S-Wave Dominant Solutions of the Pion-Pion Integral Equations*

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The integral equations for pion-pion scattering formulated by Chew and Mandelstam are put into a form suitable for numerical solution. An iteration procedure is described that is applicable when the S-wave amplitude dominates the equations, all higher partial waves being small; this paper considers only solutions for which such is the case. The requirement that the equations have consistent solutions without bound states turns out to limit the pion-pion coupling constant to the range $-0.46<\lambda\lesssim 0.3$. Results are given for various values of λ within this interval.

I. INTRODUCTION

IN the preceding paper,[1] henceforth to be referred to as CM, a set of coupled integral equations for pion-pion scattering has been derived. We describe here the solution of these equations by a numerical procedure of iteration. Such an iteration procedure follows more or less naturally from the structure of the equations, and is found to be straightforward in application and rapidly convergent. Further, checks on the approximations underlying the equations can be applied once the equations have been solved and confirm the legitimacy of these approximations.

A question arising at once is whether these solutions obtained by iteration are the only solutions of the nonlinear integral equations. The uniqueness is not at all obvious and, in fact, the structure of the equations indicates that another class of solutions is also possible. The solutions obtained here are characterized by the smallness of all amplitudes with $l>0$. However, a rough examination of the equations indicates that solutions with large P-wave amplitudes are also possible even if the S-wave amplitudes are small. It seems likely that this latter type of solution is the one actually occurring in nature, Frazer and Fulco having shown that a P-wave resonance brings the calculations of nucleon electromagnetic structure into much better agreement with experiment.[2] In this paper, however, we shall consider only the simplest type of solution, which can be obtained by the iteration procedure. The study of this type will be of help in obtaining the more complicated solutions. Moreover, we cannot be sure that these simple solutions are devoid of practical interest, the answer to the electromagnetic structure problem lying in some other direction than a P-wave resonance.

The first task is the straightforward problem of changing variables so as to achieve finite limits of integration. This is done in Sec. II, while in Sec. III the iteration procedure actually carried out with the Livermore 704 computer is described. Finally, in Sec. IV, we present our results as a function of λ, the pion-pion coupling constant.

II. CHANGE OF VARIABLES

The integral equations set up in CM involve the variable ν, the square of the three-momentum in the barycentric system, on the negative real axis in the range $-\infty$ to -1. It is convenient to introduce

$$x=(-\nu)^{-\frac{1}{2}}, \tag{II.1}$$

which will run from 0 to 1. At the same time we replace the functions $E_l^I(\nu)=D_l^I(-\nu)$ with $F_l^I(x)$, defined by

$$F_l^I(x)=xE_l^I(1/x^2)=xD_l^I(-1/x^2). \tag{II.2}$$

The S-wave equation CM-(V.14) then becomes

$$F_0^I(x)=x+a_I xQ(x)+x(1-\tfrac{2}{3}x^2)\times\frac{2}{\pi}\int_0^1 dx'\,\frac{L(x,x')}{1-\frac{2}{3}x'^2}f_0^I(x')F_0^I(x'), \tag{II.3}$$

where

$$Q(x)=(1/x^2-\tfrac{2}{3})K(1/x^2,\tfrac{2}{3})=\frac{2}{\pi}\left[\frac{1}{(1-x^2)^{\frac{1}{2}}}\ln\left(\frac{1+(1-x^2)^{\frac{1}{2}}}{x}\right)-\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right], \tag{II.4}$$

and

$$L(x,x')=\frac{1}{x^2x'^2}K\left(\frac{1}{x^2},\frac{1}{x'^2}\right)=\frac{2}{\pi}\frac{1}{x'^2-x^2}\left[\frac{1}{(1-x^2)^{\frac{1}{2}}}\ln\left(\frac{1+(1-x^2)^{\frac{1}{2}}}{x}\right)-\frac{1}{(1-x'^2)^{\frac{1}{2}}}\ln\left(\frac{1+(1-x'^2)^{\frac{1}{2}}}{x'}\right)\right]. \tag{II.5}$$

Equation (II.3) is a linear integral equation which may be solved by standard methods. As pointed out in CM, the equation is singular but has a unique solution when the S waves are dominant.

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[1] G. F. Chew and S. Mandelstam, preceding paper [Phys. Rev. **119**, 467 (1960)].

[2] W. Frazer and J. Fulco, Phys. Rev. Letters **2**, 365 (1959).

Once the functions $F_0^I(x)$ have been obtained, the S phase shifts in the physical region are given by formula CM-(V.20). It is convenient to introduce a variable that runs between 0 and 1 as ν covers the physical interval 0 to ∞. We therefore define

$$y=1/(1+\nu)^{\frac{1}{2}}, \quad \nu>0, \tag{II.6}$$

and rewrite CM-(V.20) as

$$(1-y^2)^{\frac{1}{2}}\cot\delta_0^I=\frac{1-a_IR(y)-\left(1-\frac{y^2}{3}\right)\frac{2}{\pi}\int_0^1 dx'\,\frac{M(y,x')}{1-\frac{2}{3}x'^2}f_0^I(x')F_0^I(x')}{a_I+\left(1-\frac{y^2}{3}\right)\frac{2}{\pi}\int_0^1 dx'\,\frac{f_0^I(x')F_0^I(x')}{(1-\frac{2}{3}x'^2)(y^2+x'^2-x'^2y^2)}}, \tag{II.7}$$

where

$$R(y)=\left(\frac{1}{y^2}-\frac{1}{3}\right)I\left(\frac{1}{y^2}-1,\frac{2}{3}\right)$$

$$=\frac{2}{\pi}\left[\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-(1-y^2)^{\frac{1}{2}}\ln\left(\frac{1+(1-y^2)^{\frac{1}{2}}}{y}\right)\right], \tag{II.8}$$

and

$$M(y,x')=\frac{1}{x'^2y^2}I\left(\frac{1}{y^2}-1,\frac{1}{x'^2}\right)$$

$$=\frac{2}{\pi}\frac{1}{y^2x'^2-x'^2-y^2}\left[(1-y^2)^{\frac{1}{2}}\ln\left(\frac{1+(1-y^2)^{\frac{1}{2}}}{y}\right)-\frac{1}{(1-x^2)^{\frac{1}{2}}}\ln\left(\frac{1+(1-x^2)^{\frac{1}{2}}}{x}\right)\right]. \tag{II.9}$$

Before writing down the crossing relation for f_0^I in terms of our new variables, x and y, we give the P-wave equations corresponding to (II.3) and (II.7). These are as follows:

$$F_1^1(x)=x+x\frac{2}{\pi}\int_0^1 dx'\,L(x,x')f_1^1(x')F_1^1(x'), \tag{II.10}$$

$$(1-y^2)^{\frac{3}{2}}\cot\delta_1^1=\frac{1-(1-y^2)\frac{2}{\pi}\int_0^1 dx'\,M(y,x')f_1^1(x')F_1^1(x')}{\frac{2}{\pi}\int_0^1 dx'\,\frac{f_1^1(x')F_1^1(x')}{y^2+x'^2-x'^2y^2}}. \tag{II.11}$$

Note that to calculate any higher phase shift, neglecting the right-hand cut, we have simply

$$\frac{(y^2)^{l-1}}{(1-y^2)^{l+\frac{1}{2}}}\tan\delta_l^I=(-1)^{l+1}\frac{2}{\pi}\int_0^1 dx'\,\frac{f_l^I(x')(x')^{2l-1}}{y^2+x'^2-x'^2y^2}, \tag{II.12}$$

for $l\geqslant 2$. This last formula corresponds roughly to a "Born approximation," once the left-hand cut is given, and should be valid so long as the phase shift in question is small. The right-hand cut is then of the order of magnitude of the square of the phase shift. The sum over *all* higher waves is given by Eqs. (IV.9) and (IV.10) of CM.

We turn now to the crossing relations CM-(V.8) expressing the imaginary parts of the partial-wave amplitudes on the left-hand cut in terms of the imaginary parts of S and P amplitudes on the physical cut. Introducing

$$g_l^I(y)=\mathrm{Im}A_l^I(\nu),$$
$$=(1-y^2)^{-\frac{1}{2}}/(1+\cot^2\delta_l^I), \tag{II.13}$$

for $\nu>0$, we have, from CM-(IV.4),

$$f_l^I(x)=-2x^2\int_x^1\frac{dy}{y^3}P_l\left(1-2\frac{x^2}{y^2}\right)\left[\alpha_{I0}g_0^0(y)+\alpha_{I2}g_0^2(y)+3\left(1-2\frac{x^{-2}-1}{y^{-2}-1}\right)\alpha_{I1}g_1^1(y)\right], \tag{II.14}$$

where the matrix $\alpha_{II'}$ is given by CM-(IV.6). Note that if $g_l^I(y)$ remains finite as $y\to 0$, then according to Eq. (II.14) the same is true for $f_l^I(x)$ as $x\to 0$. Such would not be the case if D and higher partial waves were included under the integral in (II.14). One must check *a posteriori* that higher partial waves do not make an important contribution to $f_l^I(x)$ in the range $\frac{1}{3}\lesssim x<1$, where the polynomial expansion converges.

The last formula required in terms of the new variables is CM-(V.18), giving the connection between λ and a_I. We find

$$a_I=\binom{-5}{-2}\lambda+\frac{2}{\pi}\int_0^1\frac{dy}{y}\left[\frac{3}{2y^2}\ln\left(\frac{3}{3-2y^2}\right)-\frac{3}{3-y^2}\right]\times\left[\alpha_{I0}g_0^0(y)+\alpha_{I2}g_0^2(y)+3\left(\frac{1-\frac{1}{3}y^2}{1-y^2}\right)\alpha_{I1}g_1^1(y)\right]. \tag{II.15}$$

As noted in CM, the integral here has a very small value and can be ignored in a first approximation.

III. THE ITERATION PROCEDURE

The general procedure for solving the pion-pion equations is evident. One chooses a value of λ and gets corresponding values for a_I from Eq. (II.15), neglecting the integral. By making some guess for $f_0{}^I$, one solves Eq. (II.3) to obtain $F_0{}^I$, which may then be used in Eq. (II.7) to give $\delta_0{}^I$. At this stage it is possible from Eq. (II.14) to calculate the part of $f_l{}^I$ coming from the S-wave terms under the integral. In particular, the $f_0{}^I$ so obtained may be used to correct the a_I through Eq. (II.15) and to solve (II.3) again in a better approximation. At the same time, $f_1{}^1$ may be used in Eq. (II.10) to give a first approximation to $F_1{}^1$ and through (II.11) to $\delta_1{}^1$.

At this point the cycle starts all over again with the formula (II.14), in which both S and P terms are kept, and the cycle is repeated until convergence is achieved. Higher partial waves are not considered until the end, at which time their phase shifts may be obtained from (II.12).

The variations in the above procedure are associated with the starting guess that is made. Clearly the rate of convergence should be faster the better the initial guess. We have adopted a program that may not give the fastest convergence, but has the advantage of being systematic and corresponds physically to the familiar notion of "turning on" the interaction adiabatically.

Note, first of all, that a solution for $\lambda=0$ is all phase shifts vanishing identically. For λ small the S phase shifts grow linearly with λ, but the quantities $g_0{}^I$ and $f_l{}^I$, being proportional to $\sin^2\delta_0{}^I$, are quadratic and therefore so are P and higher phase shifts. One may obtain the solution for small λ, therefore, by a power series or, alternatively, by a rapidly converging iteration using the first terms of the power series as a start. As λ becomes of the order of magnitude unity, the power-series approach breaks down, but at any value of λ the exact solution for a slightly smaller value may always be used as a starting guess. Even better, one may use an extrapolation based on all the lower values of λ for which solutions have been achieved.

In this way we "turn on" the pion-pion interaction slowly and develop the solution as a continuous function of λ. If λ is negative and sufficiently large in absolute value we get a bound S state by the adiabatic approach. Experimentally no bound state has been observed, so that, as explained in CM, there is a limit to the range of negative λ that needs to be studied. It was also explained in CM that λ cannot be too large and positive.

As pointed out in the introduction, the solutions of our equations obtained by this iteration procedure are probably not the only ones. There may exist other solutions even at values of λ for which consistent solutions were obtained by this method. In such a case, the solutions of the equations are not unique—or, to put it another way, the parameter λ is not a good way of specifying the solutions. Another possibility is that, for positive values of λ greater than those giving a consistent solution by iteration, a solution may still exist. In this paper we confine our attention to the simplest solutions, which can be reached by applying the procedure outlined above.

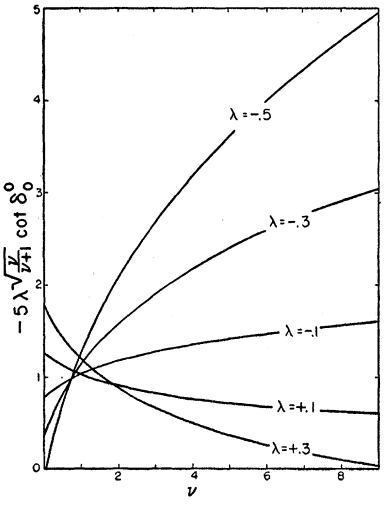

FIG. 1. The cotangent of $\delta_0{}^0$, multiplied by $-5\lambda[\nu/(\nu+1)]^{\frac{1}{2}}$, as a function of $\nu=q^2$, for various values of λ within the allowed range. As λ approaches zero the family of curves approaches a horizontal straight line with unit ordinate. Note that the function for $\lambda=-0.5$ is negative at $\nu=0$, indicating the existence of a bound state.

IV. CALCULATIONS AND RESULTS

In numerical calculations with the Livermore 704 Computer, each of the intervals 0 to 1 in the x and y variables was divided into 20 mesh points. The linear integral equations (II.3) and (II.10) were converted into simultaneous algebraic linear equations and solved by a standard matrix inversion. The most complicated operation otherwise was the evaluation of various one-dimensional integrals with finite limits and real smoothly varying integrands. The time required for one complete cycle of the equations was less than 1 minute and convergence was rapid, four or five cycles generally sufficing.

As expected, for small λ the S-wave phase shifts are roughly proportional to λ, while the P (as well as all higher) phase shifts are adequately given by the "Born approximation," (II.12), and are proportional to λ^2. What was not expected was that the P phase shift would remain small throughout the range of λ for which physically acceptable solutions were found. Our results are shown in Figs. 1, 2, and 3 where it may be seen that nowhere is the P phase shift larger than a few degrees. Phase shifts for $l\geqslant 2$ are so small as to be completely uninteresting.

The smallness of the higher phase shifts confirms the validity of the basic approximation in CM, which kept only S and P waves in calculating absorptive parts. In fact, keeping the imaginary part of the P wave is an unnecessary luxury for the type of solution found here. The contribution of g_1^1 to Eq. (II.14) is negligible.

The contribution of $g_0^{0,2}$ to Eq. (II.14) determines all phase shifts for $l \geqslant 1$, but for $l=0$ the main features of the solution are already given by CM-(V.22), which corresponds to setting f_0^I equal to zero. For example, the sign of the S phase shifts is opposite to that of λ both in the numerical solution and in CM-(V.22). The

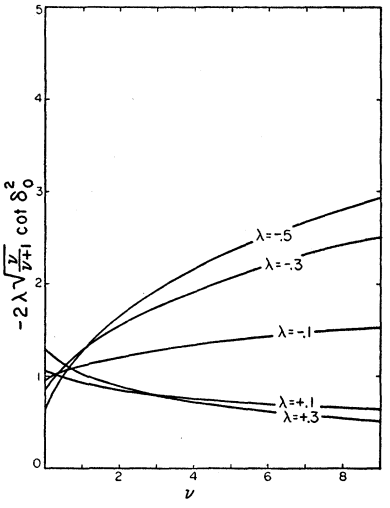

FIG. 2. The cotangent of δ_0^2, multiplied by $-2\lambda[\nu/(\nu+1)]^{\frac{1}{2}}$ for the same values of λ as shown in Fig. 1. The limit as λ approaches zero is the same as in Fig. 1.

crude approximation predicts the first bound $I=0$ state at $\lambda=-0.36$, while our complete numerical result gives the critical value, $\lambda=-0.46$. The order of magnitude of the upper limit on λ is also given correctly by the rough arguments of CM, neglecting f_0^I. In our numerical solution the spurious pole in the $I=0$ state moves past the arbitrarily chosen limit, $\nu=-10$, at $\lambda=0.34$, whereas the crude estimate yields $\lambda=0.50$.

From Eq. (II.14) it may be seen that the P-state force due to the exchange of an $I=0$ S-wave pion pair is attractive, whereas the force due to an $I=2$ S-wave pair is repulsive. For negative λ and even for small

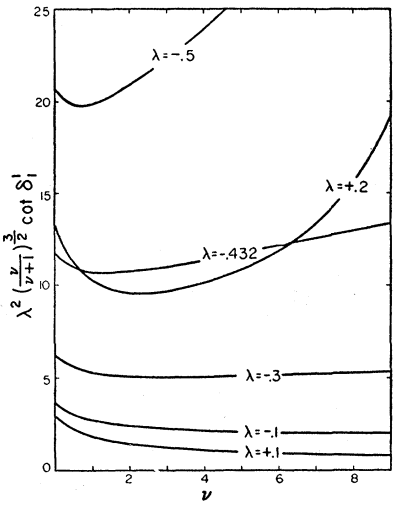

FIG. 3. The cotangent of δ_1^1, multiplied by $\lambda^2[\nu/(\nu+1)]^{\frac{3}{2}}$, for various values of λ. At $\lambda=+0.3$, a value not shown here, δ_1^1 has become negative but is still very small in absolute value.

positive λ, the $I=0$ exchanges turn out to be more important and the P phase shift is positive. For the larger positive λ range, the $I=2$ exchanges dominate and the P phase shift changes sign. The competition between these two forces helps explain the smallness of the P interaction; however, the limitation enforced by unitarity on the magnitude of the g_0^I seems to be the essential obstacle to a strong P interaction.

The only hope of getting a strong P-wave force from formula (II.14) appears to lie in making g_1^1 large and taking advantage of the large numerical factor multiplying this term. This would be a "bootstrap" mechanism; i.e., the force producing the P-wave resonance would be due to the exchange of a resonating P-wave pion pair. Such solutions of the CM equations appear entirely possible, even though they cannot be reached by the adiabatic approach; they will be dealt with in a subsequent paper.

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