Time-Dependent Impulse Approximation

SAUL T. EPSTEIN
University of Nebraska, Lincoln, Nebraska
(Received January 25, 1960)

The impulse approximation is generalized to cover cases in which a bound system is subject to a time-dependent perturbation. It is shown that the approximation is exact if the perturbation is an impulse. This result supports the supposition that the usual impulse approximation is accurate for collisions in which the collision time is short.

I. INTRODUCTION

THE impulse approximation[1] has been widely used for discussing collisions between incident particles and bound systems for which the collision time is short. However, not much has been done to show that the result is in fact accurate in such cases, although the underlying physical picture[1] certainly suggests that it is.

In this note we will present evidence that the approximation is indeed accurate. We will first generalize the collision problem by assuming a time-dependent interaction $V(t)$ [2] with the bound system and then derive for this problem the analog of the impulse approximation. We will then apply this approximation to a schematic model of a rapid collision, namely we take $V(t)$ to be an impulse, and we will show that for this situation our approximation is actually *exact*. It is this result which we offer in support of the supposition that the impulse approximation is also accurate for realistic collisions involving short collision times.

In the next section we present an intuitive derivation of the approximation and in the third section we show that it is exact for the case of impulsive forces. In the first appendix we give a formal derivation of the approximation, and in a second appendix we discuss its application to the problem of a perturbed harmonic oscillator, a problem which can be solved exactly.[3] In this note we do not discuss any practical applications of our approximation, however it is clear that it may be useful in problems involving the interaction of a bound system with time-dependent external fields, or in collision problems where the incident particle essentially follows a classical path and therefore can be represented simply as a moving source of an external field.

II. DERIVATION OF THE APPROXIMATION

First we must introduce some notation. Our Schrödinger equation is ($\hbar=1$)

$$[H_0+U+V(t)]|n,t\rangle=i\frac{\partial}{\partial t}|n,t\rangle;\quad |n,0\rangle=|n\rangle. \qquad (1)$$

Here H_0 is the kinetic operator for all the particles, U the binding potential, and $|n\rangle$ is an eigenket of H_0+U: $(H_0+U)|n\rangle=E_n|n\rangle$. We will also need the kets $|p,t\}$ and $|p\}$ defined by

$$[H_0+V(t)]|p,t\}=i\frac{\partial}{\partial t}|p,t\};\quad |p,0\}=|p\}, \qquad (2)$$

and $H_0|p\}=\epsilon_p|p\}$.

We now replace (1) by the integral equation[4]

$$|n,t\rangle=|n\rangle e^{-iE_nt}-i\int_0^\infty dt'K_U(t,t')V(t')|n,t'\rangle, \qquad (3)$$

where K_U is the time dependent Green's function for the operator H_0+U. From (3) we can derive the familiar result that the transition amplitude or S-matrix element is given by

$$S_{mn}=\delta_{mn}-i\int_0^\infty dt'\langle m|V(t')|n,t'\rangle e^{iE_mt'}. \qquad (4)$$

The derivation of the time-dependent impulse approximation now proceeds as follows: We first write down the Born approximation

$$S_{mn}{}^B=\delta_{mn}-i\int_0^\infty dt'\langle m|V(t')|n\rangle e^{i(E_m-E_n)t'},$$

and then, in the spirit of the impulse approximation, we seek to change it in such a way that if U were equal to zero, that then our approximation would be exact. The change which immediately suggests itself is to replace

$$\langle m|V(t')|n\rangle=\sum_p\langle m|V(t')|p\}\{p|n\rangle$$

by

$$\sum_p\langle m|V(t')|p,t'\}\{p|n\rangle e^{i\epsilon_pt'},$$

since one readily verifies that with this substitution we do get the correct result when U equals zero. Further, not unexpectedly, one also finds that if V is independent of time that our approximation is equivalent to the usual impulse approximation. Thus we are led to the time-dependent impulse approximation:

$$S_{mn}{}^I=\delta_{mn}-i\sum_p\int_0^\infty dt'\langle m|V(t')|p,t'\}\{p|n\rangle \times e^{i(E_m-E_n+\epsilon_p)t'}. \qquad (5)$$

[1] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952).

[2] We suppress all coordinate labels. $V(t)$ will be taken to be zero for $t<0$ and to go to zero rapidly for large values of t.

[3] G. Ludwig, Z. Physik **130**, 468 (1951).

[4] R. P. Feynman, Phys. Rev. **76**, 749 (1949).

III. IMPULSIVE INTERACTIONS

First we will derive the exact expression for the S operator and then we will show that our approximation yields the same result. Let us consider the Hamiltonian $H_0+V(t)$ where now we specialize $V(t)$ to be of the form

$$V(t)=gW \quad \text{for} \quad 0\leqslant t\leqslant T, \quad \text{zero otherwise.}$$

Here g measures the strength of the force and W is independent of time. For this Hamiltonian we then have the well known result that the S operator is given by $S=S(T)$ where

$$S(t)=\mathcal{P}\exp\left(-i\int_0^t g\bar{W}(t')dt'\right), \qquad (6)$$

with $\mathcal{P}$ denoting Dyson's chronological ordering operator and

$$\bar{W}(t)=e^{iH_0t}We^{-iH_0t}.$$

We now pass to the impulse limit by letting $g\rightarrow\infty$ and $T\rightarrow 0$ in such a way that $gT\equiv F$ stays finite. To this end we change variables in (6) from t' to $\tau=gt'$. Then letting $g\rightarrow\infty$ we find for $t=T$

$$S=\mathcal{P}\exp\left(-i\int_0^F \bar{W}(0)d\tau\right)=\exp(-iFW). \qquad (7)$$

In particular note that S is independent of H_0 and therefore is also the S operator for our actual Hamiltonian H_0+U+V. That is, as one might expect, *for impulsive forces the S operator is independent of the internal dynamics of the system.*

Now let us consider our approximation (5). Bringing $\exp i\epsilon_p t'$ inside the scalar product and replacing it with $\exp iH_0t'$, noting that $|p,t'\}=\exp(-iH_0t')S(t')|p\}$ and that $idS(t)/dt=\bar{V}(t)S(t)$ we find that

$$S_{mn}{}^I=\delta_{mn}+\int_0^T dt'\left\langle m\left|e^{-iH_0t'}\frac{dS(t')}{dt'}e^{iH_0t'}\right|n\right\rangle \times e^{-i(E_n-E_m)t'}.$$

Now again introducing τ and letting $g\rightarrow\infty$ we find

$$S_{mn}{}^I=\delta_{mn}+\int_0^F d\tau\left\langle m\left|\frac{dS(t)}{d\tau}\right|n\right\rangle =\delta_{mn}+\langle m|S(F)-S(0)|n\rangle.$$

But $S(0)=1$, $S(F)=S$, and $\langle m|n\rangle=\delta_{mn}$ so

$$S_{mn}{}^I=\langle m|S|n\rangle,$$

which is, as we have seen, the correct result.

APPENDIX I. FORMAL DERIVATION[5]

We start from (3) and the analogous equation for $|p,t\}$:

$$|p,t\}=|p\}e^{-i\epsilon_pt}-i\int_0^\infty dt'K(t,t')V(t')|p,t'\}. \qquad (8)$$

We now write $|n,t\rangle=\sum_p\alpha_n(p,t)|p,t\}$, insert this on the right-hand side of (5) and rearrange as follows:

$$\begin{aligned}-i\sum_p\int_0^\infty dt'K_U(t,t')V(t')\alpha_n(p,t')|p,t'\} &=-i\sum_p\int_0^\infty dt'K(t,t')V(t')\alpha_n(p,t)|p,t'\}\\ &\quad -i\sum_p\int_0^\infty dt'[K_U(t,t')\alpha_n(p,t')\\ &\qquad -K(t,t')\alpha_n(p,t)]V(t')|p,t'\}\\ &\equiv -i\sum_p\int_0^\infty dt'K(t,t')V(t')\alpha_n(p,t)|p,t'\}+|X\}.\end{aligned}$$

We now use (8) to write this as

$$\begin{aligned}&=\textstyle\sum_p\alpha_n(p,t)[|p,t\}-|p\}e^{-i\epsilon_pt}]+|X\}\\ &=|n,t\rangle-\textstyle\sum_p\alpha_n(p,t)|p\}e^{-i\epsilon_pt}+|X\},\end{aligned}$$

so that (3) becomes

$$0=|n\rangle e^{-iE_nt}-\textstyle\sum_p\alpha_n(p,t)|p\}e^{-i\epsilon_pt}+|X\}.$$

Finally, using the orthogonality of the $|p\}$ we derive an integral equation for $\alpha_n(p,t)$:

$$\alpha_n(p,t)=\{p|n\rangle e^{i(\epsilon_p-E_n)t}+\{p|X\}e^{i\epsilon_pt}.$$

One now readily verifies that the inhomogeneous term, when used to approximate $|n,t\rangle$ and inserted into (4), gives rise to the time dependent impulse approximation. The integral equation then permits systematic improvement of this approximation.

APPENDIX II. PERTURBED HARMONIC OSCILLATOR

We consider a one dimensional harmonic oscillator subject to the perturbation $V(x,t)=-gxf(t)$. This problem is of interest for testing the range of validity of our approximation because it can be solved exactly.[3] Here

$$\langle x|p\}=(2\pi)^{-\frac{1}{2}}\exp(ipx), \quad \epsilon_p=p^2/2M,$$

and one easily shows that

$$\langle x|p,t\}=(2\pi)^{-\frac{1}{2}}\exp\{i[R(t)x-L(t)]\},$$

$$R(t)=P+g\int_0^t f(t')dt', \quad L(t)=\frac{1}{2M}\int_0^t R^2(t')dt'. \qquad (9)$$

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[5] For an analogous derivation of the impulse approximation see S. T. Epstein, Phys. Rev. 86, 836 (1952).

Unfortunately however we have not yet been able to calculate $S_{mn}{}^{I}$ for the general case (one can do all the integrals analytically except for the time integration). We have, however, been able to carry out the calculation in the impulse limit and get the correct answer, thus providing a welcome check on the formal manipulations of Sec. III. We now briefly sketch this calculation. From (5) with (9), or directly from (7) it follows that in the impulse limit[6]

$$S_{mn}{}^{I}=\int_{-\infty}^{\infty} dx\, \phi_m{}^*(x)e^{iFx}\phi_n(x),$$

where ϕ_m and ϕ_n are harmonic oscillator wave functions. One can carry out the integration by use of the known result[7] for Hermite polynomials that

$$H_m(y)H_n(y)=\sum_{k=0}^{m} 2^k k!\binom{m}{k}\binom{n}{k}H_{m+n-2k}(y),\qquad m\leqslant n, \quad (10)$$

and the fact that $\int_{-\infty}^{\infty} dy\, H_L(y)\exp(iyz-y^2)$ is easily evaluated by performing L integrations by parts. The sum introduced by (10) is then recognized as being proportional to an associated Laguerre polynomial and in this way one derives exactly Ludwig's result.

[6] This same integral, for obvious reasons, also occurs in the theory of the infrared catastrophe. See W. Pauli and M. Fierz, Nuovo cimento **15**, 167 (1938), Eq. (18).

[7] A. Erdelyi, *et al.*, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 195.

PHYSICAL REVIEW VOLUME 119, NUMBER 1 JULY 1, 1960

Boson Furry Theorem

D. C. Peaslee
Australian National University, Canberra, A.C.T., Australia

AND

M. T. Vaughn*†
Purdue University, Lafayette, Indiana

(Received October 12, 1959; revised manuscript received January 29, 1960)

A Furry theorem for heavy mesons and photons is given for a class of highly symmetric interactions, neglecting the Ξ-N mass difference. Because of this neglect most rules are only approximately valid, but a few depend on charge conjugation alone and are absolute.

EXTENSION of the Furry theorem[1] to heavy bosons has proceeded gradually from special to more general cases,[2–6] with considerable duplication and rediscovery along the way. We here base similar remarks on a separately described[7] scheme of seven-dimensional charge space: the conclusions are not all new, but there is some generalization of previous results, and the whole exercise shows how simple and compact is the seven-dimensional scheme for such purposes.

The "antiparticulation" operator A defined in reference 7 has the property that $A^2=1$, and

$$\begin{gathered} Ap=-(\Xi^-)^c,\quad A\Sigma^+=-(\Sigma^-)^c,\\ An=(\Xi^0)^c,\quad A\Sigma^{0+}=(\Sigma^{0-})^c,\\ A\varphi=-\varphi,\\ A\gamma=\gamma, \end{gathered}\qquad (1)$$

where φ is any meson field and γ the photon. Invariance rules under A are valid only to the extent that the Ξ-N mass difference Δ can be neglected; according to the scheme of reference 7 this mass difference has an "intrinsic" basis, while there is no asymmetry in the (unrenormalized) strong boson-fermion interactions. Thus A forbiddenness may mean reducing the matrix element of a process by only about $\Delta/M\approx 20\%$; but this is sufficient to be of some practical importance, and in special cases the reduction in the matrix element could be of order $(\Delta/M)^2$. Invariance rules based on the charge conjugation operator C are of course exact

* Assisted by the Air Force Office of Scientific Research.

† Now at University of Pennsylvania, Philadelphia, Pennsylvania.

[1] W. H. Furry, Phys. Rev. **51**, 125 (1937).

[2] Particular cases were considered by H. Fukuda and Y. Miyamoto, Progr. Theoret. Phys. (Kyoto) **4**, 389 (1949); C. B. van Wyck, Phys. Rev. **80**, 487 (1950); K. Nishijima, Progr. Theoret. Phys. (Kyoto) **6**, 614 (1951); L. Michel, Progress in Cosmic-Ray Physics (Interscience Publishers, New York, 1952).

[3] General forms for pion and nucleon systems were given by A. Pais and R. Jost, Phys. Rev. **87**, 871 (1952); L. Michel, Nuovo cimento **10**, 319 (1953); T. D. Lee and C. N. Yang, Nuovo cimento **3**, 749 (1956); with applications to nucleon-antinucleon systems by D. Amati and B. Vitale, Nuovo cimento **2**, 719 (1955); C. Goebel, Phys. Rev. **103**, 258 (1956); S. Barshay, Phys. Rev. **109**, 554 (1958).

[4] D. C. Peaslee, Nuovo cimento **6**, 1 (1957) defines an analogous operator, essentially the A of reference 7, applicable to K mesons and baryons as well as pions and nucleons.

[5] R. E. Pugh, Phys. Rev. **109**, 989 (1958), gives a Furry theorem involving pions, photons, and baryons.

[6] G. Feinberg and R. E. Behrends, Brookhaven National Laboratory Report BNL-4090, 1959 (unpublished), give analogous considerations involving K mesons.

[7] D. C. Peaslee, Phys. Rev. **117**, 873 (1960).