a}$$

with

$$Y=y^2+4\gamma^2. \tag{22b}$$

Note, however, that according to the accuracy stated, we need only use

$$\Phi^2+\Gamma^2\cong\Phi^2=2^9\gamma^5/\pi Y^4; \tag{22c}$$

this is connected with the fact that the normalization factor N was obtained with sufficient accuracy from

$$4\pi N^2\int_0^\infty [f^2(r)+g^2(r)]dr\cong 4\pi N^2\int_0^\infty f^2(r)dr=1$$

alone, and the consequent normalization of the transforms (11a), (11b) is

$$\int_0^\infty y^2dy\int_{-1}^1 dw(\Phi^2+\Gamma^2)=1; \tag{22d}$$

only (22c) satisfies this condition together with the normalization factor (1c).

For obtaining $\mathrm{Re}\Theta$, the integral II has to be evaluated. This is done in Appendix I, with the result:

$$\mathrm{Re}\Theta=\left(\frac{2\gamma}{\pi}\right)^{\frac{1}{2}}\frac{\gamma\gamma_e}{xY}\times\left\{-2\left[\frac{yw}{Z}+\frac{4\gamma}{Y}\tan^{-1}\left(\frac{yw}{2\gamma}\right)\right]+\frac{Y}{x}\left(\frac{1}{Z}-\frac{2y^2w^2}{Z^2}\right)\right\}, \tag{22e}$$

with

$$Z=y^2w^2+4\gamma^2. \tag{22f}$$

In obtaining (22d) from (A5a) *via* (12a) and (18b), we have again expanded and kept terms of lowest and next lowest order in γ only, treating y as being of order γ [see the remarks after Eq. (19b)]. This can now be justified as follows: The coefficients of the W's in (19), as far as calculated, all contain $1/Y$ to at least third power, times a smooth function of y. Therefore, at $y\sim\gamma=Z/137\ll 1$, they are by roughly a factor γ^{-6} larger than at $y\sim 1$. Therefore, y can safely be taken as $\sim\gamma$. This argument will be seen to apply to all the terms in (19). It is also valid if the Φ, Γ, etc. are obtained for an extended nucleus, as in the next section.

The spectrum to accuracy γ is thus obtained from:

$$w_-^{(0)}=w_+\epsilon^5(3-2x)^{-1}\times\left[\vartheta(1-x)\int_0^{2(1-x)} y^2dy\int_{-1}^1 dw+\int_{2|1-x|}^2 y^2dy\int_{w_0}^1 dw\right]\times[(\Phi^2+\Gamma^2)W_0-2\Phi\Gamma W_1+2\Phi\,\mathrm{Re}\Theta W_{1e}]. \tag{23}$$

The property of the integrand containing at least a factor Y^{-3} that makes it negligible at $y\sim 1$, (and which will hold even for an extended nucleus), permits us already to read off several interesting facts. Let us designate the region where x lies within an interval of range γ around 1, by $\langle\gamma\rangle$, i.e., $|1-x|=O(\gamma)$, and the remaining region by $\langle 1\rangle$. We then find:

(1) In $\langle 1\rangle$,

$$w_-^{(0)}=w_++O(\gamma^2).$$

Proof: The second integral in (23) is then negligible, its lower limit not reaching down to the region where the integrand is large. In the first integral, the w integration cancels out all the terms which would be $\sim O(\gamma)$, as they are all odd in w: this is partly caused by the W factors, in the third term by the oddness of the large part of $\mathrm{Re}\Theta$ (first bracket) itself. The term $\sim O(1)$ comes from W_0 and gives just w_+ for $x<1$ (zero for $x>1$), as we can use the normalization (22d), and the factors $(3-2x)$ cancel out. Therefore, a significant deviation of the bound from the free spectrum can only occur in the region $\langle\gamma\rangle$ (this disagrees with assertions of Tenaglia[4]).

(2) In $\langle\gamma\rangle$,

$$w_-^{(0)}=\tfrac{1}{2}w_+ + [\text{terms} \sim O(1) \text{ which are odd around } x=1]+O(\gamma).$$

Proof: For the terms of order 1, it is sufficient to consider $\Phi^2+\Gamma^2$ only. The first w integration gives a factor 2, the second one a factor $1-2(1-x)/xy$, which is entirely of order 1. The two y integrals can then be combined (making use of the y parity) to give us a leading term for $x \lessgtr 1$:

$$(3-2x)\int_{2(x-1)}^{2} y^2 dy(\Phi^2+\Gamma^2)\left(1-\frac{2}{y}\frac{1-x}{x}\right);$$

now the 1 in the bracket provides the $\frac{1}{2}w_+$, using (22d) again, and the second term in the bracket is odd in $1-x$, as claimed. The next higher terms are $\sim O(\gamma)$.

(3) This allows us to draw the significant conclusion about the decay rate $\lambda_\pm$ (integral of $w_\pm$ over dx):

$$\lambda_-=\lambda_+ + O(\gamma^2). \tag{24}$$

Proof: It is obvious for the contribution from $\langle 1\rangle$; and from $\langle\gamma\rangle$, the $\frac{1}{2}w_+$ to *both* sides of $x=1$ just adds up to make λ_- completely equal to λ_+, whereas the odd terms $\sim O(1)$ cancel out, and the correction terms $\sim O(\gamma)$, by integration over a regio $dx\sim\langle\gamma\rangle$ only, are degraded to order γ^2. We see that actually the linear term in Z in λ_-/λ_+ is absent, as stated in the introduction.

Performing the integration in (23) and writing the result accurately in order 1, γ, we obtain[5]:

$$\begin{aligned} w_-^{(0)} \cong w_+ - \frac{\epsilon^5}{\pi}\Bigg\{ & \tan^{-1}\frac{1}{\gamma}+\tan^{-1}\frac{1-x}{\gamma} \\ & +\gamma\frac{2-x}{X}\left[(1-x)\left(1+\frac{2}{3}\frac{\gamma^2}{X}\right)+\frac{\gamma^2}{3}\right] \\ & +\gamma\left[2\gamma\frac{1-x}{X}\left(1+\frac{2}{3}\frac{\gamma^2}{X}\right)\tan^{-1}\frac{1-x}{\gamma}\right. \\ & \left.+\left(\tan^{-1}\frac{1-x}{\gamma}\right)^2-\frac{\pi^2}{4}+\frac{\gamma^2}{X}\left(1-\frac{2}{3}\frac{\gamma^2}{X}\right)\right]\Bigg\}, \end{aligned} \tag{25}$$

with $X=(1-x)^2+\gamma^2$. This is plotted as a broken curve in Fig. 2 for $Z=25$ and $Z=40$ (although we cannot expect it to represent the spectrum accurately for $Z=40$ any more). We see the smearing-out of the free-muon spectrum in the region $\langle\gamma\rangle$. For $x \gtrsim 1.1$, the spectrum is slightly negative in this order. We plotted against $x_0 = \epsilon x$, so that this scale be proportional to the *absolute* (Mev) energy scale.

Integration over dx of (25) and of the terms $\sim O(\gamma^2)$ in $\langle 1\rangle$, which come from (23) but have not been included in (25), and also expanding ϵ^5, gives the result

$$\lambda_-^{(0)}=\lambda_+[1-\tfrac{5}{2}\gamma_p^2-3\gamma^2+(10/3)\gamma\gamma_e]. \tag{26}$$

For a calculation of the complete decay rate, the remaining terms in (19), all of order γ^2, should be integrated. This will be done in Sec. VI. We can, however, prove here our assertion of Sec. II that the muon wave function need to be known accurately to order γ only. It enters in Φ and Γ, and γ^2 terms of the wave function could give γ^2 terms of w_- or λ_- only through the zero order term $\Phi^2+\Gamma^2$ in (19). This term, however, gives rise in region $\langle 1\rangle$ to w_+ *exactly* [statement (1)] without correction terms, only due to the fact that the wave function, whatever it is, is normalized and thus obeys (22d); in region $\langle\gamma\rangle$, there are correction terms of order 1 to $\frac{1}{2}w_+$, though [statement (2)], and these could be changed in second order if the wave function is. But if we consider the rate only, this region contributes in higher order only, as $dx\sim O(\gamma)$. Thus, the decay rate can be obtained in second order from a muon wave function accurate in first order.

V. ELECTRON SPECTRUM: CASE OF EXTENDED NUCLEUS

The lowest Bohr orbit of the mesic atom has a radius not much larger than the nuclear radius, especially for heavy atoms. It seems necessary, therefore, to take the nuclear extension into account in the present calculation. We shall do this only concerning the first order terms, and it will turn out that the ensuing change in the spectrum is rather small. To evaluate the muon wave function, we shall assume a uniform nuclear

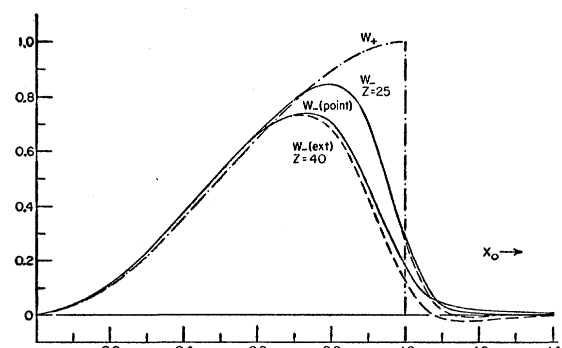

FIG. 2. Decay electron spectrum of bound muons for $Z=25$ and $Z=40$. Broken line: point nucleus; solid line: extended nucleus; $x_0=\epsilon x$.

charge distribution of radius R, which gives rise to a potential

$$V(r) = -(\gamma_e/R)(\tfrac{3}{2} - \tfrac{1}{2}r^2/R^2), \quad r<R, \qquad (27)$$
$$= -(\gamma_e/r), \quad r>R.$$

The corresponding ground-state muon wave function is worked out to some approximation in Appendix II by a power series expansion of the Dirac equation. Using these wave functions ($A9$) with $\sigma=0$, new expressions for Φ and Γ have been obtained:

$$\Phi = N\Bigg\{3\left(\frac{1}{3}+\frac{A}{5}+\frac{B}{7}\right)\frac{j_1(k)}{k} - 2\left(\frac{A}{5}+\frac{2B}{9}\right)\frac{j_3(k)}{k} + \frac{8B}{63}\frac{j_5(k)}{k} + (1+A+B)\left[\frac{\sin k}{k}\left(\frac{\delta-1}{\delta^2+k^2}+\frac{2\delta^2}{(\delta^2+k^2)^2}\right) + \cos k\left(\frac{1}{\delta^2+k^2}+\frac{2\delta}{(\delta^2+k^2)^2}\right)\right]\Bigg\},$$

$$\Gamma = \tfrac{1}{2}\gamma Nk\Bigg\{\lambda\frac{j_2(k)}{k^2} + \frac{1+A+B}{k^2} \times \left[\frac{\sin k}{k}\left(1+\frac{\delta(3-\delta)}{\delta^2+k^2} - \frac{2\delta^3}{(\delta^2+k^2)^2}\right) + \cos k\left(\frac{2-\delta}{\delta^2+k^2} - \frac{2\delta^2}{(\delta^2+k^2)^2}\right)\right]\Bigg\}, \qquad (28)$$

with $j_l(k)$ the spherical Bessel functions, $k=(\gamma_1\epsilon/2\gamma)y$, and a new $\epsilon=1-\zeta\gamma^2$ appropriate to an extended nucleus. Numerical inspection of (28) shows these functions to have a similar steeply decreasing behavior with y as the point nucleus functions, (22a).

Using Eq. (23) for the spectrum again, also ReΘ has to be worked out for the extended nucleus. It is now difficult to do this analytically with the uniform nuclear charge distribution used for calculating the muon wave function; it is possible however if we assume that the electron sees a somewhat different charge distribution, which is of the form $r^n \exp(-br)$; in this case then, a method which was devised by Budini and Furlan[19] will work. Using the expression on the extreme right of Eq. (12a), the new integral II becomes for the extended nucleus (we may set $\beta=0$ if we consider the real part only):

$$\mathrm{ReII}(\mathbf{k},\alpha) = \mathrm{Re}\int \frac{d^3s}{s^2-p^2+i\eta}\,\frac{F(|\mathbf{s}-\mathbf{p}|)}{(\mathbf{s}-\mathbf{p})^2}\,\frac{1}{(\mathbf{s}+\mathbf{k})^2+\alpha^2}, \qquad (29)$$

where

$$F(q) = -\int \rho(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3r \qquad (29a)$$

is the Fourier transform (form factor) of the nuclear charge distribution. Performing a Laplace transformation

$$r\rho(r) = \int_0^\infty \varphi(t)e^{-rt}dt, \qquad (30)$$

we find the form factor relation

$$F(q) = -4\pi\int_0^\infty t^{-2}\varphi(t)F_t(q)dt, \qquad (31)$$

where $F_t(q)$ is the Fourier transform of a Yukawa charge distribution

$$\rho_t(r) = -(t^2/4\pi)e^{-tr}/r \qquad (32)$$

(normalized to unity), and accordingly obtain

$$\mathrm{ReII}(\mathbf{k},\alpha) = -4\pi\int_0^\infty t^{-2}\varphi(t)\,\mathrm{ReII}_t(\mathbf{k},\alpha)dt, \qquad (33)$$

with

$$\mathrm{ReII}_t(\mathbf{k},\alpha) = \mathrm{ReII}_{\mathrm{point}}(\mathbf{k},\alpha;\beta=0) - \mathrm{ReII}_{\mathrm{point}}(\mathbf{k},\alpha;\beta=t), \qquad (33a)$$

using the point nucleus functions of Eq. (12c) on the right-hand side, with special values of β as indicated. If we take now a charge distribution of the aforementioned form (normalized to unity):

$$\rho(r) = -\frac{b^{n+3}r^n}{4\pi(n+2)!}e^{-br}, \qquad (34)$$

the Laplace transform is simply a derivative of the delta function,

$$\varphi(t) = \frac{b^{n+3}}{4\pi(n+2)!}\frac{\partial^{n+1}}{\partial t^{n+1}}\delta(t-b), \qquad (34a)$$

and we obtain

$$\mathrm{ReII}(\mathbf{k},\alpha) = \mathrm{ReII}_{\mathrm{point}}(\mathbf{k},\alpha;\beta=0) + (-1)^n\frac{b^{n+3}}{(n+2)!}\frac{\partial^{n+1}}{\partial b^{n+1}}\left[\frac{1}{b^2}\,\mathrm{ReII}_{\mathrm{point}}(\mathbf{k},\alpha;\beta=b)\right]. \qquad (35)$$

The question is now how well (34) can describe the actual charge distribution (which is, e.g., given by Hahn *et al.*[20]). A comparison shows that the physically significant quantity $r^2\rho(r)$ seems to be approximated fairly well by (34), especially for larger $n\gtrsim 2$, although its peak lies at smaller r than that of the actual charge distribution. For the uniform charge distribution, by contrast, the peak lies at larger r and is too high. For these estimates, we determined R and b from the requirement that each model charge distribution should have the same mean square radius as the actual one.

[19] P. Budini and G. Furlan, Nuovo cimento **13**, 790 (1959).

[20] B. Hahn, D. G. Ravenhall, and R. Hofstadter, Phys. Rev. **101**, 1131 (1956).

That gives:

$$R=1.3\times10^{-13}A^{\frac{1}{3}}\text{ cm},^{21}$$
$$n=1,\quad b^{-1}=0.225\times10^{-13}A^{\frac{1}{3}}\text{ cm},\tag{36}$$
$$n=2,\quad b^{-1}=0.184\times10^{-13}A^{\frac{1}{3}}\text{ cm}.$$

The second integral $\text{ReII}_{\text{point}}$ in (35) has been partly evaluated along the lines of Appendix I, but with β set equal to b; in view of the values (36), it has then been found to give a contribution of at least one order of γ smaller than the first term of (35), and can thus be dropped. Therefore, the expression (22e) can be used for $\text{Re}\Theta$ even for the extended nucleus. This and (28) have been inserted in (23), and the electron spectrum was calculated with the aid of an IBM-650 electronic computer. The results are shown as a solid curve in Fig. 2 for $Z=25$ and $Z=40$. We see that the influence of the nuclear extension is rather small, and increases with larger Z. The main differences as compared to the point charge results, namely those occurring in the region $x_0\gtrsim1.1$, are probably physically not significant, as terms of order γ^2 are not negligible here.[5]

VI. DECAY RATE: CASE OF POINT NUCLEUS

To obtain the muon decay rate, all the second power terms of (19) have to be evaluated and integrated over dx and the result added to (26). Note that everything is already of order γ^2; therefore only the first integration in (21) needs to be taken—the second one gives results $\sim O(\gamma^2)$ only in the region $\langle\gamma\rangle$, therefore $dx\sim O(\gamma)$, and the contribution to the rate is $\sim O(\gamma^3)$ which is negligible. The functions Ψ_{12} were worked out from (12b), (17) and (18c), using the first Appendix, with the result (to lowest order):

$$\Psi_1=-\left(\frac{2\gamma}{\pi}\right)^{\frac{1}{2}}\frac{2\gamma^2\gamma_e}{xY^2}\tan^{-1}\left(\frac{yw}{2\gamma}\right),$$
$$\Psi_2=\left(\frac{2\gamma}{\pi}\right)^{\frac{1}{2}}\frac{2\gamma^3\gamma_e}{x^2YZ}.\tag{37}$$

If, so far, we collect all terms $\sim O(\gamma^2)$ in (19) which contain no divergence and no second Born approximation,

$$w_-^{(1)}=w_+\epsilon^5(3-2x)^{-1}\int y^2dydw[4\Phi\Psi_1W_{2e}+4\Phi\Psi_2W_{2e}'$$
$$-4\Gamma\,\text{Re}\Theta W_{2e}''+4(\text{Re}\Theta)^2W_{2ee}],\tag{38}$$

then we obtain after performing the integrations:

$$\lambda_-^{(1)}=\lambda_+[(5/3)\gamma\gamma_e+(\pi^2/12)\gamma_e^2].\tag{38a}$$

For obtaining the remaining terms, the second Born approximation integrals $\text{III}(\mathbf{k},\alpha)$ and $\mathbf{III}(\mathbf{k},\alpha)$, Eq. (13c), (13d), have to be worked out. The evaluation of such seemingly complicated integrals has recently been facilitated by the work of Kacser,[22] and we have made use of the methods developed by him. The integrals are obtained in Appendix III. The remaining terms $\sim O(\gamma^2)$ are just

$$w_-^{(2)}=w_+\epsilon^5(3-2x)^{-1}\int y^2dydw[4(\text{Im}\Theta)^2W_{2ee}$$
$$-4\Phi(\Sigma+\Delta_2)\bar{W}_{2ee}-4\Phi\Delta_1\bar{W}_{2ee}'],\tag{39}$$

and the last parts of the integrand still needed are, from (A5b):

$$\text{Im}\Theta=\left(\frac{2\gamma}{\pi}\right)^{\frac{1}{2}}\frac{2^3\gamma^2\gamma_e}{xY^2}\left(\ln\frac{\lambda x}{Y}Z^{\frac{1}{2}}+1-\frac{Y}{2Z}\right),\tag{40}$$

with $\lambda=\beta/p\to0$, and from (A17), (A20) and (A22):

$$\Sigma=\left(\frac{2\gamma}{\pi}\right)^{\frac{1}{2}}\frac{2\gamma^2\gamma_e^2}{x^2Y^2}\left[\ln^2\frac{\lambda x}{Y}Z^{\frac{1}{2}}+\frac{\pi^2}{6}-\left(\tan^{-1}\frac{yw}{2\gamma}\right)^2\right.$$
$$\left.+2\left(1-\frac{Y}{2Z}\right)\ln\frac{\lambda x}{Y}Z^{\frac{1}{2}}-\frac{Y}{Z}\frac{yw}{2\gamma}\tan^{-1}\frac{yw}{2\gamma}\right],\tag{41}$$

$$\Delta_2=\Sigma,\quad\Delta_1\cong0.\tag{42}$$

Inserting all this in (39), we see that all the terms containing the logarithmic divergence as $\lambda\to0$ cancel out, as it must be. Performing the integrations, we obtain

$$\lambda_-^{(2)}=-\lambda_+(\pi^2/12)\gamma_e^2,\tag{43}$$

and our final result for the total rate $\lambda_-=\lambda_-^{(0)}+\lambda_-^{(1)}+\lambda_-^{(2)}$ is from (26), (38a), and (43), accurate to the second order in γ:

$$\lambda_-=\lambda_+(1-\tfrac{5}{2}\gamma_p^2-3\gamma^2+5\gamma\gamma_e),\tag{44a}$$

or if we now disregard the origin of the terms:

$$\lambda_-=\lambda_+(1-\tfrac{1}{2}\gamma^2),\tag{44b}$$

the decay rate of a muon bound by a point nucleus.

VII. DISCUSSION

The experimental results for the decay rate, given by the various authors,[6–11] are shown in Fig. 3, and (44b) is entered also. We may then make three main remarks concerning our results:

(1) As already predicted in (24), there is no term linear in γ in the theoretical rate. Indeed, it would be hard to fit all the lowest points of Fig. 3, up to $Z=26$, by a straight line. In particular, such a line fitting Fe would already give an excess of λ_- of 3% for carbon ($Z=6$).[23] However, $\lambda_-(C)$ may be shown to equal λ_+ more accurately. The total muon disappearance rate

[21] D. G. Ravenhall, Revs. Modern Phys. **30**, 430 (1958).

[22] C. Kacser, Nuovo cimento **13**, 303 (1959); Proc. Roy. Soc. (London) **A253**, 103 (1959).

[23] At such a small Z, λ_- should still deviate very little from λ_+, and some results of the measurements, e.g., of references 6, 7, and 10, were actually obtained by comparing with carbon under the assumption $\lambda_-(C)=\lambda_+$.

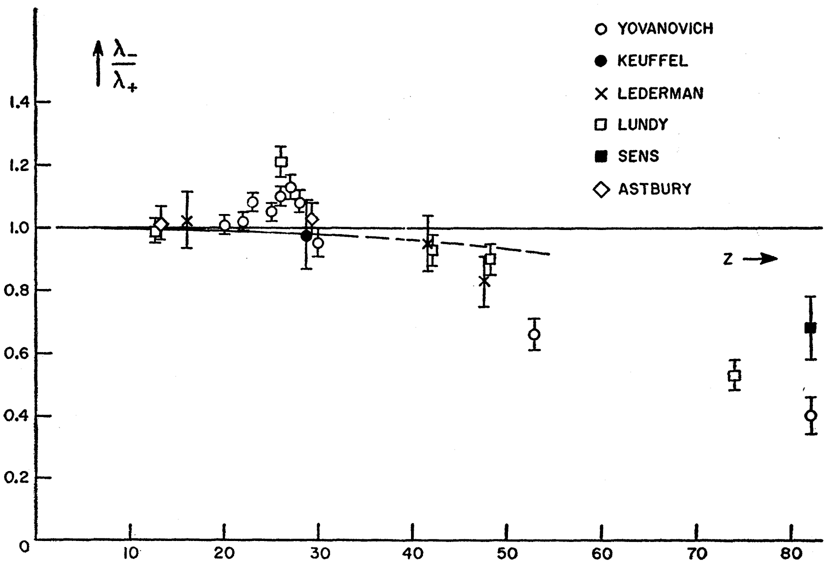

FIG. 3. Experimental points and theoretical curve of muon decay rate as a function of stopping material. The points were obtained by Yovanovitch *et al.* (reference 8), Holmstrom and Keuffel, (reference 11). L. Lederman and M. Weinrich [*Proceedings of the CERN Symposium on High-Energy Accelerators and Pion Physics, Geneva, 1956* (CERN, Geneva, 1956), Vol. 2, p. 427], Lundy *et al.* (reference 7), Sens *et al.* (reference 6), and Astbury *et al.* (reference 10). Note. Yovanovich should read "Yovanovitch."

$\lambda_{tot}=\lambda_-+\lambda_{cap}$ contains also the nuclear capture rate, and we have

$$\frac{\lambda_-}{\lambda_+}=\frac{\lambda_{tot}/\lambda_+}{1+(\lambda_{cap}/\lambda_-)}. \tag{45}$$

The ratio λ_{tot}/λ_+ is measured[24] for carbon as 1.10 ± 0.02, and the ratio λ_{cap}/λ_- is 0.10 ± 0.01 from a propane bubble chamber measurement,[25] which gives $\lambda_-/\lambda_+ = 1.00\pm0.02_5$.

(2) Our results do not fit the points around $Z\sim26$. The experiments are therefore not understood in this region within the assumptions of this paper, if one believes that an expansion in powers of Z should still be valid here. This assumes also that nuclear extension should cause no drastic change in the second order terms, just as it did not in the first order terms in the spectrum, Sec. V. The experimental points around Fe seem however to indicate a certain singularity, and thus an expansion in powers of Z is perhaps not permissible, or slowly convergent.

(3) Equation (44b) is a monotonically decreasing function of Z, and a result of this kind has also recently and independently been obtained by Mathews.[26] For $30\lesssim Z\lesssim 48$, the experiments are fitted rather inconclusively by our formula, which cannot pretend to be very accurate for these large Z. More important, however, is that (44b) represents a quite flat curve, very different from the expression

$$\lambda_+(1-5\gamma^2/2)$$

(in second order), which the phase space alone would give,[8] or from the expression

$$\lambda_+(1-5.15\gamma^2)$$

obtained by Khuri[27] (presumably without electron-nucleus interaction). The nuclear extension would cause (44b) to become even flatter due to a reduction of γ (the muon, moving partly within the nucleus for larger Z, sees less of its charge). It is therefore indicated that the drop of experimental points for large values of Z, which was thought to be well understood from the phase space argument, becomes much less so after all the terms of order γ^2 were obtained.

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Helpful correspondence with Dr. Haakon Olsen, Professor H. Primakoff, and Dr. Claude Kacser is appreciated. I wish to thank Professor L. Wolfenstein for frequent discussions and much encouragement, and Professor V. Telegdi and Professor J. W. Keuffel for communication of their unpublished results.

APPENDIX I. EVALUATION OF $\Pi(\mathbf{k},\alpha)$[28,29]

The integral

$$\Pi(\mathbf{k},\alpha)=\int\frac{d^3s}{s^2-p^2+i\eta}\frac{1}{(\mathbf{s}-\mathbf{p})^2+\beta^2}\frac{1}{(\mathbf{s}+\mathbf{k})^2+\alpha^2} \tag{A1}$$

is first transformed by combining the last two denominators, using the Feynman identity

$$(ab)^{-1}=\int_0^1 dz[az+b(1-z)]^{-2}. \tag{A2}$$

At the same time, the angular integrations may be

[24] J. C. Sens, Phys. Rev. **113**, 679 (1959).
[25] T. H. Fields, R. L. McIlwain, and J. G. Fetkovich, Bull. Am. Phys. Soc. Ser. II, 4, 81 (1959).
[26] J. Mathews, Bull. Am. Phys. Soc. Ser. II, 4, 446 (1959).
[27] N. D. Khuri and A. S. Wightman, often quoted[7,11] as "private communication."
[28] Dr. Haakon Olsen contributed helpful suggestions concerning this calculation.
[29] See the work of Dalitz, reference 14.

performed. This gives

$$\mathrm{II}=\pi\int_0^1 \frac{dz}{A}\int_{-\infty}^{\infty} \frac{sds}{s^2-p^2+i\eta}\frac{1}{s^2-2sA+Bz+p^2+\beta^2}, \quad \text{(A3)}$$

with $\mathbf{A}=\mathbf{q}z-\mathbf{p}$, $B=k^2-p^2+\alpha^2-\beta^2$. The second integrand has two poles in the lower and two in the upper half plane, and if we close the contour (to either side), Cauchy's theorem gives

$$\mathrm{II}=\pi^2\int_0^1 \frac{dz}{C}\frac{1}{Bz+\beta^2+2ipC}, \quad \text{(A4)}$$

with $C=[q^2z(1-z)+\alpha^2z+\beta^2(1-z)]^{\frac{1}{2}}$; positive root is understood. This can now be decomposed into a real and an imaginary part, and the remaining integrals can be found in tables. The result is:

$$\mathrm{ReII}(\mathbf{k},\alpha)=\frac{\pi^2}{p(q^2+\alpha^2)}\tan^{-1}\frac{k^2-p^2+\alpha^2}{2p\alpha}, \quad \text{(A5a)}$$

$$\mathrm{ImII}(\mathbf{k},\alpha)=\frac{\pi^2}{p(q^2+\alpha^2)}\ln\frac{\beta[(k^2-p^2+\alpha^2)^2+4p^2\alpha^2]^{\frac{1}{2}}}{2p(q^2+\alpha^2)}; \quad \text{(A5b)}$$

in both expressions, β has been set equal to zero wherever possible. We see that ImII diverges for $\beta\to 0$. Also, the fact that $\mathbf{k}=\mathbf{q}-\mathbf{p}$ has been used from the beginning.

APPENDIX II. GROUND-STATE WAVE FUNCTION OF THE MUON IN THE FIELD OF AN EXTENDED NUCLEUS WITH UNIFORM CHARGE DISTRIBUTION

We shall use the potential (27). Taking the expression (1a) for the wave function as an ansatz, the Dirac equation splits up in a well known way into two coupled equations for $f(r)$, $g(r)$:

$$\left(\frac{d}{dr}-\frac{1}{r}\right)f=(\mu_0+\mu-V)g,$$
$$\left(\frac{d}{dr}+\frac{1}{r}\right)g=(\mu_0-\mu+V)f, \quad \text{(A6)}$$

with μ_0 the muon rest mass, μ the muon total energy; they are already written down for a ${}^1S_{\frac{1}{2}}$ state. The binding energy is diminished as compared to the point nucleus case, and we write instead of (2):

$$\mu=\mu_0(1-\zeta\gamma^2). \quad \text{(A7a)}$$

Here, ζ is a function of Z which was calculated by Fitch and Rainwater,[30] essentially by fitting calculated wave functions at the nuclear boundary, and checked against experiments involving mesonic x rays. Their values can be fitted by an expression of the form

$$\zeta=0.5-1.659\gamma^2+1.685\gamma^3. \quad \text{(A7b)}$$

[30] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 782 (1953); see also E. M. Henley, Revs. Modern Phys. **30**, 438 (1958).

For $z\equiv r/R<1$, (A6) will be solved by an expansion

$$f=\sum_{k=0}^{\infty} a_k z^{2k+1}, \quad g=\sum_{k=0}^{\infty} b_k z^{2k+2}. \quad \text{(A8a)}$$

For $z>1$, the asymptotic behavior of the wave function for large r is determined by the binding energy: neglecting $1/r$ in (A6), we find that f, g have to go like $\exp[-\gamma\mu_0(2\zeta)^{\frac{1}{2}}r]$, and accordingly we try a semiconvergent series

$$f=e^{-\delta z}\sum_{k=0}^{\infty} c_k z^{s-k}, \quad g=e^{-\delta z}\sum_{k=0}^{\infty} d_k z^{s-k}, \quad \text{(A8b)}$$

with

$$\delta=\gamma_1(2\zeta)^{\frac{1}{2}}, \quad \gamma_1=(\mu_0 R)\gamma, \quad \text{(A8c)}$$

and s to be determined from the characteristic equation of the system (A6). We obtain recurrence relations for the coefficients $a_k\cdots d_k$, and according to our accuracy requirements, we discard those containing γ^2 or higher powers. We have to note, however, that the parameter γ_1 is of order 1 rather than order γ, and has to be kept throughout. The results are:

$$f=z+Az^3+Bz^5, \quad g=-\tfrac{1}{2}\gamma\lambda z^2, \quad z<1, \quad \text{(A9a)}$$

$$f=Cz^{1+\sigma}\left(1-\frac{\sigma}{2\delta}\frac{1+\sigma}{z}\right)e^{-\delta z}, \quad g=-\tfrac{1}{2}\gamma\frac{C}{1+\sigma}z^{1+\sigma}e^{-\delta z}, \quad z>1 \quad \text{(A9b)}$$

with

$$A=-\tfrac{1}{2}\lambda\gamma_1, \quad B=(1/20)\gamma_1(1+\tfrac{3}{2}\lambda^2\gamma_1), \quad \lambda=1-\tfrac{2}{3}\zeta\gamma_1, \quad \sigma=(2\zeta)^{-\frac{1}{2}}-1; \quad \text{(A9c)}$$

actually, in the interior g, two terms have been dropped which are found to be numerically small (below 5% for $Z=50$, and less for smaller Z). The C is determined from matching the large components f at the boundary, $z=1$, to be

$$C=e^{\delta}\frac{1+A+B}{1-0.5\sigma(1+\sigma)/\delta}. \quad \text{(A9d)}$$

Now, in principle, matching of the small components would determine the eigenvalues of the binding energy; we prefer however to adopt the values (A7a), (A7b) which are experimentally confirmed, and find that then the small components g match at $z=1$ with an accuracy of 6% of their value, or less than 1% of the value of the large components at $Z=50$, and even better for smaller Z. The normalization factor is found to be

$$N=\left\{(4\pi R)\left[\frac{1}{3}+\frac{2A}{5}+\frac{A^2+2B}{7}+\frac{2AB}{9}+\frac{B^2}{11}+\frac{C^2}{4\delta^3}e^{-2\delta}\times(1+2\delta+2\delta^2+2\sigma[1-e^{2\delta}\,\mathrm{Ei}(-2\delta)]-\sigma^2[1+2\delta+4e^{2\delta}\,\mathrm{Ei}(-2\delta)])\right]\right\}^{-\frac{1}{2}}, \quad \text{(A9e)}$$

with

$$\mathrm{Ei}(-z)=-\int_z^\infty t^{-1}e^{-t}dt.$$

Everything is accurate in powers of 1 and γ. The wave functions are plotted in Fig. 4 for $Z=50$, together with the nonrelativistic, point nucleus muon wave function $N_{NR}\exp(-\gamma\mu r)$, $N_{NR}=(\mu_0{}^3\gamma^3/\pi)^{\frac{1}{2}}$. For the purposes of our calculation, we will still set $\sigma=0$ in comparison with the other terms, because when using the empirical relation (A7b), σ can be thought of being of order γ^2 only.

APPENDIX III. EVALUATION OF III(k,α), III(k,α)[22]

In the integral

$$\mathrm{III}(\mathbf{k},\alpha)=\int\frac{d^3s}{s^2-p^2+i\eta}\frac{1}{(\mathbf{s}+\mathbf{k})^2+\alpha^2}\times\int\frac{d^3t}{t^2-p^2+i\eta}\frac{1}{(\mathbf{t}-\mathbf{p})^2+\beta^2}\frac{1}{(\mathbf{t}-\mathbf{s})^2+\beta^2},\quad\text{(A10)}$$

the third and fourth denominator are combined by the Feynman identity (A2) to give

$$\int_0^1 dx\frac{\partial}{\partial A_x{}^2}\frac{1}{(\mathbf{t}-\mathbf{a})^2-A_x{}^2},\quad\text{(A11)}$$

with $\mathbf{a}=\mathbf{p}x$, $A_x{}^2=p^2(1-x)^2-\beta^2-i\eta(1-x)$, and by another application of (A2) to this and the fifth denominator, we obtain for the d^3t integral:

$$\int_0^1 dx\frac{\partial}{\partial A_x{}^2}\int_0^1 dz\int\frac{d^3T}{(T^2+F)^2},\quad\text{(A12)}$$

with $F=z(1-z)B^2-zA_x{}^2+(1-z)\beta^2$, $\mathbf{B}=\mathbf{a}-\mathbf{s}$. The d^3T integral now gives $\pi^2/F^{\frac{1}{2}}$, where we define all square roots to have positive real parts. Performing further the dz integration, we have for (A12):

$$i\pi^2\int_0^1 dx\frac{\partial}{\partial A_x{}^2}\frac{1}{B}\ln\frac{A_x-i\beta-B}{A_x-i\beta+B}=i\pi^2\int_0^1\frac{dx}{A_x}\frac{1}{(A_x-i\beta)^2-B^2},\quad\text{(A13)}$$

where all the logarithms will be understood to be principal value, i.e., $-\pi<\arg\leqslant\pi$. The first two denominators of (A10) are combined to

$$\int_0^1 dy\frac{\partial}{\partial A_y{}^2}\frac{1}{(\mathbf{s}-\mathbf{b})^2-A_y{}^2},\quad\text{(A14)}$$

with $\mathbf{b}=-\mathbf{k}y$,

$$A_y{}^2=k^2(1-y)^2+(p^2-k^2)(1-y)-\alpha^2y-i\eta(1-y),$$

and finally combining (A13) and (A14) and using steps similar to those leading from (A12) to the right-hand side of (A13), we obtain

$$\mathrm{III}(\mathbf{k},\alpha)=-\pi^4\int_0^1\frac{dx}{A_x}\int_0^1\frac{dy}{A_y}\frac{1}{D^2-(A_x+A_y-i\beta)^2},\quad\text{(A15)}$$

with $\mathbf{D}=\mathbf{p}x+\mathbf{k}y$. The integral

$$\mathbf{III}(\mathbf{k},\alpha)=\int\frac{d^3t}{t^2-p^2+i\eta}\frac{\mathbf{t}}{(\mathbf{t}-\mathbf{p})^2+\beta^2}\int\frac{d^3s}{s^2-p^2+i\eta}\times\frac{1}{(\mathbf{s}+\mathbf{k})^2+\alpha^2}\frac{1}{(\mathbf{s}-\mathbf{t})^2+\beta^2}\quad\text{(A16)}$$

is treated in a similar manner to give the result

$$\mathbf{III}(\mathbf{k},\alpha)=\mathbf{p}\mathrm{III}(\mathbf{k},\alpha)+\mathbf{IV}(\mathbf{k},\alpha),\quad\text{(A17)}$$

$$\mathbf{IV}(\mathbf{k},\alpha)=\pi^4\left(\mathbf{p}\int_0^1\frac{(1-x)dx}{A_x}\int_0^1\frac{dy}{A_y}\frac{1}{D^2-(A_x+A_y-i\beta)^2}+\nabla_{\boldsymbol{\varrho}}\int_0^1\frac{dy}{A_y}\int_0^1\frac{dx}{2D}\ln\frac{A_x+A_y-i\beta-D}{A_x+A_y-i\beta-D}\right)\quad\text{(A17a)}$$

(for a definition of $\boldsymbol{\varrho}$, see (A19) below). Separate regions of integration will now be introduced, and A_x, A_y evaluated in each region (A_x to lowest order in β only)

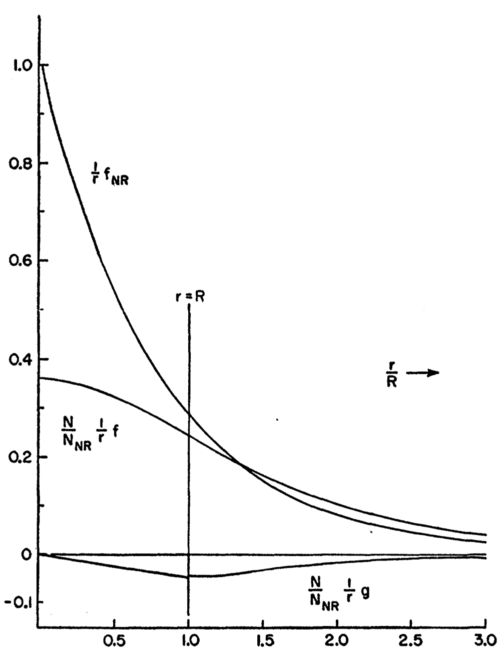

FIG. 4. Large and small components of the ground state wave function of a muon bound by a nucleus with uniform extended charge distribution, for $Z=50$. For comparison, nonrelativistic ground-state wave function with point nucleus.

and a transformation of variables made:

$$
\begin{aligned}
&(1')\ 0\leqslant x\leqslant 1-\lambda, && A_x=p[(1-x)^2-\lambda^2]^{\frac{1}{2}}, \\
& && 1-x=\lambda\cosh u, \\
&(2')\ 1-\lambda\leqslant x\leqslant 1, && A_x=-ip[\lambda^2-(1-x)^2]^{\frac{1}{2}}, \\
& && 1-x=\lambda\sin u, \\
&(1)\ 0\leqslant y\leqslant 1-\mu+\nu, && A_y=k[(1-y+\nu)^2-\xi^2]^{\frac{1}{2}}, \\
& && 1-y+\nu=\xi\cosh v, \\
&(2)\ 1-\mu+\nu\leqslant y\leqslant 1, && A_y=-ik[\xi^2-(1-y+\nu)^2]^{\frac{1}{2}}, \\
& && 1-y+\nu=\xi\sin v
\end{aligned}
\tag{A18}
$$

(the transition $\eta\rightarrow +0$ has been made after evaluation of the A's), with

$$
\lambda=\frac{\beta}{p},\quad \nu=\frac{p^2-k^2+\alpha^2}{2k^2},\quad \xi=\frac{1}{2k^2}[(p^2-k^2+\alpha^2)^2+4\alpha^2k^2]^{\frac{1}{2}}. \tag{A18a}
$$

The vector $\mathbf{D}$ can then be written

$$
\mathbf{D}=\boldsymbol{\rho}-\mathbf{p}(1-x)-\mathbf{k}(1-y+\nu),\qquad \boldsymbol{\rho}=\mathbf{p}+\omega\mathbf{k},\quad \omega=1+\nu. \tag{A19}
$$

Using an obvious notation, the part of III coming from the regions (2), (2′) is

$$
\begin{aligned}
\mathrm{III}_{22'}&=\frac{\pi^4}{pk}\int_0^{\pi/2}du\int_{\sin^{-1}(\nu/\xi)}^{\pi/2}\frac{dv}{D^2-(A_x+A_y-i\beta)^2} \\
&=\left(\frac{2}{\mu}\right)^4\frac{\pi^5}{2x^2Y}\left(\frac{\pi}{2}-\tan^{-1}\frac{yw}{2\gamma}\right);
\end{aligned}
\tag{A20}
$$

here we were able to set $\lambda=0$ immediately, and we introduced our notation (16) and kept the lowest order in γ only. In the other three parts of III, we encounter integrals of the form

$$
\begin{aligned}
&\int_0^{\cosh^{-1}(1/\lambda)}\frac{du}{a+b\cosh u+c\sinh u} \\
&\qquad=(a^2-b^2+c^2)^{-\frac{1}{2}}\left(\ln\frac{g_+}{a+b}-\ln\frac{g_-}{a+b}\right)
\end{aligned}
\tag{A21}
$$

(or also with upper limit $\cosh^{-1}(\omega/\xi)$; principal value logarithm as usual). Here,

$$
\begin{aligned}
g_\pm&=a+b+[c\pm(a^2-b^2+c^2)^{\frac{1}{2}}]T, \\
T&=\tanh(\tfrac{1}{2}\cosh^{-1}1/\lambda).
\end{aligned}
\tag{A21a}
$$

Invariably, g_- will vanish in lowest order, and expansions in powers of λ or order γ have to be made. In region (1), higher powers of λ must be kept due to the vanishing of the quantity $\rho^2+k^2\xi^2-2\boldsymbol{\rho}\cdot\mathbf{k}\xi\cosh v$ at the upper limit $v=\cosh^{-1}(\omega/\xi)$. Eventually, we obtain in lowest order in γ:

$$
\begin{aligned}
\mathrm{Re\,III}_{1'2}&=-\left(\frac{2}{\mu}\right)^4\frac{\pi^4}{x^2Y}\left[\frac{\pi}{2}\left(\frac{\pi}{2}-\tan^{-1}\frac{yw}{2\gamma}\right)\right. \\
&\qquad\left.+\frac{1}{2}\left(\frac{\pi^2}{4}-\left(\tan^{-1}\frac{yw}{2\gamma}\right)^2\right)\right], \\
\mathrm{Re\,III}_{2'1}&=-\left(\frac{2}{\mu}\right)^4\frac{\pi^4}{x^2Y}\frac{5\pi^2}{16}, \\
\mathrm{Re\,III}_{1'1}&=-\left(\frac{2}{\mu}\right)^4\frac{\pi^4}{x^2Y}\left[\tfrac{1}{2}\ln^2\frac{\lambda x}{Y}Z^{\frac{1}{2}}-\frac{17\pi^2}{48}\right].
\end{aligned}
\tag{A22}
$$

For the last integral, we were obliged to use

$$
\int\frac{dx}{x-a}\ln(x-b)=\tfrac{1}{2}\ln^2(x-a)+\mathfrak{L}_2\left(\frac{b-a}{x-a}\right),
$$

with the Euler dilogarithm

$$
\mathfrak{L}_2(z)=-\int_0^z t^{-1}\ln(1-t)dt,\quad \mathrm{Re}\,\mathfrak{L}_2(-i)=-\pi^2/48.
$$

If we evaluate $\mathbf{IV}(\mathbf{k},\alpha)$ by the same method, we observe that the result turns out to be by one power of γ smaller than $\mathrm{III}(\mathbf{k},\alpha)$, and is therefore not needed for our purposes.