High-Energy Limit of Form Factors

S. D. DRELL AND F. ZACHARIASEN
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California
(Received February 2, 1960)

This theorem is proved: For finite charge renormalization constant Z_3^{-1}, the form factors describing any vertex with two particles on the mass shell must vanish at infinite momentum transfer. The relation of this result to the work of Lehmann, Symanzik, and Zimmermann is discussed.

I. INTRODUCTION

In constructing dispersion relations it is necessary to answer, among other things, the question of how many subtraction constants are required. Mandelstam's[1] representation tells us unambiguously what constants must appear for any process involving four external particles. However, for the simpler three-particle vertex function this question still remains unanswered. Here we wish to point out the following simple statement: For finite charge renormalization constant Z_3^{-1}, the form factors describing any vertex with two particles on the mass shell must vanish at infinite momentum transfer. Consequently, no subtraction constants are required, and the unsubtracted form of dispersion relations may be used for such form factors.

Lehmann, Symanzik, and Zimmermann[2] have shown that the usual vertex function of quantum field theory must satisfy a condition which implies that it vanishes at infinite momentum transfer, independent of any assumption on Z_3^{-1}. Their result differs from the one presented here in that the form factors of dispersion theory are not the same as the vertex functions of field theory. The usual vertex function is defined as the sum of all *proper* vertex diagrams, while the conventional form factor includes *all* diagrams—including *improper* ones—contributing to the blob of Fig. 1. The form factor, therefore, includes all graphs of the type shown in Fig. 2. These additional self-energy blobs reduce to unity for external particles on the mass shell; hence, for the vertex usually of physical interest, where particle No. 1, for example, is virtual, only blobs on the No. 1 leg remain. Since these contribute a factor Z_3^{-1} when particle No. 1 carries infinite momentum transfer, the LSZ result implies simply that

$$F(q^2)/Z_3 \to 0 \quad \text{as} \quad q^2 \to \infty,$$

where F is the form factor and q^2 the momentum transfer. Thus, in order to obtain a statement about the physically measurable function F, a statement on Z_3^{-1} is required. If, however, one is willing to make an assumption about Z_3^{-1}, the result that $F \to 0$ can be obtained directly with much greater ease than required by the development of LSZ.

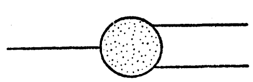

FIG. 1. Dispersion theory diagram for form factors; the blob includes *proper* and *improper* graphs.

II. PROOF BY SCHWARTZ INEQUALITY

In order that any given form factor satisfy an unsubtracted dispersion relation, it is necessary that the imaginary part of it vanish at infinite momentum transfers; consequently, we shall begin with a study of the properties of these imaginary parts. We shall concern ourselves explicitly with the nucleon electromagnetic form factors, since at present these are receiving the most attention in the literature; but it will be evident that the same discussion may be applied to the form factors associated with any three particle vertex.

The nucleon form factors, F_1 and F_2, are defined by

$$\frac{1}{(4E_pE_{p'})^{\frac{1}{2}}}(\bar{u}_p|\gamma_\mu F_1(q^2)+\sigma_{\mu\nu}q_\nu F_2(q^2)|v_{p'}) = \langle pp'^{(-)}|j_\mu(0)|0\rangle, \quad (1)$$

where $q=p+p'$ is the four momentum of a virtual photon in the reaction $\gamma \to N+\bar{N}$. p and p' are the four momenta of the nucleon and antinucleon, respectively; E_p and $E_{p'}$ are the corresponding energies. $\langle pp'^{(-)}|$ denotes an ingoing Heisenberg state of the indicated nucleon-antinucleon pair, $|0\rangle$ is the Heisenberg vacuum, and finally

$$j_\mu(x) = \Box A_\mu(x),$$

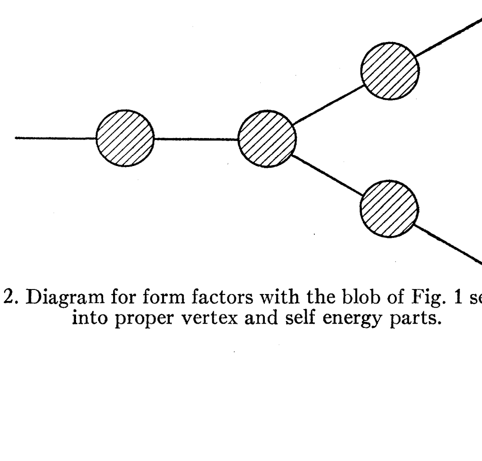

FIG. 2. Diagram for form factors with the blob of Fig. 1 separated into proper vertex and self energy parts.

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958), and to be published.
[2] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **2**, 425 (1955). Their proof has recently been carried through explicitly for the specific case of photons and electrons by L. E. Evans (to be published).

where $A_\mu(x)$ is the renormalized Heisenberg electromagnetic field operator.[3]

As we have stated above, what will concern us now are the functions $\mathrm{Im}F_1$ and $\mathrm{Im}F_2$; these are defined as follows[3]:

$$\frac{1}{(4E_pE_{p'})^{\frac{1}{2}}}(\bar{u}_p|\mathrm{Im}F_1\gamma_\mu+\mathrm{Im}F_2\sigma_{\mu\nu}q_\nu|v_{p'})$$
$$=-\tfrac{1}{2}\sum_n(2\pi)^4\delta^4(p_n-q)\frac{\langle p|\bar{j}_N(0)v_{p'}|n^{(-)}\rangle}{(2E_{p'})^{\frac{1}{2}}}\times\langle n^{(-)}|j_\mu(0)|0\rangle; \quad (2)$$

here

$$j_N(x)=(i\gamma_\mu\nabla_\mu-M)\psi(x),$$

where $\psi(x)$ is the renormalized Heisenberg nucleon field operator. The sum runs over a complete set of Heisenberg states $|n\rangle$; we have for convenience chosen those with ingoing boundary conditions. Equation (2) may be rewritten as

$$\frac{1}{(4E_pE_{p'})^{\frac{1}{2}}}(\bar{u}_p|\mathrm{Im}F_1\gamma_\mu+\mathrm{Im}F_2\sigma_{\mu\nu}q_\nu|v_{p'})=-\tfrac{1}{2}\sum_n a_n^*b_n, \quad (3)$$

with the help of the definitions

$$a_n^*=[(2\pi)^4\delta^4(P_n-q)]^{\frac{1}{2}}\frac{1}{(2E_{p'})^{\frac{1}{2}}}\langle p|\bar{j}_N(0)v_{p'}|n^{(-)}\rangle,$$
$$b_n=[(2\pi)^4\delta^4(P_n-q)]^{\frac{1}{2}}\langle n^{(-)}|j_\mu(0)|0\rangle. \quad (4)$$

From Eq. (3) it is evident that a bound may be placed on $\mathrm{Im}F_1$ and $\mathrm{Im}F_2$ by the use of Schwartz's inequality, viz.,

$$\left(\frac{1}{(4E_pE_{p'})^{\frac{1}{2}}}\right)^2|(\bar{u}_p|\mathrm{Im}F_1\gamma_\mu+\mathrm{Im}F_2\sigma_{\mu\nu}q_\nu|v_{p'})|^2\leqslant(\sum_n|a_n|^2)(\sum_n|b_n|^2)\equiv\|a\|^2\|b\|^2. \quad (5)$$

We shall obtain some conditions on the asymptotic behavior of $\mathrm{Im}F_1$ and $\mathrm{Im}F_2$ by examining the inequality (5) as $q^2\to\infty$. Equation (5) presents a useful bound because of the possible limits on the right-hand side. By (4) we see that $\|a\|^2$ is proportional to the total cross section for a nucleon-antinucleon collision in specified angular momentum channels according to the selection rules and is limited by unitarity; $\|b\|^2$ is proportional to the weight function for the photon propagator and is limited by what we say about Z_3. First, however, it will be convenient to rid ourselves of the spinors and Dirac matrices by summing Eq. (5) over the spins of p and p', and the index μ. This operation replaces the left side of (5) by[4]

$$\left(\frac{1}{(4E_pE_{p'})^{\frac{1}{2}}}\right)^2\mathrm{Sp}[(\not{p}+M)(\mathrm{Im}F_1\gamma_\mu+\mathrm{Im}F_2\sigma_{\mu\nu}q_\nu)\times(\not{p}'-M)(\mathrm{Im}F_1\gamma_\mu+\mathrm{Im}F_2\sigma_{\mu\nu}q_\nu)]$$
$$=-(4/q^2)\{q^2(\mathrm{Im}F_1-4M\,\mathrm{Im}F_2)^2+2(M\,\mathrm{Im}F_1-q^2\,\mathrm{Im}F_2)^2\}, \quad (6)$$

where we have chosen the center-of-mass system $\mathbf{p}=-\mathbf{p}'$, and $\not{p}\equiv p_\mu\gamma_\mu$.

On the right-hand side of (5) it is necessary to evaluate $\sum_{\mathrm{spins}}\|a\|^2$ and $\sum_\mu\|b\|^2$; we first discuss the a term. Explicitly, we have

$$\sum_{\mathrm{spins}}\|a\|^2=\sum_{\mathrm{spins}}\sum_n(2\pi)^4\delta^4(P_n-q)\frac{|\langle n^{(-)}|\bar{v}_{p'}j_N(0)|p\rangle|^2}{2E_{p'}}; \quad (7)$$

since the scattering matrix for the reaction $N+\bar{N}\to n$ is

$$T_{n,pp'}=\frac{1}{(2E_{p'})^{\frac{1}{2}}}\langle n^{(-)}|\bar{v}_{p'}j_N(0)|p\rangle,$$

this reduces to

$$\sum_{\mathrm{spins}}\|a\|^2=\sum_{\mathrm{spins}}\sum_n(2\pi)^4\delta^4(P_n-q)|T_{n,pp'}|^2=v\sigma_T((q^2)^{\frac{1}{2}}) \quad (8)$$

where $\sigma_T((q^2)^{\frac{1}{2}})$ is the total nucleon-antinucleon annihilation cross section at a total c.m. energy $(q^2)^{\frac{1}{2}}$, and $v=[(q^2-4M^2)/q^2]^{\frac{1}{2}}$ is the relative velocity of the nucleon and antinucleon in the c.m. system. One further observation on Eq. (8) will be useful—that is, that in Eq. (2) it is evident that the only states which contribute are those with total angular momentum one and even parity in the c.m. system. The same is therefore true in Eq. (7); thus, $\sigma_T((q^2)^{\frac{1}{2}})$ is actually just the nucleon-antinucleon annihilation cross section for the $^3S_1$ and $^3D_1$ states of the $N\bar{N}$ system.

Next, consider the b term: Here we have to evaluate

$$\sum_\mu\|b\|^2=\sum_\mu\sum_n(2\pi)^4\delta^4(P_n-q)|\langle n^{(-)}|j_\mu(0)|0\rangle|^2. \quad (9)$$

Now $\langle n^{(-)}|j_\mu(0)|0\rangle=P_n^2\langle n^{(-)}|A_\mu(0)|0\rangle$; furthermore, recall that the definition of the weight function in the spectral representation of the photon propagator is just

$$\rho(q^2)=-\tfrac{1}{3}\sum_{n\neq1\gamma}(2\pi)^3\delta^4(P_n-q)\sum_\mu|\langle n|A_\mu(0)|0\rangle|^2. \quad (10)$$

[3] We follow here the notation and spinor convention of S. D. Drell and F. Zachariasen, Phys. Rev. **111**, 1727 (1958).

[4] Note that while it is true that the sum over μ is to be made in accordance with the usual convention

$$A_\mu A_\mu=A_tA_t-A_xA_x-A_yA_y-A_zA_z,$$

the inequality which was expressed in Eq. (5) is not destroyed by summing over μ. This is because both sides of Eq. (5) are positive definite, while both sides of Eq. (6) are negative definite after summing over μ; hence, the direction of the inequality is reversed upon summing on μ and reversed again upon multiplying by the minus sign this sum produces in each side.

Putting this together, we see that

$$\sum_\mu \|b\|^2 = -(3q^4)[2\pi\rho(q^2)]. \tag{11}$$

Finally, inserting Eqs. (11), (8), and (6) into (3) we have the result

$$(4/q^2)\{q^2(\mathrm{Im}F_1 - 4M\,\mathrm{Im}F_2)^2 + 2(M\,\mathrm{Im}F_1 - q^2\,\mathrm{Im}F_2)^2\} \leq 6\pi q^4[(q^2-4M^2)/q^2]^{\frac{1}{2}}\sigma_T((q^2)^{\frac{1}{2}})\rho(q^2). \tag{12}$$

Consider what happens in (12) when q^2 is allowed to approach infinity. On the right-hand side $q^2\sigma_T((q^2)^{\frac{1}{2}}) \leq |L|$ where $|L| < \infty$, since σ is the cross section for two partial waves only, and is therefore bounded by unitarity in the physical region $q^2 > 4M^2$. Furthermore, we have

$$Z_3^{-1} = 1 + \int \rho(q^2)dq^2; \tag{13}$$

therefore, since ρ is positive definite, if Z_3^{-1} is finite, $q^2\rho(q^2) \to 0$. As $q^2 \to \infty$, then, (12) becomes

$$[(\mathrm{Im}F_1)^2 + 2q^2(\mathrm{Im}F_2)^2 - 12M\,\mathrm{Im}F_1\,\mathrm{Im}F_2] \leq 0,$$

which means $(q^2)^{\frac{1}{2}}\,\mathrm{Im}F_2 \to 0$ and $\mathrm{Im}F_1 \to 0$. To conclude, then, the finiteness of Z_3^{-1} forces the vanishing of $\mathrm{Im}F_1$ and $\mathrm{Im}F_2$ at ∞.[5]

The statement that $\mathrm{Im}F_1$ vanishes as $q^2 \to \infty$, incidentally, does not of itself require that the entire form factor F_1 vanishes as well, although we shall show in IV that this result is also a consequence of the finiteness of Z_3^{-1}.

III. LSZ PROOF

At this point it is convenient to insert a brief discussion of the work of LSZ[2] on the vertex function of field theory. In Sec. II we have shown, and in Sec. IV we shall show directly that in order to ensure the vanishing of the form factor, for example in quantum electrodynamics, at infinity, it is necessary that Z_3^{-1} be finite. LSZ show that the vertex function vanishes at infinity with no restrictions on Z_3^{-1}.[6] The difference between these results is easily stated; the vertex function Γ of field theory is related to either form factor F_1 or F_2 of dispersion relations by

$$F_{1,2}(q^2) = \frac{D_{F1}(q^2)}{D_F(q^2)}\Gamma_{1,2}(q^2).$$

Here D_{F1} is the complete renormalized propagator of the photon, and $D_F = 1/q^2$. Thus, as $q^2 \to \infty$, we have

$$\lim_{q^2\to\infty} F(q^2) = Z_3 \lim_{q^2\to\infty} \Gamma(q^2), \tag{14}$$

since $D_{F1}/D_F \to Z_3$ as $q^2 \to \infty$. Therefore, a vanishing Γ *only* insures a vanishing F provided Z_3 is finite.

The LSZ proof of the vanishing of Γ uses the fact that the spectral function ρ of Eq. (9) is bounded below by the contribution of only the two-nucleon states in Eq. (9), and the contribution of the two-nucleon state is proportional to $D_{F1}{}^2\Gamma^2$, so that if D_{F1} is replaced by its expression in terms of the spectral function ρ an integral condition results for Γ which requires $\Gamma \to 0$ as $q^2 \to \infty$.

However, this proof, which is a bit involved, is involved only because of the reappearances of D_{F1} in the two nucleon contribution to ρ. If $D_{F1}\Gamma$ is replaced by just $D_F F$, then conditions may be obtained on F, which is after all the interesting quantity, with much greater ease.

IV. SIMPLE PROOF

By adapting the LSZ approach to the dispersion theory form factors, we can obtain an even stronger statement on their infinite q^2 behavior than that achieved in II. Recall the form of the spectral function for the photon propagator; namely,

$$\rho(q^2) = -\tfrac{1}{3}\sum_{n\neq 1\gamma}(2\pi)^3\delta^3(\mathbf{P}_n)2E_n\delta(E_n{}^2-q^2) \times \sum_\mu |\langle n^{(-)}|A_\mu(0)|0\rangle|^2. \tag{15}$$

This is a sum of positive terms, in spite of the indefinite metric associated with the polarization sum. It is, therefore, bounded below by the fermion antifermion term alone. The contribution of this state in $\sum_n$ is

$$\rho^{(2)}(q^2) = \frac{1}{12\pi^2}\frac{1}{q^2}\left(1-\frac{4M^2}{q^2}\right)^{\frac{1}{2}}\left\{(F_1-4MF_2)^2 + \frac{2M^2}{q^2}\left(F_1 - \frac{q^2}{M}F_2\right)^2\right\}, \tag{16}$$

in terms of the electromagnetic form factors of the fermion, defined as usual by

$$\frac{1}{(4E_pE_{p'})^{\frac{1}{2}}}(\bar{u}_p|F_1\gamma_\mu + F_2\sigma_{\mu\nu}q_\nu|v_{p'}) = \langle pp'^{(-)}|j_\mu(0)|0\rangle. \tag{17}$$

Here j_μ is the photon current, given by

$$j_\mu(x) = \Box A_\mu(x), \tag{18}$$

where $A_\mu(x)$ is the renormalized Heisenberg field operator of the electromagnetic field.

---

[5] For the three-boson vertex it is necessary to make the stronger assumption that the self mass,

$$\delta\mu^2 = \int \rho(q^2)q^2dq^2,$$

is finite in order to assure the vanishing of the form factor as $q^2 \to \infty$. This is easily seen because Eq. (6) lacks the q^2 from the spur for the three-boson case, and, therefore, takes the form $(1/q^2)[\mathrm{Im}F(q^2)]^2$ when F is the three-boson form factor.

[6] They must assume the absence of "ghosts," i.e., the positive definiteness of the spectral function ρ; this is an assumption we used implicitly in Sec. II also. S. Weinberg has recently shown (private communication to be published) that their proof is valid even if subtractions are required in the representations of the propagator.

Now since

$$Z_3^{-1}=1+\int\rho(q'^2)dq'^2\geqslant 1+\int\rho^{(2)}(q'^2)dq'^2, \qquad (19)$$

it is clear that if $Z_3^{-1}<\infty$, then it is necessary[7] that $F_1\to 0$ and $F_2\to 0$ as $q^2\to\infty$.

This condition is actually stronger than the one obtained in II, in that here we get the entire form factor $F\to 0$, not just the imaginary part.

V. CONCLUSION

Thus far our discussion has been limited to the statement, "if $Z_3^{-1}<\infty$, then . . ." The critical question then is whether Z_3^{-1} is or is not finite. We shall certainly not answer this question; nevertheless, a few comments may be appropriate.

The form factors were defined by Eq. (1); application of the contraction rule to the matrix element on the right side of (1) yields

$$\langle pp'^{(-)}|j_\mu(0)|0\rangle=\frac{1}{(4E_pE_{p'})^{\frac{1}{2}}}\left(\bar{u}_p\left|\frac{e}{Z_3}\gamma_\mu\right|v_{p'}\right)-i\frac{1}{(2E_p)^{\frac{1}{2}}}\int d^4x\, e^{ip'\cdot x}\eta(x_0)\times\langle p|[\bar{j}_N(x),j_\mu(0)]|0\rangle v_{p'}. \qquad (20)$$

The first term in (20) gives a contribution of e/Z_3 to $F_1(q^2)$. If it could be shown that the integral in (20) vanished as $q^2\to\infty$, then one would find $F_1\to e/Z_3$ as $q^2\to\infty$. This is consistent with our result that a finite Z_3^{-1} implies $F_1\to 0$ for $q^2\to\infty$ only if $Z_3=\infty$, i.e., $Z_3^{-1}=0$.

[7] As remarked in S. G. Gasiorowicz, D. R. Yennie, and H. Suura, Phys. Rev. Letters **2**, 513 (1959), this is not a sufficient condition for finite renormalization constants.

The integral in (20) can, of course, be evaluated explicitly:

$$-\frac{i}{(2E_{p'})^{\frac{1}{2}}}\int d^4x\, e^{ip'\cdot x}\eta(x_0)\langle p|[\bar{j}_N(x)v_{p'},j_\mu(0)]|0\rangle$$
$$=\frac{1}{(2E_{p'})^{\frac{1}{2}}}\sum_n\left\{\frac{(2\pi)^3\delta^3(\mathbf{P}_n-\mathbf{q})\langle p|\bar{j}_N(0)v_{p'}|n\rangle\langle n|j_\mu(0)|0\rangle}{E_n-E_p-E_{p'}-i\epsilon}+\frac{(2\pi)^3\delta^3(\mathbf{P}_n-\mathbf{p}')\langle p|j_\mu(0)|n\rangle\langle n|\bar{j}_N(0)v_{p'}|0\rangle}{E_n-E_{p'}}\right\}. \qquad (21)$$

One could now choose a coordinate system with $\mathbf{p}=0$. Then

$$E_{p'}=(q^2-2M^2)/2M, \qquad (22)$$

so as $q^2\to\infty$, the denominators in (21) go to infinity. This, of course, does not show that the entire expression vanishes as $q^2\to\infty$, for two reasons. First, the numerator depends on q^2 through the momentum delta functions, and it could be that the numerators diverge as $q^2\to\infty$ also. Second, there are an infinite number of terms in Eq. (28) for $q^2=\infty$, and even if each term in $\sum_n$ did vanish as $q^2\to\infty$, there would be no guarantee that the sum on n also vanished. Such a conclusion would require the uniform convergence of the series in (21), a property which is by no means obvious.

Attempts to show that at least one renormalization constant is infinite have been made by making a second contraction in (20). This question is presently under active discussion.[7,8]

[8] Looking just at (20) and (21) there is nothing to prevent the integral in (20) from approaching a finite limit $-e/Z_3$ as $q^2\to\infty$. This means $F_1(q^2)\to 0$, which is consistent with the possibility of the finiteness of Z_3^{-1}. We were unable to see any further light being cast on this problem by a second contraction. See G. Källén, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. V, Part 1, Sec. 47.