Charged-Scalar Strong-Coupling Theory*

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A treatment of the charged-scalar strong-coupling theory is given which employs a somewhat different choice of variables than that usually used; one which is more convenient for a discussion of the effects of quantum mechanical field fluctuations. The expansion parameter of the strong-coupling theory is shown to be $(1/g^2)\ln(1/Ka)$, where a is the source radius. The isobar energy is calculated to order $1/g^4$, and terms of order $(1/g^2)\ln(1/Ka)$ relative to the leading $1/g^2$ term are found to appear. Similar terms occur in the charge-renormalization factor. The logarithmic term in the isobar energy is found to be precisely that required to renormalize the charge; that is, the isobar energy, if expressed in terms of the renormalized coupling constant, is explicitly independent of the source radius.

IN a previous paper,[1] Pais and one of the present authors began a reexamination of strong-coupling meson theory. This reexamination was motivated by the belief that some features suggested by the strong-coupling model might be relevant and suggestive to the actual structure of nucleons, by the hope that the methods used for strong coupling might be extended towards the intermediate coupling range, and by the circumstance that the strong-coupling theories were interesting examples in field theory.

One way in which the treatment of the charged-scalar theory given in (I) went beyond previous work was in attempting to understand better the effects of quantum mechanical field fluctuations. These appeared to contribute to the usual isobar energy $2KP_\theta^2/g^2$, terms of order P_θ^2/g^4a, depending explicitly on the reciprocal of the source radius. In fact, it appeared that the perturbation expansions being used led to series in powers of $1/g^2Ka$. Since it is easily demonstrated that the true eigenvalues of the theory are not more singular than $1/a$ (the self-energy singularity), it was apparent that the series being developed could be valid, if valid at all, only in the region $g^2 \gg 1/Ka$. The question was thus raised whether the usual condition $g^2 \gg 1$ was sufficient, indeed, of whether the strong-coupling results could be maintained at all in the limit of vanishing source size.

Looking back at the methods employed, but keeping this situation in mind, one is struck by the circumstance that attention was focussed on the largeness of g, but no account was taken of the possible smallness of a. Thus the terms in the Hamiltonian called small in (I) and deemed suitable for perturbation treatment were of order $1/g^2$ and higher, but some were actually and explicitly proportional to $1/g^2Ka$.

The present paper gives a somewhat different treatment of the problem, which avoids the difficulty just described. It will be recalled that in (I) it was shown that the transformations expressing the rotational symmetry of the Hamiltonian in isotopic spin space could be described in terms of an arbitrary distribution function. Subsequently, in (I), the distribution function was chosen in the way which has become conventional in strong-coupling theory. Our new method will make a different choice of the distribution function, and hence will be employing different dynamical variables. When the Hamiltonian is written in terms of these variables, the small terms are nonsingular in the limit of vanishing source size. It is easily seen that, as a result of quantum field fluctuations, the convergence of the perturbation series depends on source size through logarithmic terms, and the essential parameter is $(1/g^2)\ln(1/Ka)$.

Compared with the conventional procedure, our new treatment has the additional advantage that it follows more closely the simple intuitive picture of the physical situation in the case of strong-coupling. According to this picture, the field is split into two parts: the quasi-classically determined self-field plus the field of free mesons (and fluctuations). Since the magnitude of the self-field is proportional to the coupling constant, the strong-coupling approximation consists in an expansion in the ratio of free field to self-field. One thus begins by considering the motion of the self-field, which, because of the invariance of the theory to rotations in the isotopic spin plane, can be expected to be a uniform rotation in this plane (the rotational energy being the isobar energy). The direction of the self-field vector is defined as the direction of the vector

$$\int f(r)\boldsymbol{\phi}d\mathbf{r}/F^{\frac{1}{2}}, \tag{1}$$

where $F=\int f^2 d\mathbf{r}$, $\boldsymbol{\phi}=(\phi_1,\phi_2)$ is the total meson field, and $f(r)$ (the distribution function) is a function of the space coordinates which describes the radial dependence of the self-field (i.e., the Yukawa potential). One now introduces a rotating coordinate system, with the new $1'$ axis in the direction of the vector given by Eq. (1). This transformation introduces a pair of canonical variables: the ignorable angle variable θ, the angle

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[1] A. Pais and R. Serber, Phys. Rev. **105**, 1636 (1957). This paper will be referred to as (I).

between the 1′ and 1 axes, and its canonical angular momentum, P_θ, which is the total isotopic spin of the system and is a constant of the motion.

As a result of the transformation, the original Hamiltonian[2]

$$H=H_{\text{free}}+H_{\text{int}},$$
$$H_{\text{free}}=\tfrac{1}{2}\int(\pi_\alpha{}^2+\phi_\alpha\omega^2\phi_\alpha), \tag{2}$$
$$H_{\text{int}}=g(2\pi)^{\frac{1}{2}}\tau_\alpha\int U\phi_\alpha,$$

becomes

$$H=H_{\text{free}}'+H_{\text{int}}'+\frac{F}{2}\frac{(P_\theta-\Sigma'-\frac{1}{2}\tau_3)^2-\frac{1}{4}}{\left(\int f\phi_1'\right)^2}, \tag{3}$$

where the primes indicate that all quantities are now measured in the rotating system, and Σ' is the meson isotopic angular momentum

$$\Sigma'=\int(\phi_1'\pi_2'-\phi_2'\pi_1'). \tag{4}$$

The last term in Eq. (3) represents the isobar energy.

The condition that the 2′ axis is perpendicular to the self-field direction is expressed by

$$\int f\phi_2'=\int f\pi_2'=0. \tag{5}$$

In consequence of Eq. (5), the 2′ fields do not obey the usual commutation relations, but instead satisfy

$$[\phi_2'(\mathbf{r}),\pi_2'(\mathbf{r}')]=i[\delta(\mathbf{r}-\mathbf{r}')-f(r)f(r')/F]. \tag{6}$$

The physical consequence of Eq. (6) is that the normal modes of the 2′ field contain scattered waves.

We have described the transformations in terms of a physical picture of their meaning [e.g., identifying $f(r)$ with the shape of the self-field]. But the usual treatments of strong-coupling theory, including the treatment given in (I), do not follow this picture. Insofar as the formal procedure is concerned, $f(r)$ is an arbitrary function, which can be chosen by any desired criterion. The usual idea in strong-coupling theory is to first diagonalize H_{int}. In view of Eq. (5), this can be accomplished by choosing $f(r)$ to be the source function, $U(r)$, rather than the self-field function, which henceforth we shall call $v(r)$. The τ_2 term then drops out of H_{int}, and it can be diagonalized by taking τ_1 diagonal (i.e., one returns to the neutral scalar problem).

With the choice $f=U$, the physical meaning of the transformations is less transparent, for example the last term in Eq. (3) is no longer the isobar energy. As a matter of fact F, which, with f the self-field, would represent the moment of inertia of the nucleon and would be independent of the source size a as $a\rightarrow 0$, now becomes $F=\int U^2\sim 1/a^3$. Another transformation is required [given by Eq. (45) of (I)], which separates from the last term in Eq. (3) a piece, independent of g, which is part of the kinetic energy of free mesons, and leaves the usual $1/g^2$ isobar energy.

The source of our previous difficulties lies in the inadequacies of the last mentioned transformation. If ϕ_1' is separated into the self-field and free meson contributions, $\phi_1'=v+\phi_1''$, the denominator of the last term in Eq. (3) becomes $(q+\int U\phi_1'')^2$, where $q=\int Uv$ (the self-energy integral) is proportional to g/a. On expanding the denominator in powers of $\int U\phi_1''/q$ (which is the usual strong-coupling expansion in powers of $1/g$), we see that the entire isobar term has a factor $F/q^2\sim 1/g^2a$. The transformation given by Eq. (45) of (I) eliminates this factor only to the extent that $\int U\phi_1''$ is neglected; as a consequence the terms resulting from the expansion of the denominator remain proportional to F/q^2. An explicit calculation of the isobar energy to order $1/g^4$ purported to show that the $1/a$ singularity did not disappear as a result of any cancellations with perturbation terms coming from the numerator, which also are of order $1/a$. However it has since been realized that there is an additional term, the analog of Eq. (35) of this paper, which was forgotten in (I), and which could lead to the cancellation of the $1/a$ terms.

THE CHOICE $f=v$

We shall now describe a different way of treating the problem, which will lead us back to the choice $f=v$. For the moment, however, f may be left unspecified. The method is to diagonalize the interaction energy in Eq. (3) by rotating the secular component of the nucleon spin (by definition the 1 component) to the direction of the vector

$$\int U\boldsymbol{\phi}d\mathbf{r}, \tag{7}$$

that is, through the angle between the directions given by Eqs. (7) and (1). This is accomplished by the transformation function

$$S=\exp\left[\tfrac{1}{2}i\tau_3\tan^{-1}\left(\int U\phi_2'\Big/\int U\phi_1'\right)\right]. \tag{8}$$

In addition to rotating the nuclear spin, S generates new momenta π_1'', π_2'' which are given by the relations

$$\begin{aligned}\pi_1'&=\pi_1''+\tfrac{1}{2}\tau_3U\int U\phi_2'/Q^2,\\ \pi_2'&=\pi_2''-\tfrac{1}{2}\tau_3(U-qfF^{-1})\int U\phi_1'/Q^2,\end{aligned} \tag{9}$$

[2] Where variables of integration are omitted they are understood to be the three-dimensional volume element $d\mathbf{r}$.

where

$$Q^2=\left(\int U\phi_1'\right)^2+\left(\int U\phi_2'\right)^2,$$
$$q=\int fU. \tag{10}$$

In terms of the new variables, the angular momentum, given by Eq. (4), becomes

$$\Sigma'=\Sigma''-\tfrac{1}{2}\tau_3+\tfrac{1}{2}\tau_3 qF^{-1}\int f\phi_1'\int U\phi_1'/Q^2, \tag{11}$$

and the Hamiltonian, Eq. (3), becomes

$$H=H_{\text{free}}''+g(2\pi)^{\frac{1}{2}}\tau_1 Q-\frac{\tau_3}{4}\left(P\frac{1}{Q^2}+\frac{1}{Q^2}P\right)+\frac{1}{8}\frac{N}{Q^2}-\frac{1}{8}\frac{q^2}{F}\frac{\left(\int U\phi_1'\right)^2}{Q^4}$$
$$+\frac{F}{2}\frac{\left(P_\theta-\Sigma''-\frac{1}{2}\tau_3 qF^{-1}\int f\phi_1'\int U\phi_1'\Big/Q^2\right)^2-\frac{1}{4}}{\left(\int f\phi_1'\right)^2}, \tag{12}$$

where

$$P=\int U\phi_1'\int U\pi_2''-\int U\phi_2'\int U\pi_1'', \tag{13}$$

and

$$N=\int U^2.$$

Equation (12) can also be written

$$H=H_{\text{free}}''+g(2\pi)^{\frac{1}{2}}\tau_1 Q-\frac{\tau_3}{4}\left[P\frac{1}{Q^2}+\frac{1}{Q^2}P+(P_\theta-\Sigma'')\frac{q\int U\phi_1'}{Q^2\int f\phi_1'}+\frac{q\int U\phi_1'(P_\theta-\Sigma'')}{Q^2\int f\phi_1'}\right]+\frac{1}{8}\frac{N}{Q^2}+\frac{F}{2}\frac{(P_\theta-\Sigma'')^2-\frac{1}{4}}{\left(\int f\phi_1'\right)^2}. \tag{14}$$

This Hamiltonian contains terms proportional to τ_1 and τ_3. To see the magnitudes involved, we remember that ϕ_1' contains the self-field and is proportional to g, while the other field variables may be considered of order unity. Thus $Q\sim\int U\phi_1'\sim g$ and the interaction energy (τ_1 term) is proportional to g^2. The dominant term in the coefficient of τ_3 comes from the term in P proportional to ϕ_1': $P\sim\int U\phi_1'\int U\pi_2''$. The τ_3 term, to highest order in g, is

$$-\tfrac{1}{2}\tau_3\int U\pi_2''\Big/\int U\phi_1'; \tag{15}$$

and is thus of order $1/g$. This term can be eliminated by the rotation generated by

$$S=\exp\left[i\frac{\tau_2}{2}\tan^{-1}\frac{\int U\pi_2''}{2(2\pi)^{\frac{1}{2}}g\left(\int U\phi_1'\right)^2}\right]. \tag{16}$$

It will be observed that this rotation is through an angle proportional to $1/g^3$, whereas the previous rotation [Eq. (8)] is through an angle of order $1/g$.

We thus see how terms proportional to τ_2 and τ_3 can be successively eliminated, to any desired order, by a series of infinitesimal rotations. This technique avoids any explicit discussion of the upper states, such as was given in (I).

If we only require the Hamiltonian to order $1/g^4$, it is unnecessary to carry out the transformation given by Eq. (16) completely; only the result equivalent to treating the τ_3 term by second order perturbation theory is needed. To order $1/g^4$, we can take the Hamiltonian to be

$$H=H_{\text{free}}''+g(2\pi)^{\frac{1}{2}}\tau_1 Q+\frac{\tau_1\left(\int U\pi_2''\right)^2}{8g(2\pi)^{\frac{1}{2}}Q^3}+\frac{1}{8}\frac{N}{Q^2}+\frac{F}{2}\frac{(P_\theta-\Sigma'')^2-\frac{1}{4}}{\left(\int f\phi_1'\right)^2}, \tag{17}$$

and furthermore, since we shall restrict our consideration to the lower states, we can take $\tau_1=-1$.

The next step is to separate ϕ_1' into self-field and free field parts,

$$\phi_1'=v+\phi_1. \tag{18}$$

[In the subsequent equations we shall, for simplicity in writing, omit all primes on the new field variables, i.e., we shall use ϕ_1 as defined by Eq. (18) and introduce $\phi_2=\phi_2'$, $\pi_1=\pi_1''$, $\pi_2=\pi_2''$.] The Hamiltonian will then involve two arbitrary functions, f and v. These are chosen so that the variation of the Hamiltonian vanishes when the free field variables $(\phi_1,\phi_2,\pi_1,\pi_2)$ are varied from zero [subject to the condition given by Eq. (5)]. The equations of motion for the free field variables, if regarded as classical equations between c numbers, will then have an exact solution obtained simply by putting the free field variables equal to zero. The self-field v is thus defined as being the exact solution of the classical theory with no free field present. The resulting Hamiltonian will contain the free field variables only in quadratic and higher powers. When we vary H with respect to π_2, the condition

$$\delta H/\delta\pi_2=-i[\phi_2,H]=0,$$

is satisfied (for zero values of the variables) if

$$[\phi_2,\Sigma]=\left[\phi_2,\int v\pi_2\right]=i\left(v-fF^{-1}\int fv\right)=0,$$

which is solved by

$$f=v.$$

It should be observed that for f significantly different from v, $\Sigma\sim\int v\pi_2\sim g$, and the last term in Eq. (17), instead of representing the isobar energy, is of order g^2 larger. With $f=v$, the orthogonality condition, Eq. (5), eliminates this term, and Σ in fact represents the isotopic angular momentum of the free mesons.

The equation for v, obtained from

$$(\delta H/\delta\phi_1')_{(\phi_1'=v)}=0,$$

is

$$\omega^2v-g(2\pi)^{\frac{1}{2}}U-\frac{1}{4}\frac{N}{q^3}U-\frac{(P_\theta^2-\frac{1}{4})}{V^2}v=0, \tag{19}$$

where $V=\int v^2$. Equation (19) can be written

$$\bar{\omega}^2v=g'(2\pi)^{\frac{1}{2}}U, \tag{20}$$

where

$$\bar{\omega}^2=K'^2-\Delta,\quad K'^2=K^2-(P_\theta^2-\tfrac{1}{4})/V^2,$$
$$(2\pi)^{\frac{1}{2}}g'=(2\pi)^{\frac{1}{2}}g+(1/4)(N/q^3). \tag{21}$$

Equation (20) differs from the corresponding Eq. (51) of (I) in two respects. First, g is replaced by g', which is corrected by a term of relative order $1/g^4$. This term, N/q^3, depends on the shape of the source function, but is regular as $a\to 0$. Second, in the definition of K'^2, $P_\theta^2-\frac{1}{4}$ now appears where (I) had only P_θ^2. Equation (55) of (I) is thus modified to

$$K'=K[1+16(P_\theta^2-\tfrac{1}{4})/g'^4]^{-\frac{1}{4}}. \tag{22}$$

For the ground state, $P_\theta=\pm\frac{1}{2}$, Eq. (22) gives $K'=K$, independent of the value of g', and only for the higher isobars does $K'\to 0$ as $g'^2/(P_\theta^2-\frac{1}{4})^{\frac{1}{2}}\to 0$.

If we now make the substitution given by Eq. (18) in Eq. (17) and expand in powers of the free fields, we find, keeping all terms to order $1/g^4$,

$$\begin{aligned}H=&-\frac{g'^2}{4a}+\frac{g'^2K}{4}\left[\left(1+\frac{16(P_\theta^2-\frac{1}{4})}{g'^4}\right)^{\frac{1}{2}}-1\right]\\&+\frac{3}{8}\frac{N}{q^2}+H_{\text{free}}-\frac{1}{2}\frac{g'(2\pi)^{\frac{1}{2}}}{q}\left(\int U\phi_2\right)^2\\&+\frac{g(2\pi)^{\frac{1}{2}}}{2}q\left[\frac{xy^2}{1+x}+\frac{1}{4}\frac{y^4}{(1+x)^3}-\frac{1}{8}\frac{y^6}{(1+x)^5}\right]\\&+\frac{3}{8}\frac{N}{q^4}\left(\int U\phi_1\right)^2-\frac{1}{8g(2\pi)^{\frac{1}{2}}q^3}\left(\int U\pi_2\right)^2\\&-\frac{(2P_\theta\Sigma-\Sigma^2)}{2V}\left(1-2\int v\phi_1\Big/V\right)\\&+\frac{3}{2}\frac{[(P_\theta-\Sigma)^2-\frac{1}{4}]}{V^3}\left(\int v\phi_1\right)^2.\end{aligned} \tag{23}$$

In Eq. (23) we have written

$$x=\int U\phi_1/q,\quad y=\int U\phi_2/q,$$

in terms of cubic and higher powers in the free fields.

The change of P_θ^2 into $P_\theta^2-\frac{1}{4}$ has the interesting consequence that the isobar levels given by the second term of Eq. (23) are bound only for $g^2>2$, and become unstable against π emission for $g^2<2$.[3]

NORMAL MODES OF THE FREE FIELDS

Normal modes are chosen to diagonalize the free field terms in H of zero order in g:

$$H_0=H_{\text{free}}-\tfrac{1}{2}(g'\sqrt{2\pi}/q)\left(\int U\phi_2\right)^2. \tag{24}$$

The equations of motion for the 2 field, resulting from H_0 and the commutation relations, Eq. (6), are[4]

$$\begin{aligned}\dot{\phi}_2&=\pi_2,\\ \dot{\pi}_2&=-\omega_{op}^2\phi_2+(v/V)\int v\omega_{op}^2\phi_2\\&\quad+(g'\sqrt{2\pi}/q)\int U\phi_2[U-qv/V].\end{aligned} \tag{25}$$

[3] In the strong coupling limit, the ratio g_c^2/g^2 has the value $\frac{1}{4}$. Thus the critical value of the renormalized coupling constant is $g_c^2=\frac{1}{2}$. This agrees with the result of H. Jahn, Forschr. Physik **7**, 451 (1959).

[4] To avoid ambiguity in Eqs. (25)–(28) we write $\omega_{op}^2=K^2-\Delta$. In these and subsequent equations ω^2 designates (k^2+K^2) and $\bar{\omega}^2=k^2+K'^2$.

On using Eqs. (20) and (5), we obtain the field equation

$$\ddot{\phi}_2 = -\omega_{op}^2\phi_2 + (g'\sqrt{2\pi}/q)U\int U\phi_2. \quad (26)$$

Since we need only consider spherically symmetric solutions in this fixed-source theory, we write

$$\phi_2(\mathbf{x},t) = \phi_k^{(2)}(r)e^{-i\omega t}.$$

It is readily verified from Eq. (26) that the functions $\phi_k^{(2)}$ form an orthogonal set. We take the normalization to be

$$\int \phi_k^{(2)}\phi_{k'}^{(2)}d\mathbf{x} = \delta(k-k')/4\pi k^2. \quad (27)$$

It can also be shown from Eq. (26) that

$$\int \phi_k^{(2)}\omega_{op}^2\phi_{k'}^{(2)}d\mathbf{x} - g'(\sqrt{2\pi}/q)\int U\phi_k^{(2)}d\mathbf{x} \times \int U\phi_{k'}^{(2)}d\mathbf{x} = \omega^2\delta(k-k')/4\pi k^2. \quad (28)$$

Thus if we develop the fields according to

$$\phi_\alpha = \int d\mathbf{k}\ (2\omega)^{-\frac{1}{2}}[a_\alpha(k)+a_\alpha^+(k)]\phi_k^{(\alpha)}(r),$$
$$\pi_\alpha = -i\int d\mathbf{k}\ (\omega/2)^{\frac{1}{2}}[a_\alpha(k)-a_\alpha^+(k)]\phi_k^{(\alpha)}(r), \quad (29)$$

with $\phi_k^{(2)}(r)$ the function introduced above, $\phi_k^{(1)}(r) = (2\pi)^{-\frac{3}{2}}\sin kr/kr$, and the $a_\alpha(k)$, $a_\alpha^+(k)$ annihilation and creation operators satisfying the usual commutation relations, H_0 takes the form

$$H_0 = \int d\mathbf{k}\ [a_1^+(k)a_1(k)+a_2^+(k)a_2(k)]\omega - (1/2\pi)\int \delta d\omega - \tfrac{1}{2}K, \quad (30)$$

where δ is the phase-shift of the $\phi_k^{(2)}(r)$ solutions. The last two terms represent the change in the zero-point energy of the 2 field due to the presence of the source. This change can most easily be calculated by quantizing in a spherical box of radius R. The energy change in question is

$$\Delta E = \tfrac{1}{2}\sum_{n=2}^{\infty}\omega(k_n') - \tfrac{1}{2}\sum_{n=1}^{\infty}\omega(k_n),$$

where $k_n = n\pi/R$, $k_n' = (n\pi/R)-\delta/R$. As we shall see, $\delta(k)$ has the value π for $k=0$, and decreases to 0 as $k\to\infty$; it is because $\delta(0)=\pi$ that the k_n' series begins with $n=2$. [The reason for the "missing state" is that the $\phi_k^{(2)}$'s are not a complete set, in virtue of Eqs. (5) and (6). The missing state is, of course, just v itself, which is the solution of Eq. (26) for $\omega=0$.]

Thus

$$\Delta E = \tfrac{1}{2}\sum_{n=2}^{\infty}[\omega(k_n-\delta/R)-\omega(k_n)] - \tfrac{1}{2}K.$$

On replacing the sum by an integral and allowing $R\to\infty$ we obtain the result given in Eq. (30).

If we put

$$\phi_k^{(2)}(r) = [1/(2\pi)^{\frac{3}{2}}]\int d\mathbf{k}'\ \phi_{kk'}e^{i\mathbf{k}'\cdot\mathbf{r}},$$

the normalized solutions of Eq. (26) are readily found to be

$$\phi_{kk'} = \cos\delta\left[\frac{\delta(k-k')}{4\pi k^2} - \frac{v_k u_{k'}}{\gamma(\omega'^2-\omega^2)}\right], \quad (31)$$

where

$$\gamma = \int [v_{k'}u_{k'}/(\omega'^2-\omega^2)]d\mathbf{k}', \quad (32)$$

and v_k and u_k are the Fourier amplitudes of $v(r)$ and $U(r)$. In Eqs. (31) and (32) the principal value is to be taken at the pole $\omega'=\omega$. The phase shift, δ, is given by

$$\tan\delta = -2\pi^2 k v_k u_k/\gamma. \quad (33)$$

In the limit $U(r)\to\delta(r)$, one finds $\gamma = g'(2\pi)^{\frac{1}{2}}K'/4\pi\bar{\omega}^2$ and thus

$$\tan\delta = -k/K', \quad k\ll 1/a.$$

It is readily checked that the solution given by Eq. (31) is indeed orthogonal to v. With the choice $f=U$ of (I), the normal modes differ from Eq. (31) by the fact that U and v are interchanged (aside from some small terms, of relative order $1/g^4$). The phase shift is unaltered, and the discussion of the scattering given in (1) needs no essential change. If one fixes one's attention on the determination of the self-field and the lowest order isobar energy, it is not evident that there is any great difference in the variables used in (I) and the present paper, e.g., the transformation, Eq. (8), is only through an angle of order $1/g$. It should be remarked, however, that the normal modes of the free fields are quite different for the different choices of f.

CORRECTIONS TO THE ISOBAR ENERGY

For the purpose of evaluating the energy shift in Eq. (30) we shall make a specific choice of U:

$$u_k = (2\pi)^{-\frac{3}{2}}/(1+\Lambda^2k^2). \quad (34)$$

The length Λ is connected to the source size a [defined as in (I) under Eq. (59)] by the relation

$$a = 2\Lambda(1+\Lambda K)^2.$$

With this choice of U the integral appearing in the

energy shift can be evaluated exactly,[5] and one finds

$$\Delta E=-(1/2\pi)\int\delta d\omega-\tfrac{1}{2}K$$
$$=(1/\pi\Lambda)\{(1-K^2\Lambda^2)^{\frac{1}{2}}\tanh^{-1}(1-K^2\Lambda^2)^{\frac{1}{2}}$$
$$-[(1+\tfrac{1}{2}K'\Lambda)^2-\tfrac{1}{4}K^2\Lambda^2]^{\frac{1}{2}}$$
$$\times\tanh^{-1}[1-\tfrac{1}{4}K^2\Lambda^2/(1+\tfrac{1}{2}K'\Lambda)^2]^{\frac{1}{2}}\}$$
$$-\frac{1}{2\pi}\frac{(P_\theta^2-\frac{1}{4})^{\frac{1}{2}}}{V}\tan^{-1}\frac{(P_\theta^2-\frac{1}{4})^{\frac{1}{2}}}{VK'}$$
$$=-\frac{\ln 2}{\pi\Lambda}-\frac{K'}{2\pi}\left[\ln\left(\frac{4}{K\Lambda}\right)+1\right]-\frac{1}{2\pi K'}\frac{(P_\theta^2-\frac{1}{4})}{V^2}. \quad (35)$$

In the final form we have neglected terms of order Λ, and terms of order $1/g^8$ coming from the expansion of the arctangent.

We thus obtain correction terms to the self-energy of order $1/a$ and $K\ln(1/Ka)$. There is also a $\ln Ka$ correction term in the isobar energy, due to the dependence of K' on P_θ.

On expanding K' we obtain, to order $1/g^4$,

$$\Delta E=-\frac{\ln 2}{\pi\Lambda}-\frac{K}{2\pi}\left[\ln\left(\frac{4}{K\Lambda}\right)+1\right]+\frac{1}{4\pi K}\frac{(P_\theta^2-\frac{1}{4})}{V^2}\times\left[\ln\left(\frac{4}{K\Lambda}\right)-1\right]. \quad (36)$$

Remembering that $V=g^2/4K'$, we see that, to order $1/g^4$, the usual isobar energy,[6] $2K(P_\theta^2-\frac{1}{4})/g^2$, is corrected to

$$\frac{2K(P_\theta^2-\frac{1}{4})}{g^2}\left\{1+\frac{2}{\pi g^2}\left[\ln\left(\frac{4}{K\Lambda}\right)-1\right]\right\}. \quad (37)$$

An interpretation of the correction factor in Eq. (37) may be found in terms of the coupling constant renormalization factor. The ratio of the renormalized coupling constant, g_c, to g is the matrix element of $\frac{1}{2}(\tau_1+i\tau_2)$ taken between physical neutron and proton states,[7]

$$g_c/g=\langle P|\tfrac{1}{2}(\tau_1+i\tau_2)|N\rangle. \quad (38)$$

Under the transformation from the original coordinate system to the rotating system

$$\tfrac{1}{2}(\tau_1+i\tau_2)\rightarrow\tfrac{1}{2}(\tau_1+i\tau_2)e^{i\theta},$$

where θ is the angle variable canonically conjugate to P_θ. The factor $e^{i\theta}$ of course just provides the factor in the matrix element $\delta_{P_\theta+1,P_\theta}$.

The next transformation, given by Eq. (8), leads to the rotation

$$\tfrac{1}{2}(\tau_1+i\tau_2)\rightarrow\tfrac{1}{2}(\tau_1+i\tau_2)\times\exp\left\{\tan^{-1}\left[\int U\phi_2'\Big/\int U\phi_1'\right]\right\}$$
$$=\tfrac{1}{2}(\tau_1+i\tau_2)\left(\int U\phi_1+i\int U\phi_2\right)\Big/Q$$
$$=\tfrac{1}{2}(\tau_1+i\tau_2)\left(1+\frac{iy}{1+x}-\frac{1}{2}\frac{y^2}{(1+x)^2}+\cdots\right).$$

Taking the expectation value, for a state of $P_\theta^2=\frac{1}{4}$ (with no free mesons), and remembering that

$$\langle\tau_1\rangle=-1,\quad\langle\tau_2\rangle=0,$$

we find, to order $1/g^2$,

$$g_c/g=-\tfrac{1}{2}(1-\tfrac{1}{2}\langle y^2\rangle). \quad (39)$$

With the source function given by Eq. (34), one finds[5]

$$\langle y^2\rangle=\frac{2}{\pi g'^2}(1+K\Lambda)^3\left\{\frac{(1+\frac{1}{2}K\Lambda)}{(1+K\Lambda)^{\frac{1}{2}}}\tanh^{-1}\left[\frac{(1+K\Lambda)^{\frac{1}{2}}}{1+\frac{1}{2}K\Lambda}\right]-1\right\}$$
$$=\frac{2}{\pi g'^2}\left[\ln\left(\frac{4}{K\Lambda}\right)-1\right],$$

plus terms of order Λ, and, to order $1/g^2$,

$$\frac{g_c}{g}=-\frac{1}{2}\left\{1-\frac{1}{\pi g^2}\left[\ln\left(\frac{4}{K\Lambda}\right)-1\right]\right\}. \quad (40)$$

Comparing Eqs. (40) and (37) we see that the logarithmic correction term in Eq. (37) is just that involved in the charge renormalization, that is, Eq. (37) can be rewritten

$$K(P_\theta^2-\tfrac{1}{4})/2g_c^2, \quad (41)$$

and when so expressed in terms of g_c, the isobar energy is explicitly independent of source size.

Another way of getting the same result is to restrict oneself to $|P_\theta^2-\frac{1}{4}|/g^4\ll 1$ from the outset, and define v to have its value for $P_\theta^2=\frac{1}{4}$, i.e., omit the final term in Eq. (19). The only changes in the Hamiltonian, Eq. (23), will be that the isobar energy takes the simpler form

$$(1/2V)(P_\theta^2-\tfrac{1}{4}). \quad (42)$$

and the linear term arising from the expansion of the denominator of the centrifugal potential term will still be present. This term is

$$-[(P_\theta^2-\tfrac{1}{4})/V^2]\int v\phi_1. \quad (43)$$

[5] Details of this calculation are given in the Appendix.
[6] Since the Hamiltonian, Eq. (23), contains perturbation terms proportional to $|P_\theta|/g^2$, we restrict our treatment to small values of this parameter, and accordingly expand the isobar term in Eq. (23) in powers of this parameter.
[7] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

With this choice of v, the term proportional to $(P_\theta^2-\frac{1}{4})$ in Eq. (36) for ΔE will of course be missing. The $1/g^4$ correction to the isobar energy instead comes from the second order effects of the $1/g^3$ term given by Eq. (43), and the $1/g$ term in Eq. (23),

$$\tfrac{1}{2}g(2\pi)^{\frac{1}{2}}qxy^2. \tag{44}$$

Perturbation theory immediately gives for the energy shift

$$g\frac{(2\pi)^{\frac{1}{2}}}{2}\frac{(P_\theta^2-\frac{1}{4})}{V^2}\langle y^2\rangle\int\frac{u_kv_k}{\omega^2}d\mathbf{k}$$
$$=(1/2V)(P_\theta^2-\tfrac{1}{4})\langle y^2\rangle, \tag{45}$$

on using Eq. (20).

Adding Eq. (45) to Eq. (42) gives for the isobar energy

$$[(P_\theta^2-\tfrac{1}{4})/2V](1+\langle y^2\rangle), \tag{46}$$

and, on comparison with Eq. (39) (and remembering $V\sim g^2$), the desired connection between isobar correction and charge renormalization. This derivation demonstrates that the connection in question is independent of the form of U.

There are additional contributions to the $1/g^4$ isobar energy coming from the small terms in the Hamiltonian, but these are finite in the limit of vanishing source size. The term $-P_\theta\Sigma/V$ in Eq. (23) gives a second order perturbation contribution[5]

$$-(1/2\pi KV^2)(P_\theta)(1-\pi/4), \tag{47}$$

and the last term in Eq. (23) gives in first order

$$(1/\pi KV^2)(P_\theta^2-\tfrac{1}{4}). \tag{48}$$

The isobar energy to order $1/g^4$ is the sum of Eqs. (37), (47), and (48):

$$E=\frac{2K}{g^2}(P_\theta^2-\tfrac{1}{4})\left\{1+\frac{2}{\pi g^2}\left[\ln\left(\frac{4}{K\Lambda}\right)+1+\frac{\pi}{2}\right]\right\}. \tag{49}$$

If we use as the criterion for the coupling constant just large enough to produce stable isobars

$$(\partial E/\partial P_\theta)_{(P_\theta=\frac{1}{2})}=K,$$

we find that the critical value, which was $g^2=2$ without inclusion of the $1/g^4$ corrections, becomes $g^2=3.9$, if we take $K\Lambda=1/7$. (The critical value of the renormalized coupling constant is $g_c^2=0.64$.) For the same value of $K\Lambda$, the intermediate coupling calculation of Christian and Lee[8] gave for the critical value $g^2=4.0$.

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APPENDIX

A number of symbols have been introduced to denote various integrals of functions of U and v. Here we shall evaluate these integrals for the specific choice of a Yukawa source and give explicit formulas for N, a, V, and γ. We shall also determine the phase shift as a function of k and evaluate the energy shift ΔE. The Yukawa source function is given by

$$U(r)=(1/4\pi\Lambda^2)r^{-1}e^{-r/\Lambda}, \tag{A1}$$

which is normalized so that

$$\int U(r)d\mathbf{r}=1. \tag{A2}$$

The norm of $U(r)$ is

$$N=(1/4\pi\Lambda^4)\int_0^\infty e^{-2r/\Lambda}dr=1/8\pi\Lambda^3. \tag{A3}$$

The Fourier amplitude of the Yukawa source is

$$u_k=[1/(2\pi)^{\frac{3}{2}}]\int d\mathbf{r}\; U(r)e^{-i\mathbf{k}\cdot\mathbf{r}}$$
$$=[1/(2\pi)^{\frac{3}{2}}][1+(\Lambda k)^2]. \tag{34}$$

The definition of the source radius a given in reference (I) is equivalent to

$$1/a=4\pi\int d\mathbf{k}\; u_k^2/\omega^2.$$

With u_k as given by Eq. (34) one finds

$$a=2\Lambda(1+K\Lambda)^2. \tag{A4}$$

Using Eq. (20), the norm of v can be written

$$V=g'^2(2\pi)\int u_k^2d\mathbf{k}/\bar{\omega}^4=(g'^2/4K')(1+K'\Lambda)^{-3}. \tag{A5}$$

The choice of a Yukawa source leads to the following expression for γ [defined by Eq. (32)]:

$$\gamma=\frac{g'}{\pi(2\pi)^{\frac{1}{2}}}\frac{1}{\Lambda^4}P\int_0^\infty dk'\frac{k'^2}{(k'^2-k^2)(k'^2+K'^2)(k'^2+1/\Lambda^2)^2}.$$

Here P denotes principal value. Since the integrand is an even function of k', the integral can readily be evaluated by residues, and one finds

$$\bar{\omega}^2\gamma=\frac{g'\{K'(2+K'\Lambda)-\Lambda[3+2K'\Lambda+(K'\Lambda)^2]k^2-\Lambda^3k^4\}}{4(2\pi)^{\frac{1}{2}}(1+K'\Lambda)^2(1+\Lambda^2k^2)^2}. \tag{A6}$$

[8] T. D. Lee and R. Christian, Phys. Rev. 94, 1760 (1954).

Substituting this expression for $\bar{\omega}^2\gamma$ into Eq. (33), we find that the phase shift is given by

$$\tan\delta=-\frac{2(1+K'\Lambda)^2k}{K'(2+K'\Lambda)-\Lambda[3+2K'\Lambda+(K'\Lambda)^2]k^2-\Lambda^3k^4}. \tag{A7}$$

The phase shift δ has the value π for $k=0$, goes through $\pi/2$ for $k\sim(K'/\Lambda)^{\frac{1}{2}}$, and for large Λk is approximately

$$\delta=2(1+K'\Lambda)^2/(\Lambda k)^3.$$

Next we evaluate the energy shift

$$\Delta E=-(1/2\pi)\int_0^\infty dk\ (k/\omega)\delta-\tfrac{1}{2}K.$$

It is convenient to write $\delta=\tan^{-1}[-f(k)]$. Then

$$\Delta E=(1/2\pi)\int_0^\infty dk\ (k/\omega)\tan^{-1}[f]-\tfrac{1}{2}K.$$

The arctangent can be eliminated by partial integration:

$$\Delta E=-(1/2\pi)\omega\delta(k)\Big|_0^\infty-\tfrac{1}{2}K-(1/2\pi)\int_0^\infty dk\,[\omega/(1+f^2)]\partial f/\partial k.$$

Since $\delta(0)=\pi$, and δ decreases faster than $1/k$ in the limit $k\to\infty$, the first two terms cancel, and we are left with

$$\Delta E=-(1/2\pi)\int_0^\infty dk\,[\omega/(1+f^2)]\partial f/\partial k. \tag{A8}$$

For the Yukawa source

$$f(k)=\frac{2(1+K'\Lambda)^2k}{K'(2+K'\Lambda)-\Lambda[3+2K'\Lambda+(K'\Lambda)^2]k^2-\Lambda^3k^4}.$$

To avoid writing the lengthy denominator of $f(k)$, let

$$d\equiv K'(2+K'\Lambda)-\Lambda[3+2K'\Lambda+(K'\Lambda)^2]k^2-\Lambda^3k^4.$$

It is easy to verify that

$$\frac{\partial f}{\partial k}=\frac{2(1+K'\Lambda)^2}{\Lambda d^2}[(\Lambda k)^2+1][3(\Lambda k)^2+K'\Lambda(2+K'\Lambda)],$$

$$1+f^2=(\bar{\omega}^2/d^2)[(\Lambda k)^2+1][(\Lambda k)^2+(2+K'\Lambda)^2].$$

Therefore Eq. (A8) becomes

$$\Delta E=-\frac{(1+K'\Lambda)^2}{\pi\Lambda}\times\int_0^\infty dk\frac{\omega[3(\Lambda k)^2+K'\Lambda(2+K'\Lambda)]}{\bar{\omega}^2[(\Lambda k)^2+1][(\Lambda k)^2+(2+K'\Lambda)^2]}.$$

Let us separate the integrand into partial fractions:

$$\frac{(1+K'\Lambda)^2[3(\Lambda k)^2+K'\Lambda(2+K'\Lambda)]}{[(\Lambda k)^2+1]\bar{\omega}^2[(\Lambda k)^2+(2+K'\Lambda)^2]}=\frac{1}{k^2+h^2}+\frac{1}{2}\frac{K'\Lambda}{\bar{\omega}^2}-\frac{1}{2}\frac{(2+K'\Lambda)}{k^2+(2+K'\Lambda)^2h^2},$$

where h denotes the reciprocal of Λ. Thus

$$\Delta E=-\frac{1}{\pi\Lambda}\left(\int_0^\infty dk\frac{\omega}{k^2+h^2}+\tfrac{1}{2}K'\Lambda\int_0^\infty dk\frac{\omega}{\bar{\omega}^2}-(1+\tfrac{1}{2}K'\Lambda)\int_0^\infty dk\frac{\omega}{k^2+(2+K'\Lambda)^2h^2}\right). \tag{A9}$$

One can easily verify by differentiation that

$$\int dk\frac{\omega}{k^2+A^2}=\sinh^{-1}\left(\frac{k}{K}\right)-\frac{c}{A}\tanh^{-1}\left[\frac{c}{A}\frac{k}{\omega}\right], \tag{A10}$$

where

$$c^2=A^2-K^2.$$

Here we have assumed $c^2>0$. If $A^2<K^2$, let $c'=ic$ and remember that $\tanh^{-1}(ix)=i\tan^{-1}x$. Thus

$$\int dk\frac{\omega}{\bar{\omega}^2}=\sinh^{-1}\left(\frac{k}{K}\right)+\frac{(P_\theta^2-\frac{1}{4})^{\frac{1}{2}}}{VK'}\times\tan^{-1}\left[\frac{(P_\theta^2-\frac{1}{4})^{\frac{1}{2}}}{VK'}\frac{k}{\omega}\right], \tag{A11}$$

where we have used Eq. (21), namely

$$K^2-K'^2=(P_\theta^2-\tfrac{1}{4})/V^2.$$

Using (A10) and (A11) to evaluate the three definite integrals in Eq. (A9), one obtains the result given in Eq. (35).

The expectation value of y^2 is calculated as follows: The normal mode solutions of the 2 field give

$$\int U\phi_k^{(2)}d\mathbf{r}=-qu_k\cos\delta/\bar{\omega}^2\gamma.$$

$\bar{\omega}^2\gamma$ is given by Eq. (A6). Using Eq. (A7) and the trigonometric identity

$$\cos\delta=[1+\tan^2\delta]^{-\frac{1}{2}},$$

one finds

$$\cos\delta=\frac{K(2+K\Lambda)-\Lambda[3+2K\Lambda+(K\Lambda)^2]k^2-\Lambda^3k^4}{\omega[1+(\Lambda k)^2][(\Lambda k)^2+(2+K\Lambda)^2]^{\frac{1}{2}}}.$$

Taking $K'=K$ (we wish to determine the expectation value for a state with $P_\theta^2=\frac{1}{4}$) we obtain

$$\int U\phi_k^{(2)}d\mathbf{r}=-\frac{2}{\pi}\frac{q(1+K\Lambda)^2}{g'\omega[(\Lambda k)^2+(2+K\Lambda)^2]^{\frac{1}{2}}},$$

and it follows immediately that

$$\langle y^2\rangle=\frac{8}{\pi}\frac{(1+K\Lambda)^4}{g'^2\Lambda^2}\int_0^\infty\frac{dkk^2}{\omega^3[k^2+(2+K\Lambda)^2/\Lambda^2]}.$$

Let us introduce $m=(2+K\Lambda)/\Lambda$. It is convenient to write

$$\int\frac{dkk^2}{\omega^3(k^2+m^2)}=\frac{(1+\frac{1}{2}K\Lambda)}{(1+K\Lambda)}\int\frac{dk}{\omega(k^2+m^2)}-\frac{(K\Lambda)^2}{4(1+K\Lambda)}\int\frac{dk}{\omega^3}.$$

The indefinite integrals are given by

$$\int dk/\omega^3=k/K^2\omega,$$

$$\int\frac{dk}{\omega(k^2+m^2)}=\frac{\Lambda^2}{4(1+\frac{1}{2}K\Lambda)(1+K\Lambda)^{\frac{1}{2}}}\times\tanh^{-1}\left[\frac{(1+K\Lambda)^{\frac{1}{2}}}{(1+\frac{1}{2}K\Lambda)}\frac{k}{\omega}\right].$$

Therefore

$$\langle y^2\rangle=\frac{2}{\pi}\frac{(1+K\Lambda)^3}{g'^2}\left\{\frac{(1+\frac{1}{2}K\Lambda)}{(1+K\Lambda)^{\frac{1}{2}}}\times\tanh^{-1}\left[\frac{(1+K\Lambda)^{\frac{1}{2}}}{1+\frac{1}{2}K\Lambda}\right]-1\right\}.$$

The result of a second order perturbation calculation given by Eq. (47) requires evaluation of the integral[9]

$$I\equiv\int_0^\infty dkdk'\frac{(kk')^2}{\omega\omega'^3(\omega+\omega')^3}=\frac{\pi}{2K}\left(1-\frac{\pi}{4}\right).$$

[9] A second order perturbation calculation on $-P_\theta\Sigma/V$ gives

$$-(2\pi)^2\frac{P_\theta^2}{V^2}\int_0^\infty\frac{dkdk'\,k^2k'^2v_k^2u_{k'}^2\cos^2\delta}{\omega\omega'\gamma^2(\omega+\omega')^3}.$$

Since this correction to the isobar energy is finite in the point source limit, we can set $\Lambda=0$ first and then integrate. In the limit of a point source

$$\cos\delta=K'/\bar{\omega},\quad \bar{\omega}^2\gamma=g'K'/2(2\pi)^{\frac{1}{2}}.$$

we also take $\bar{\omega}=\omega$ (since we are only looking for corrections of order $1/g^4$) and this contribution becomes $-P_\theta^2I/\pi^2V^2$.

Let us perform the k integration first. The indefinite integral is

$$\int dk\frac{k^2}{\omega(\omega+\omega')^3}=\frac{1}{2}\frac{k(\omega\omega'+K^2)}{k'^2(\omega+\omega')^2}-\frac{K^2}{k'^3}\tanh^{-1}\left[\frac{k'k}{(\omega'+K)(\omega+K)}\right],$$

and the definite integral is

$$\int_0^\infty dk\frac{k^2}{\omega(\omega+\omega')^3}=\frac{1}{2}\frac{\omega'}{k'^2}-\frac{K^2}{k'^3}\tanh^{-1}\left(\frac{k'}{K+\omega'}\right).$$

Dropping primes

$$I=\frac{1}{2}\int_0^\infty\frac{dk}{\omega^2}-K^2\int_0^\infty\frac{dk}{k\omega^3}\tanh^{-1}\left(\frac{k}{K+\omega}\right)=\frac{1}{2}\left(\frac{\pi}{2K}\right)-K^2J,$$

where we still have to evaluate

$$J\equiv\int_0^\infty\frac{dk}{k\omega^3}\tanh^{-1}\left(\frac{k}{K+\omega}\right)=\frac{1}{2}\int_0^\infty\frac{dk}{k\omega^3}\sinh^{-1}\left(\frac{k}{K}\right).$$

Instead of evaluating J directly, we shall calculate

$$M\equiv\int_0^\infty\frac{dk}{k\omega}\sinh^{-1}\left(\frac{k}{K}\right).$$

Knowing M, J is easily determined since

$$\frac{\partial M}{\partial K}=-2KJ-\frac{1}{K}\int_0^\infty\frac{dk}{\omega^2}=-2KJ-\frac{\pi}{2K^2}.$$

Note that $\sinh^{-1}[k/K]=\ln[k+\omega/K]$. It is convenient to introduce a new integration variable x such that $(1+x)/(1-x)=(k+\omega)/K$ or $x=k/(\omega+K)$. The x integration goes from $x=0$ to $x=1$. Also $dx=Kdk/\omega(\omega+K)$. Therefore

$$M=\frac{1}{K}\int_0^1\frac{dx}{x}\ln\left(\frac{1+x}{1-x}\right)=\frac{\pi^2}{4K}.$$

We find that

$$J=\frac{\pi}{4K^3}\left(\frac{\pi}{2}-1\right),$$

and hence

$$I=\frac{\pi}{2K}\left(1-\frac{\pi}{4}\right).$$