the Foley theory since practically all line-broadening theories yield a nearly linear relation. The Fourier integral theory predicts, for interactions with $p=6$ [Eq. (57) of reference 5], that the difference in interaction energy constant for the ground state ^6S and the excited states ^6P of Mn in argon is 3×10^{-58} erg cm^6.

An extension of the theory of the fine structure pressure effect[6] is applied to this triplet. A rather elaborate theoretical calculation of the broadening of the three fine structure components has been presented as a separate paper.[7] The results lead to the relative values of half-widths or shifts of the lines in the ratio

$$\text{Mn } (j=\tfrac{3}{2}):(j=\tfrac{5}{2}):(j=\tfrac{7}{2})=0.36:0.36:0.34.$$

(Experimentally the ratio is 0.33:0.36:0.30.) Although not perfect, the agreement is quite good.

The difference in the behavior of the shift for argon and helium may be of theoretical significance. The lack of shift in Mn lines broadened by helium is evidently very fascinating. According to Foley there should be a shift related in a definite way to the width. However, subsequent theoretical developments have shown that the shift is generally smaller than that predicted by adiabatic theories of the Foley type. The effect of collision-induced (real or virtual) nonadiabatic transitions is to cause interferences which can result in a very small shift even though the broadening is appreciable. Under the present experimental conditions the collision-induced transitions between discrete states were not likely to occur.

One should bear in mind that the theoretical assumptions for the analysis of pressure effects produced by light gases are somewhat open to question. The interaction force between He and Mn could be of the type $1/r^p$, p being a large value. Consequently, the short-range force may give a very small shift. Another cause may be the velocity effect. The important force is probably repulsive. The motion of the He atom passing by the Mn atom will probably be slowed down at close collisions but will not be appreciably affected if the collision takes place at large parameters.

[6] M. Takeo and S. Y. Ch'en, Phys. Rev. **93**, 420 (1954).
[7] M. Takeo, (to be published).

PHYSICAL REVIEW VOLUME 119, NUMBER 3 AUGUST 1, 1960

Nonlocal Optical Model for Nucleon-Nuclear Interactions*†

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An attempt is made to achieve a unified potential description of the gross structure of the nucleon-nuclear interactions in bound states and in states of scattering. A model is employed with a nonlocal complex diffuse potential with spin-orbit coupling and surface absorption. This represents a relatively simple nonlocal generalization of the usual static models which might reasonably be expected to describe the nucleon-nuclear interaction in the low-energy range (say from −25 Mev to 25 Mev). Choosing the range of the nonlocal forces as suggested by considerations of the properties of infinite nuclear matter, the real parameters are fixed largely on the basis of neutron and proton separation energies. Two absorption parameters are then adjusted to provide agreement with total reaction and differential elastic cross-section data for neutrons. It is found that the successes of local optical models with energy-dependent parameters are largely preserved. Contrary to expectations, it is found that nonlocality tends to accentuate rather than wash out diffraction patterns. Although a diverse variety of experimental phenomena are treated, a range of parameter choices remains. Because of theoretical uncertainties as to the size of the "rearrangement energy," an effort is made to establish limits as to its magnitude on phenomenological grounds. The influence of several choices upon the physical phenomena used in adjusting the parameters of this model are shown. It would appear that this study does allow for a rearrangement energy but that it is rather small (<6 Mev) and comparable to the probable fluctuations of the potential from element to element.

I. INTRODUCTION

IN recent years, considerable success has been achieved in accounting for the gross features of neutron and proton cross sections with the aid of a complex potential well model. During the same period,

* Portions of this work were carried out at the Los Alamos Scientific Laboratory and Florida State University with the support of the U. S. Atomic Energy Commission. Some of the final calculations were carried out at Eglin Air Force Proving Grounds through kind arrangements made by the Air Force Office of Scientific Research.

† This work is based in part on a dissertation (by P. J. Wyatt) submitted to the Graduate School of the Florida State University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

comparable success has been achieved in parallel studies in accounting for neutron and proton binding energies with the aid of a similar model with a real potential. The close similarity of the potentials used in these studies have suggested the possibility that perhaps a single potential well description (at least for the real part) might serve to handle both negative and positive energy states of nucleons interacting with nuclei.

It is clear, however, that a local or static potential could not provide a complete description. Many studies, directed toward accounting for the saturation properties of nuclei by the use of exchange forces, have long implied that the nucleon-nuclear potential would be velocity dependent, or nonlocal. Recent intensive efforts directed toward utilizing our current detailed knowledge of the two-body forces by Brueckner,[1] Bethe,[2] and others[3] have reinforced this earlier view and at the same time are now leading to predictions concerning nuclear matter properties; such as, total energy per particle, density, compressibility, etc., in reasonable accord with experimental information. From a phenomenological standpoint, recent studies[4–8] of the optical model, over broad energy ranges, also point to the dependence of the nuclear potentials upon energy or velocity.

In this paper, an attempt is made to unify the description of the positive and negative energy states of nuclei in the region of concern in classical nuclear physics (± 25 Mev) by the use of the Schrödinger equation which has been generalized to the form

$$\frac{\hbar^2}{2\mu}[\nabla^2+E]\psi(\mathbf{r})=\int K(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'. \tag{1}$$

A kernel, $K(\mathbf{r},\mathbf{r}')$ is chosen which is characterized in a relatively simple way, and for convenience of computation the "effective mass approximation"[9] is used. The consequences of Eq. (1) for the bound states of nucleons in diffuse realistic potential wells have already been examined.[10] The extension of this earlier work to include scattering necessarily requires that $K(\mathbf{r},\mathbf{r}')$ be complex. On the basis of the following three assumptions, the unification of the bound states and scattering is attempted.

(1) The imaginary part of the optical potential is essentially discontinuous at zero energy (see Sec. IV).

(2) All other nuclear quantities are continuous at zero energy.

(3) The primary energy dependence of the nuclear potential is embodied in its nonlocality.

Several calculations are made which are aimed at elucidating the "rearrangement energy" question.

II. THE PRESENT DESCRIPTION

In the bound-state investigations it is assumed that the potential matrix, $K(\mathbf{r},\mathbf{r}')$ can be written as follows: $K(\mathbf{r},\mathbf{r}')=V[(\mathbf{r}+\mathbf{r}')/2]\delta_a(\mathbf{r}-\mathbf{r}')$. Here $\delta_a(\mathbf{r}-\mathbf{r}')$ is a sharply peaked even function of its argument with "range" a ($\simeq$range of two-body forces) and whose integral properties are described by

$$\int\delta_0(\varrho)a^3d\varrho=1, \quad \int\rho^2\delta_0(\varrho)a^3d\varrho=\tfrac{3}{2}. \tag{2}$$

In the limit of zero range, $\delta_a(\mathbf{r}-\mathbf{r}')$ behaves as the Dirac delta function and Eq. (1) immediately reduces to the usual Schrödinger equation.

If $ka\ll 1$, then (neglecting terms of fourth order and higher) Eq. (1) may be reduced to[9]:

$$-\frac{\hbar^2}{8}\left[\nabla^2\frac{1}{M(\mathbf{r})}+\nabla\frac{2}{M(\mathbf{r})}\nabla+\frac{1}{M(\mathbf{r})}\nabla^2\right]\psi(\mathbf{r})+V(\mathbf{r})\psi(\mathbf{r})=E\psi(\mathbf{r}), \tag{3}$$

where $\mu=mA/(A+1)$, and

$$M(\mathbf{r})=\frac{\mu}{1-[a^2\mu/\hbar^2]V(\mathbf{r})}=\frac{\mu}{1+[a^2\mu V^*/2\hbar^2]\xi(r)} \tag{4}$$

is the so-called "effective mass." $V(\mathbf{r})$ now corresponds to the usual static potential which, in this paper, is taken to be spherically symmetric, i.e., $V(\mathbf{r})=V(r)=-V^*\xi(r)$ where $\xi(r)$ is a conveniently chosen structure function. Incorporating a spin-orbit term of the Thomas type,[11] one has in the center-of-mass system:

$$-\frac{\hbar^2}{8}\left[\nabla^2\frac{1}{M(r)}+\nabla\frac{2}{M(r)}\nabla+\frac{1}{M(r)}\nabla^2\right]\psi(\mathbf{r})-V^*\xi(r)\psi(\mathbf{r})+a_{so}{}^2V^*\frac{1}{r}\frac{d\xi}{dr}\frac{\mathbf{l}\cdot\mathbf{S}}{\hbar^2}\psi(\mathbf{r})=E\psi(\mathbf{r}). \tag{5}$$

This equation is readily separated in spherical coordinates to yield the following radial equation:

[1] K. A. Brueckner, Phys. Rev. **103**, 1121 (1956).
[2] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).
[3] K. A. Brueckner, C. A. Levinson, and H. M. Mahmoud, Phys. Rev. **95**, 217 (1954).
[4] *Proceedings of the International Conference on the Nuclear Optical Model, Florida State University Studies Number 32*, edited by A. E. S. Green, C. E. Porter, and D. S. Saxon (Florida State University, Tallahassee, 1959).
[5] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).
[6] J. R. Beyster, M. Walt, and E. W. Salmi, Phys. Rev. **104**, 1319 (1956).
[7] S. Fernbach, Revs. Modern Phys. **30**, 414 (1958).
[8] H. Feshbach, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, California, 1958), Vol. 8, p. 49.
[9] W. E. Frahn and R. H. Lemmer, Nuovo cimento **5**, 1564 (1957); **6**, 664 (1957).
[10] A. E. S. Green and P. C. Sood, Phys. Rev. **111**, 1147 (1958).
[11] F. E. Bjorklund and S. Fernbach, Phys. Rev. **109**, 1295 (1958). Despite the slight energy dependence associated with the spin-orbit potential by Bjorklund and Fernbach, the size of the resultant correction terms, were the spin-orbit contribution included in the nonlocal kernel, is of such a small magnitude that their consideration can safely be postponed for a later study.

$$\frac{d^2G_l}{dr^2}-\frac{1}{M}\frac{dM}{dr}\frac{dG_l}{dr}+\left[\frac{1}{2M^2}\left(\frac{dM}{dr}\right)^2-\frac{1}{4M}\frac{d^2M}{dr^2}+\frac{1}{2rM}\frac{dM}{dr}+\frac{2M}{\hbar^2}(V^*\xi+E)-\frac{2MV^*}{\hbar^2}a_{so}{}^2\frac{\mathfrak{L}}{r}\frac{d\xi}{dr}-\frac{l(l+1)}{r^2}\right]G_l=0, \quad (6)$$

where

$$\psi(\mathbf{r})=\sum_l F_l(\theta,\phi)G_l(r)/r,$$

and

$$\mathfrak{L}=l/2 \quad \text{for} \quad j=l+\tfrac{1}{2}$$
$$=-(l+1)/2 \quad \text{for} \quad j=l-\tfrac{1}{2}.$$

Formulation of the Scattering Problem

The transition from bound states to scattering via the nonlocal formalism might at first suggest the simple replacement of $V(r)$ by $V(r)+iW(r)$ in Eq. (1). This would result in "complex" effective mass terms of the type:

$$M(r)=\frac{\mu}{1-[a^2\mu/2\hbar^2][V(r)+iW(r)]}. \quad (7)$$

Bjorklund[4] and others have found that an imaginary spin-orbit term is not necessary to fit the scattering data for incident energies up to about 50 Mev, although above this energy such a term seems to provide improved fits. As the present investigation is concerned with low energies (<25 Mev), the replacement by $V(r)+iW(r)$ is taken in all terms in Eq. (6) except the spin-orbit term which is kept real. For an attractive potential [$V(r)<0$, $W(r)<0$ for all r] the resulting $iW(r)G(r)$ term would be a source of the usual "local" absorption in the customary optical model description where $a=0$. The additional terms, however, would be due entirely to the nonlocality. It is not difficult to show that the additional imaginary terms arising from nonlocality essentially would produce *emission*. In fact, it may be seen from the continuity equation[12] that the nonlocal imaginary terms would at best *diminish* the absorption with increasing energy which would contradict the observation that the absorption increases up to about 100 Mev.[13]

In an effort to simulate the known variation of absorption with energy, a reduced mass of the form

$$M(r)=\mu/\{1+(\mu V^*g^2/2\hbar^2)[\xi(r)+i\zeta_1\eta(r)]\}, \quad (8)$$

has been chosen but the local refractive and absorptive term has been taken as $-V^*[\xi(r)+i\zeta_0\eta(r)]$ where ζ_0 and ζ_1 are two free parameters introduced in scattering, $\eta(r)$ is a form function associated with the absorption term and g is a "generalized" nonlocality parameter replacing a. This introduction of two free parameters can be associated with the possibility that the true kernel $K(\mathbf{r},\mathbf{r}')$ might actually represent a sum of complex kernels with different ranges of nonlocality.[12,14]

III. THE PHENOMENOLOGICAL ANALYSIS

It is now possible to examine the specific consequences of the formalism established in Sec. II. Prior to this, however, the form factors $\xi(r)$ and $\eta(r)$ must be considered as well as the complete set of parameters characterizing the nucleus.

From the earlier phenomenological analysis of proton and neutron bound states and scattering,[4,7,15–17] as well as from the Stanford electron scattering experiments,[18] it is clear that the form of the real part of the nuclear potential is probably best described as having an approximately uniform interior region and a diffuse surface which falls off rapidly within 2 to 4 fermis beyond the rms radius. A great variety[17,19–23] of potential shapes satisfying these criteria have been used in the past. For computational simplicity as well as because of its continuity properties, the following form was used in the present work:

$$\xi(r)=1, \quad r<a$$
$$=\frac{1}{2}-\frac{15}{16}Z+\frac{10}{16}Z^3-\frac{3}{10}Z^5, \quad a<r<b \quad (9)$$
$$=0, \quad r>b,$$

where

$$Z=[r-\tfrac{1}{2}(a+b)]/\tfrac{1}{2}(b-a).$$

The surface limits a and b may be expressed in terms of more familiar constants: the surface thickness, T, and the half-falloff radius R by the simple relations

$$a=R-0.99868T\simeq R-T,$$
and (10)
$$b=R+0.99868T\simeq R+T.$$

It should be noticed that $\xi(r)$, $\xi'(r)$, and $\xi''(r)$ are continuous for all values of r subject to the above definitions. Letting $T=2d\ \ln 9$ an almost perfect match may be obtained to the form factor which is given by

[12] Y. C. Tang, R. H. Lemmer, P. J. Wyatt, and A. E. S. Green, Phys. Rev. **116**, 402 (1959).

[13] See, e.g., curves shown on page 422 of A. E. Glassgold, Revs. Modern Phys. **30**, 419 (1958).

[14] P. J. Wyatt, Ph.D. dissertation, Florida State University, 1959 (unpublished). Also, private communications from Y. C. Tang and R. H. Lemmer.

[15] A. Green, Bull. Am. Phys. Soc. **2**, 25 (1957).

[16] A. E. S. Green, Phys. Rev. **99**, 1410 (1955).

[17] P. C. Sood, Ph.D. dissertation, Florida State University, 1958 (unpublished).

[18] R. Hofstadter, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, California, 1957), Vol. 7, p. 231.

[19] A. E. S. Green and Kuick Lee, Phys. Rev. **99**, 772 (1955).

[20] A. A. Ross, H. Mark, and R. D. Lawson, Phys. Rev. **102**, 1613 (1956).

[21] A. Schroeder, Nuovo cimento **7**, 461 (1958).

[22] J. Riese, Ph.D. thesis, Massachusetts Institute of Technology, 1958 (unpublished).

[23] A. V. Lukyanov, Y. V. Orlov, and V. V. Turovtsev, Nuclear Phys. **8**, 325 (1958).

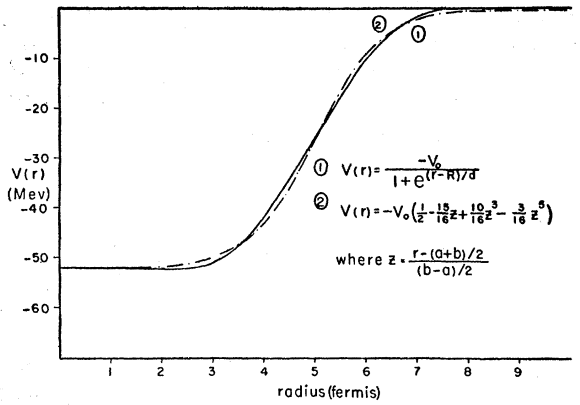

FIG. 1. Comparison of the present well shape (solid line) with that of Woods and Saxon (dashed line) for a hypothetical nucleus. Note the slightly greater slope in the surface region for the latter model.

(see Fig. 1)

$$\xi(r)=\frac{1}{\{1+\exp[(r-R)/d]\}}. \tag{11}$$

Recent theoretical investigations[24,25] as well as the significant improvements of the Bjorklund-Fernbach[11] fits suggest that the imaginary part of the optical potential (for incident energies up to 25 Mev) should be peaked in the surface region. Although the Gaussian form factor used by Bjorklund and Fernbach is probably a reasonable approximation in this energy region, rather than introduce a new form factor (and with it additional parameters) it is possible to provide a similar surface absorption simply by letting

$$\eta(r)=-d\xi/dr. \tag{12}$$

The concept of a surface derivative absorption is not entirely new and has indeed been tried by others[26,27] in the past. There are, however, several interesting features of the present form. In view of the continuity of $\xi(r)$ and its derivatives, $\eta(r)$ and $\eta'(r)$ are also continuous for all r. The desire for computational simplicity as well as functional continuity stems from the formidable differential equations to be solved as well as the higher derivatives of the form factors which occur frequently due to the mass term derivatives [see Eq. (6)].

For the cases of proton bound states and scattering, a Coulomb potential must be included in Eq. (6). In the present investigation a Coulomb potential of the Ford-Hill[28] family II type with reasonable parameters was chosen. The Coulomb half-falloff radius was chosen on the average to be about 0.5 fermi less than the nuclear half-falloff radius, and the falloff parameter $n=5$ was accepted. Although both neutron and proton bound states were investigated, the extension to scattering has been completed only for neutrons at this time.

Characteristic Nuclear Parameters

In view of the observed proportionality between $A^{\frac{1}{3}}$ and the nuclear rms radius, R is usually described by $R=r_0A^{\frac{1}{3}}+r_1$ where r_0 and r_1 are constants. The work of Green and Sood[10,17] concerning the proton potential anomaly, as well as the work of Nemirovsky,[29] indicates that the central well depth should be dependent upon the neutron excess, i.e.,

$$V^*=V_0\pm[(N-Z)/A]V_1,$$

where the positive sign refers to the proton case, the negative one to neutrons,[30] and A, Z, and N refer to mass, proton, and neutron number, respectively. The remaining parameters T, a_{so}^2, ζ_0, and ζ_1 were assumed for the most part to be A independent. Departures from this may indeed occur and are discussed in the final section.

Methods of Solution

The present investigation has been substantially dependent upon the results obtained with two codes constructed for the IBM 704. The bound-state code determines the eigenvalues and the derivatives of the eigenvalues with respect to the various parameters for a given set of input parameters. The scattering code extracts the complex transmission coefficient from which the various cross sections are calculated following the usual optical model analysis[5] with the necessary modifications due to the spin-orbit and nonlocality terms.

Adjustment of Parameters

Because of the two additional parameters occurring in the scattering formulation (ζ_0,ζ_1) the problem of finding a complete set of suitable parameters was approached from the bound-state solutions. Thus once a set of parameters V_0, V_1, r_0, r_1, a_{so}^2, T, and g^2 were found which would produce satisfactory last-particle binding energies[31,32] these parameters were fixed and the scattering predictions for various combinations of ζ_0 and ζ_1 were examined.

It should be evident from Table I that a wide range of possibilities existed, each set producing nearly equivalent bound-state fits. The procedure of finding a "good" set of bound-state parameters rested almost entirely on the use of the eigenvalue derivatives in

[24] K. Harada and N. Oda, Progr. Theoret. Phys. (Kyoto) **21**, 260 (1959).
[25] H. J. Amster, Phys. Rev. **113**, 911 (1959).
[26] W. S. Emmerich, Westinghouse Research Laboratory Report 60-94511-6R17 (unpublished).
[27] W. S. Emmerich and N. J. Amster, Physics **22**, 1163 (1956).
[28] D. L. Hill and K. W. Ford, Phys. Rev. **94**, 1617 (1954).
[29] P. Nemirovsky, preprint (kindly supplied by Professor H. H. Barshall).
[30] Reference 4 contains many interesting remarks and comments on the subject.
[31] A. H. Wapstra, Physics **21**, 367 (1956).
[32] J. R. Huizenger, Physics **21**, 410 (1956).

conjunction with a least squares fitting routine involving an overdetermined set of conditions (i.e., the last particle separation energies of 14 nuclei). The scattering predictions, however, irrespective of the values of ζ_0 and ζ_1, showed marked differences. The scattering quantities compared with experiment consisted of the total, reaction, and differential elastic cross sections.

By virtue of the success of other investigations (see, e.g., references 4 and 11) using the simple radius characterization, $R=r_0A^{\frac{1}{3}}$, it was decided to put $r_1=0$; the additional parameter (or degree of freedom) appearing unnecessary at this time. If all the remaining parameters were taken as free, the usual ambiguity $Vr_0^2\simeq$constant[4,33] arose. It was observed that for any r_0 an equivalently suitable set of parameters could be found. It was found also that near $r_0=1.25$ if g^2 (the nonlocality) were allowed to vary, the least squares procedure favored $g^2=0$; i.e., the local case. However, the scattering predictions were then very poor.

TABLE I. Comparison of theoretical and experimental last particle eigenvalues for various sets of nuclear parameters. The experimental values are from Wapstra[a] and Huizenger.[b]

Set	V_0	V_1	r_0	T	g^2	a_{so}^2
1	73	26.0	1.25	2.85	1.00	0.70
2	65	36.8	1.25	3.12	0.65	0.73
3	52	26.0	1.25	2.85	0.00	0.70
4	66	26.0	1.20	2.85	0.50	0.70
5	61.5	26.0	1.25	2.85	0.50	0.70
6	69.3	21.2	1.20	2.55	0.65	0.66

Isotope	State	E_{exp} (Mev)	Theoretical eigenvalues (Mev) Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
O^{17}	$d_{5/2}$	4.14	5.43	5.52	4.68	5.21	5.13	4.92
F^{17}	$d_{5/2}$	0.60	2.00	2.04	1.01	1.56	1.57	1.26
Si^{31}	$d_{3/2}$	6.60	6.72	6.55	6.90	6.44	6.94	6.65
Cl^{33}	$d_{3/2}$	2.50	2.68	2.65	2.67	2.11	2.79	2.30
Ca^{41}	$f_{7/2}$	8.37	8.14	8.34	8.44	8.43	8.39	8.29
Sc^{41}	$f_{7/2}$	1.83	1.53	1.62	1.36	1.37	1.55	1.26
Zn^{67}	$f_{5/2}$	7.00	8.23	8.01	9.36	8.42	9.03	8.90
Ge^{73}	$g_{9/2}$	6.64	6.79	6.45	7.25	7.19	7.21	7.35
Rb^{85}	$f_{5/2}$	6.83	7.14	7.81	8.17	7.01	7.81	7.28
Nb^{93}	$g_{9/2}$	6.24	4.91	5.85	5.76	5.20	5.44	4.93
Sn^{115}	$s_{1/2}$	7.91	6.52	7.43	8.74	7.98	7.96	7.20
Xe^{137}	$f_{7/2}$	3.57	···	2.74	4.35	3.44	3.56	2.72
Pb^{207}	$p_{1/2}$	6.73	6.05	6.52	8.23	7.51	7.57	7.01
Bi^{209}	$h_{9/2}$	3.72	···	1.81	3.28	0.47	1.80	0.15

[a] See reference 31.
[b] See reference 32.

The final adjustment of the parameters was therefore accomplished as follows: (1) g^2 was fixed to provide approximate agreement with the range of nuclear forces.[34] This choice, together with the potential well depth which it subsequently implied, in turn produced an effective mass at the nuclear center compatible with studies of the properties of infinite nuclear matter. (See, e.g., Table IV in footnote reference 17 for a summary of effective mass values appearing in the literature.) (2) r_0 was fixed at a value between 1.15 and 1.30. (3) The remaining parameters were adjusted via the least squares technique (both neutron and proton states were examined but the scattering calculations were applied only to neutrons). (4) With all the parameters now fixed, the scattering predictions were examined for a large range of ζ_0 and ζ_1. If the fits were poor, r_0 was adjusted to another value and steps 3 and 4 repeated. (5) Once a set of fixed parameters V_0, V_1, r_0, a_{so}^2, T (and ζ_0 and ζ_1 for scattering) were established, the level structures of several isotopes were examined in detail. If the structures produced were in good agreement with the expected shell structure, then, subject to the previous bound-state and scattering calculations, the set was considered to be consistent.

[33] A. E. Glassgold, W. B. Cheston, M. L. Stein, S. B. Schuldt, and G. W. Erickson, Phys. Rev. **106**, 1207 (1957).
[34] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. phys. **3**, 241 (1958).

TABLE II. Parameters constituting our best basis for unification of bound states and scattering.

Parameter	Best value	Probable variation
V_0 (Mev)	70	±3
V_1 (Mev)	22	±6
r_0 (fermi)	1.20	±0.02
T (fermi)	2.85	±0.40
a_{so}^2 (fermi^2)	$\frac{2}{3}$	±0.05
g^2 (fermi^2)	$\frac{2}{3}$	±0.03
ζ_0	0.2	±0.1
ζ_1	−0.7	±0.3

IV. RESULTS, DISCUSSION, AND CONCLUSION

As was evident from Table I, the bound states alone are insufficient to determine a unique set of parameters, especially in the nonlocal description. By introducing positive energy states into the problem, one can narrow down the possibilities substantially. A description of the myriad of machine and hand calculations pertinent to the selection of the best set of parameters has been omitted from the present paper as their inclusion would be of little value or interest. Each result was either accepted or rejected, not on its own specific

TABLE III. Comparison of last-particle eigenvalues (in Mev) based on parameter values of Table II. The experimental values are from Wapstra[a] and Huizenger.[b] The last column indicates the theoretical values of Nemirovsky.[c]

Isotope	State	E_{expt}	E_{theory}	$E_{P.N.}$
O^{17}	$d_{5/2}$	4.14	4.81	5.40
F^{17}	$d_{5/2}$	0.60	1.65	
Si^{31}	$d_{3/2}$	6.60	6.21	
Cl^{33}	$d_{3/2}$	2.50	1.12	
Ca^{41}	$f_{7/2}$	8.37	7.64	8.53
Sc^{41}	$f_{7/2}$	1.83	0.95	
Zn^{67}	$f_{5/2}$	7.00	8.16	
Ge^{73}	$g_{9/2}$	6.64	6.58	7.05
Rb^{85}	$f_{5/2}$	6.83	7.09	
Nb^{93}	$g_{9/2}$	6.24	3.97	
Sn^{115}	$s_{1/2}$	7.91	8.01	9.05
Xe^{137}	$f_{7/2}$	3.57	2.94	4.32
Pb^{207}	$p_{1/2}$	6.73	7.07	
Bi^{209}	$h_{9/2}$	3.72	0.31	

[a] See reference 31.
[b] See reference 32.
[c] See reference 29.

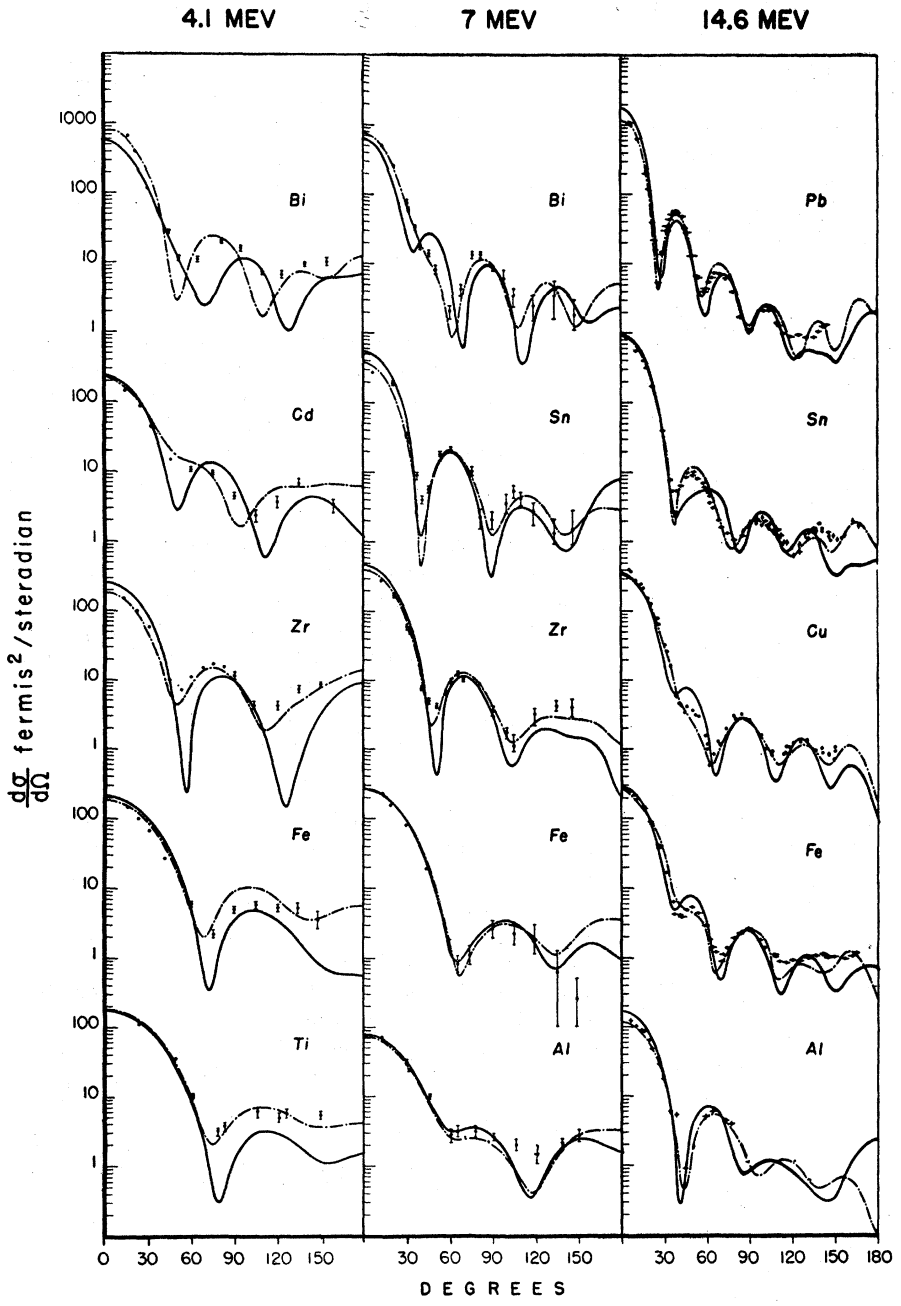

FIG. 2. Representative neutron differential scattering cross sections based on the best nuclear parameters indicated by the present preliminary work (Tab II). The dashed curves represent the "local" predictions of Bjorklund and Fernbach.

merits, but rather on the basis of the manner in which it fitted into the over-all picture, i.e., the consistency of bound-state and scattering results.

Numerical Results

Table II shows the set of parameters considered at this time to be a best basis for the unification of the bound states and scattering within the framework of the formalism presented in Sec. II. The probable range of values (based on the analysis) is also indicated. Table III shows the comparison of the eigenvalue results with experiment and some of Nemirovsky's[29] calculations. It is interesting to note that the values of V_1, obtained in this study for both local and nonlocal cases, are in good agreement with those inferred by Fulmer[35] in recent experiments concerned with proton scattering at 9.5 Mev. Figure 2 shows the neutron scattering results based on these parameters (solid curves). The dashed curves represent the local fits of Bjorklund and Fernbach.[4,7] The sources of the data are indicated in reference.[7]

[35] C. B. Fulmer (private communication).

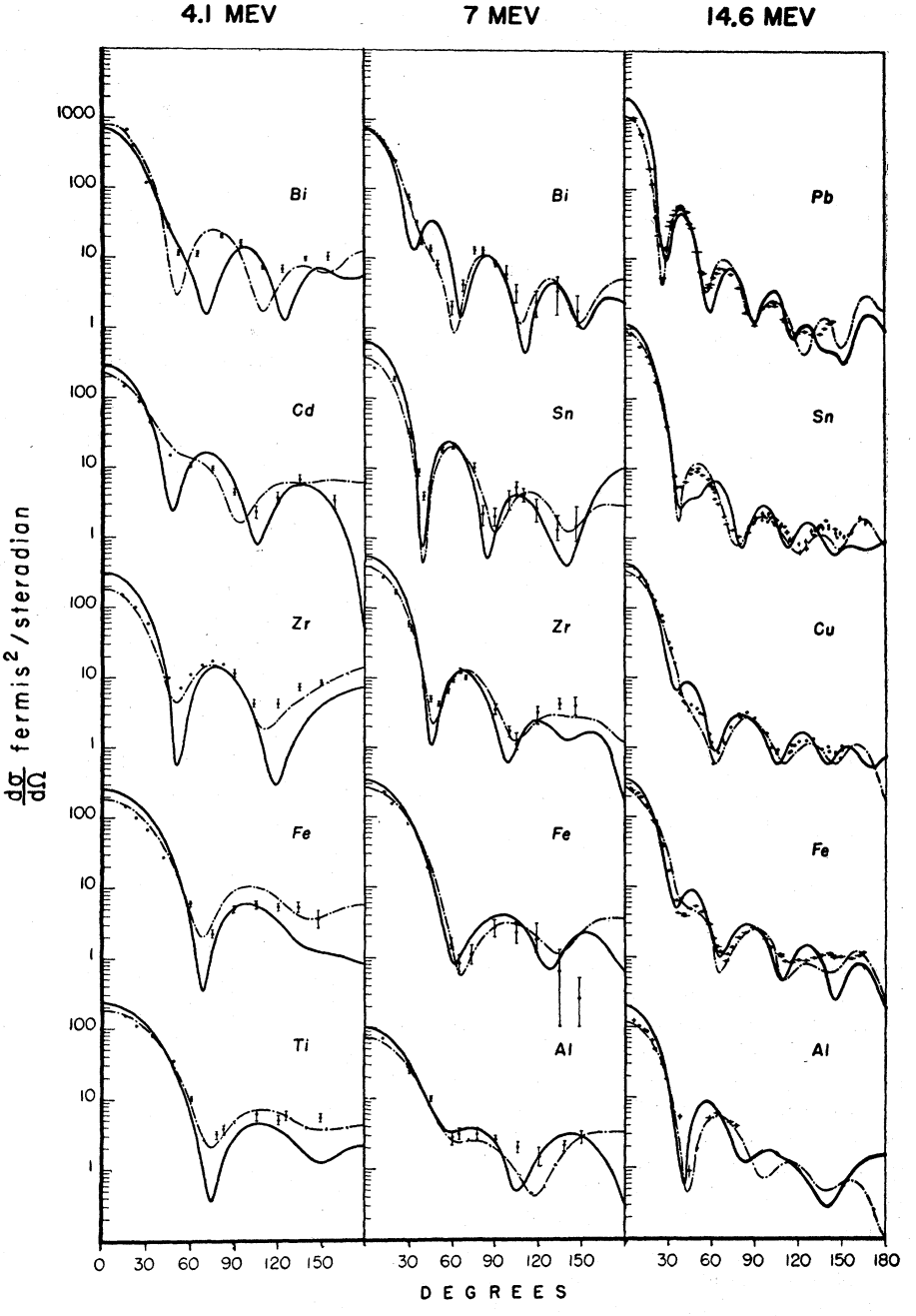

FIG. 3. Differential scattering cross sections illustrating the effect of a larger radius constant (r_0) also consistent with bound-state data. Note the higher values at small scattering angles.

Figure 3 is similar to Fig. 2, but illustrates the results of Set 2 of Table I. Note the effect of the larger radius constant ($r_0=1.25$ fermis) is to increase the scattering in the forward direction. The other differences (in parameters g^2, V_1, a_{so}^2, and T) have been found to produce no noticeable effects on the forward scattering.

Table IV presents the total and reaction cross section predictions compared with experiment[36] for various elements at 4.1, 7 and 14.6 Mev and are based on the parameters in Table II.

Figure 4 illustrates the bound state energy levels for neutrons based on the parameters of Table II. The Hf nucleus is in reality, of course, deformed. It was hypothetically chosen as spherical, however, in order to show the characteristic shell structure predicted by the present model.

[36] *Neutron Cross Sections*, compiled by D. J. Hughes and J. A. Harvey, Brookhaven National Laboratory Report BNL-325 (Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1955).

TABLE IV. Comparison of experimental and theoretical values of neutron total and reaction cross sections.

Element	Energy (Mev)	Cross sections (barns) σ_T (theory)	σ_T (expt)	σ_R (theory)	σ_R (expt)
Al	7.0	2.01	2.0	1.14	1.0
	14.6	1.92	1.7	1.04	1.0
Ti	4.1	3.67	3.4	1.51	1.3
Fe	4.1	3.89	3.7	1.56	1.5
	7.0	3.43	3.6	1.50	1.5
	14.6	2.50	2.5	1.41	1.4
Cu	14.6	2.71	2.9	1.51	1.5
Zr	4.1	4.17	4.0	1.94	1.6
	7.0	4.40	4.2	1.82	1.7
Ag	4.1	4.20	4.0	2.04	2.1
	7.0	4.64	4.2	1.99	2.0
	14.5	4.08	4.3	1.90	1.9
Cd	4.1	4.24	4.1	2.06	2.1
Sn	7.0	4.71	4.2	2.05	2.1
	14.6	4.37	4.4	1.99	2.0
Ta	4.1	5.77	6.0	2.51	···
	7.0	5.03	5.0	2.48	2.5
	14.5	5.69	5.4	2.40	2.0
Au	4.1	6.25	6.7	2.55	2.8
	7.0	5.18	5.5	2.54	2.5
	14.5	5.89	5.5	2.48	2.5
Pb	14.6	5.99	5.7	2.53	2.5
Bi	4.1	6.58	7.9	2.61	2.0
	7.0	5.33	5.6	2.60	2.5

Possibility of a Rearrangement Energy

A tacit assumption throughout much of the previous bound-state work of Green et al. and should be mentioned at this time.[37] This assumption concerns the association of "last-particle separation energies" with "energy eigenvalues." A considerable amount of controversy exists in regard to the "rearrangement energy" which may occur when a nucleon is removed from a nucleus. Thus, conceivably, the energy eigenvalue of a last nucleon may be greater than its experimentally observed separation energy; the difference between the two quantities being ascribed to the so-called rearrangement energy. The rearrangement energy would thus assist in the removal of a particle once such a *gedanken* process has started. Some maintain[38] that the two quantities differ by a "rearrangement" energy of 10 to 25 Mev while others[4,39] consider them essentially equal. Experiments by Schiffer[40] and Maris[41] et al. point to the approximate equality of the two, while calculations by Mittelstaedt[42] et al. suggest the former view.

The results discussed thus far indicate that a fair description of nucleon scattering and bound states can be made without recourse to the rearrangement energy concept. However, in view of some of the shortcomings in the fits shown in Fig. 2, it was considered desirable to examine in detail the consequences of such an hypothesis.

[37] See also pertinent remarks in reference 4.
[38] See comments by Brueckner and Weisskopf in reference 4.
[39] A. E. S. Green, Phys. Rev. **104**, 1617 (1956).
[40] J. P. Schiffer, L. L. Lee, Jr., and B. Zeidman, Phys. Rev. **115**, 427 (1959).
[41] See talk and references by Maris in reference 4.
[42] P. Mittelstaedt, Nuclear Phys. **9**, 116 (1958).

The derivative of the last-particle eigenvalue with respect to the parameter V_0 for the isotopes in Table III is on the average about 0.45. Thus by deepening the wells by 10 Mev, the last-particle eigenvalues would be augmented by about 4.5 Mev. The rearrangement energy referred to in Sec. II is believed (by its proponents) to be 10 to 20 Mev. Thus the well studied here would have to be deepened by 22 to 44 Mev to produce so large an effect.

To test the rearrangement energy hypothesis, V_0 was increased by 10 Mev. This amounted to increasing the magnitude of the last-particle eigenvalues on the average by about 4.5 Mev. The scattering results are shown in Fig. 5, and Table V compares the predicted total and reaction cross sections with experiment.[36] As is immediately evident, some of the results are considerably improved (e.g., Bi) while others become noticeably worse (e.g., Fe). It is interesting to note that a similar augmentation of the V_0 in Set 2 of Table I, produced consistently worse results. (This case, corresponding to $r_0=1.25$, was reported in reference 4.) Were V_0 increased by an amount sufficient to produce a rearrangement energy of 10 to 20 Mev, it is evident from Fig. 5 that *all* the diffraction patterns would be shifted too far towards the smaller angles to produce any reasonable fits.

The results of the rearrangement energy investigation may therefore be summarized briefly as follows: (i) A rearrangement energy *may* well exist and (ii) if so,

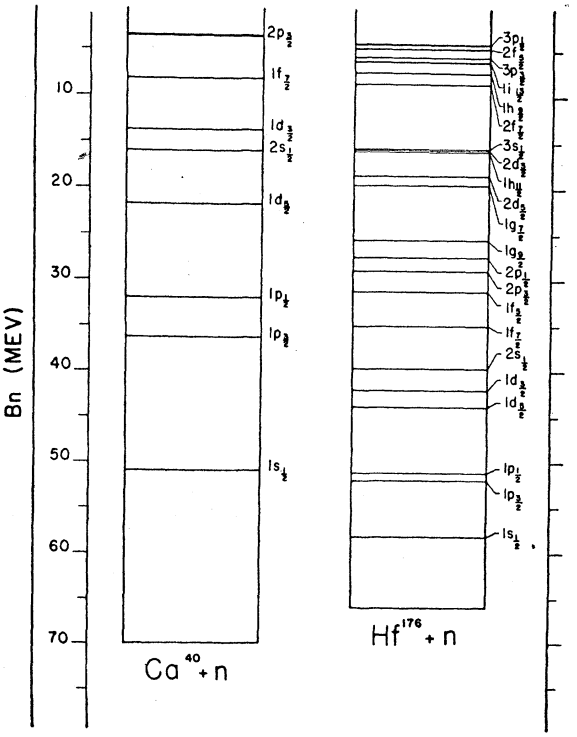

FIG. 4. Neutron bound states in Ca^{41} and the hypothetically spherical nucleus Hf^{177} based on the parameters of Table II.

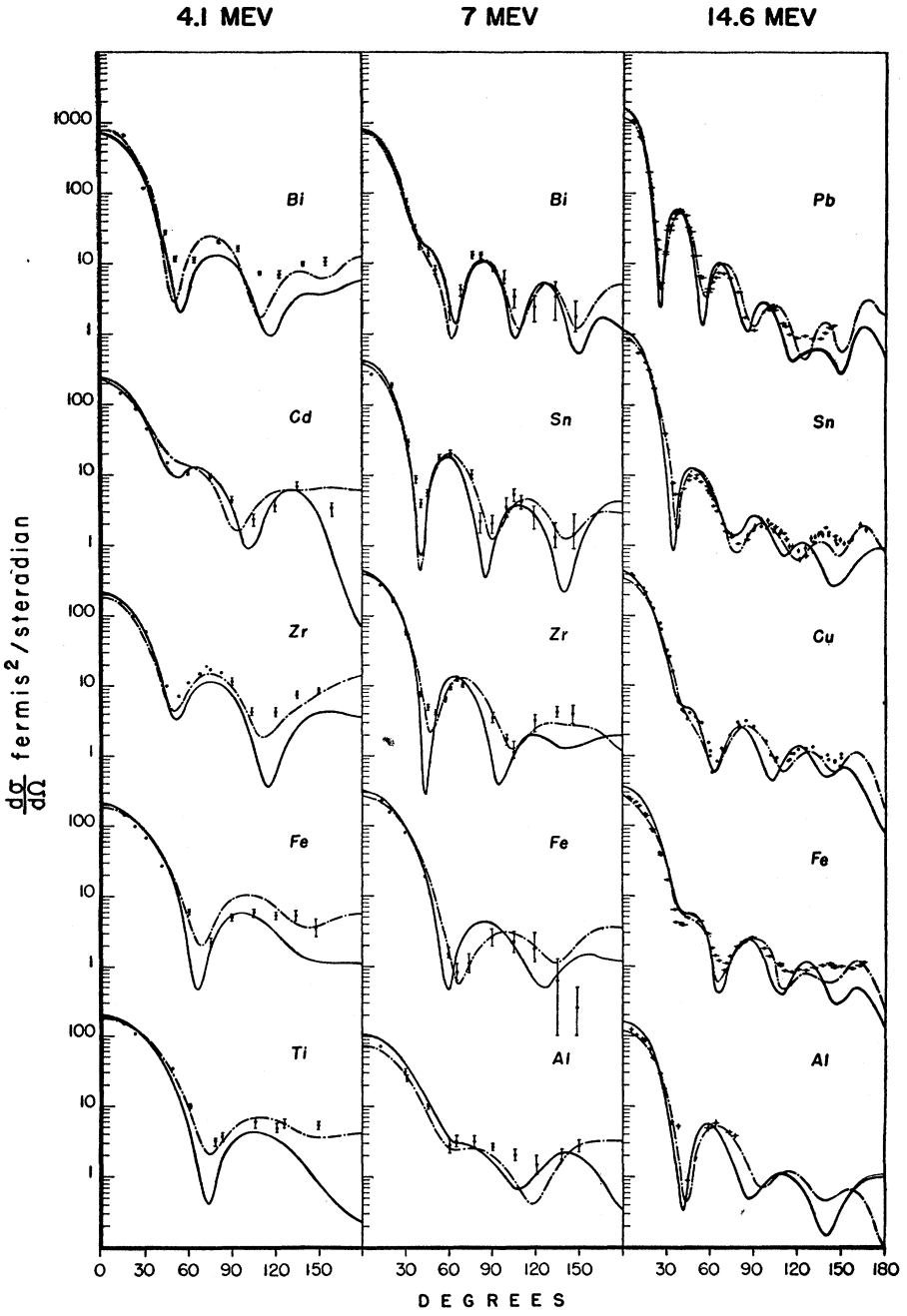

FIG. 5. The effect of deepening the potential well (nonlocal) by 10 Mev. The average corresponding "rearrangement" energy would be 4.5 Mev. Other nuclear parameters are identical to these used in Fig. 2.

then it must vary substantially from isotope to isotope. (iii) Its magnitude is sensitive to the characteristic radius constant, r_0, but (iv) cannot be larger in any event than about 6 Mev.

Concluding Remarks

Although the fits to experimental values are not perfect, the over-all success of the present description is evident. Even without the introduction of a rearrangement energy, a unification of the bound states and scattering has been achieved to a substantial degree. Were one to include the possibility of local fluctuations of parameters such as T and r_0 in going from isotope to isotope, then some improvements would naturally result. As an example, a decrease in the surface thickness in Sn of about 8% is found to shift the maximum at 60° from 10° to 15° towards smaller angles for Set 2 in Table I. Other sets produce similar results, but often at the expense of the fits for different elements. A further examination of such possible local parameter fluctuations, should therefore provide interesting re-

TABLE V. Comparison of experimental and theoretical values of neutron total and reaction cross sections based on an average rearrangement energy of 4.5 Mev.

Element	Energy (Mev)	Cross sections (barns) σ_T (theory)	σ_T (expt)	σ_R (theory)	σ_R (expt)
Al	7.0	2.22	2.0	1.19	1.0
	14.6	1.87	1.7	1.07	1.0
Ti	4.1	3.60	3.4	1.50	1.3
Fe	4.1	3.72	3.7	1.58	1.5
	7.0	3.62	3.6	1.56	1.5
	14.6	2.70	2.5	1.46	1.4
Cu	14.6	2.97	2.9	1.55	1.5
Zr	4.1	3.88	4.0	1.94	1.6
	7.0	4.21	4.2	1.89	1.7
Cd	4.1	4.26	4.1	2.08	2.1
Sn	7.0	4.29	4.2	2.08	2.1
	14.6	4.60	4.4	2.02	2.0
Pb	14.6	5.63	5.7	2.60	2.5
Bi	4.1	6.97	7.9	2.69	2.0
	7.0	5.87	5.6	2.71	2.5

sults—especially those relating to the closing of major shells.

The surface concentration associated with the shape of the imaginary potential used here is probably exaggerated and a modification in the light of recent theoretical studies might prove interesting.[24,25] The assumed discontinuity of the imaginary part of the optical potential, although somewhat unphysical, is not entirely so. The number of excited nuclear states below zero energy is usually small while the number of compound states available rises rapidly above zero energy. Thus, in the vicinity of zero energy, the imaginary part of the optical potential should be expected to rise rapidly—almost discontinuously on a linear energy scale.

The rather deep minima predicted in the theoretical differential elastic scattering cross sections (see Fig. 2) might represent a real effect even though they are not indicated by experiment, for the following reasons: (1) The errors associated with the angle measurements have the effect of smoothing out the angular distributions. This effect may be easily seen by folding a Gaussian of about 5-degrees width with one of the theoretical curves. The theory, of course, shows perfect resolution and must therefore be sharper. (2) The fact that the scattering calculations were based on an average A, rather than an isotope weighted one, also contributes to the sharpening of the theoretical curves. Were the calculations made for individual isotopes and then weighted, the angular distributions would be smeared out somewhat. (3) The possibility of a small contribution of compound elastic scattering, even at energies as high as 14 Mev,[43] could substantially affect the minima. If an isotropic component of compound elastic differential cross section of the order of, say, 7×10^{-3} barn per steradian were added to the theoretical curves at 14.6 Mev for example, the minima would be considerably smoothed out while the maxima would remain virtually unchanged. The contributions from compound elastic scattering at lower energies is considered for the most part to be responsible for the low values of differential scattering cross sections obtained at larger angles in some cases (e.g., at 4.1 Mev).

Nevertheless, the fact that the minima are as deep as they appear, even in this nonlocal formulation, seems to be contrary to the expectation[8] that the inclusion of nonlocality should tend to damp out the diffraction pattern. It is extremely difficult to assign a reason for this calculated effect. Even in the case of local optical models it has been difficult to establish in detail what features of the complex potential influence the various aspects of the scattering predictions. A similar difficulty is encountered in high-energy electron scattering in attempting to infer nuclear charge densities. The traditional approach here has been to assume a reasonable form for the density function and to adjust the parameters in this function to achieve optimum agreement. A recent study of p-shell nuclei, however, has shown that markedly different density functions can give rise to the same scattering predictions.[44] In this case of the nonlocal complex potential the problem is complicated by the fact that one does not simply introduce nonlocality (i.e., let g increase from zero) and hold all other parameters constant. Instead, one renormalizes the other parameters of the potential (particularly the well depth which is increased) to maintain fits to the key experimental data used in the parameter adjustments (e.g., binding energies and location of diffraction minima). After this is accomplished it is difficult to know whether such secondary effects as the depth of diffraction minima are due to the introduction of nonlocality or the alteration of the other potential parameters. A preliminary examination of this point (i.e., g^2 was varied and V_0 adjusted to maintain the last particle binding energy for a particular isotope) indicates that the diffraction minima are augmented (deepened) with increasing g^2 *especially* in in the background angles where the effect is most pronounced.

With these reservations one might for heuristic reasons suggest two aspects of the introduction of nonlocality which might account for the observed accentuation of the diffraction patterns. In the effective mass approximation considered here the introduction of nonlocality leads in the final radial equation [Eq. (6)] to a series of additional terms involving the variable reduced mass function $M(r)$. Several of these terms are effective only in the region of the nuclear surface. These terms have the tendency of reinforcing the centrifugal potential and tending to push the higher angular momentum states outwardly to the region of the surface. In addition the combination $M[V\xi(r)+E]$

[43] See reference 4 for comments on this possibility.

[44] V. Meyer-Berkhout, K. W. Ford, and A. E. S. Green, Ann. Phys. 8, 119 (1959).

has the tendency to sharpening the effective outer edge of the nuclear potential. The combined action of these effects might when "averaged" over angular momentum states be viewed as equivalent to adding a potential tending to concentrate nucleons to the surface. Now in the Born approximation the form function $F(q)$ where q is the momentum transfer is simply the Fourier transform of the radial potential function.[44] Here one can show that potential localization tends to enhance the oscillations in the form factor $F(q)$ which occasions an enhancement of the oscillations in the cross sections. One might expect a similar phenomena to be operative at lower energies.

This interpretation is supported somewhat by the observed influence of the innovations which Kisslinger and Rainwater[4] use in interpreting the scattering of π mesons by nuclei in which the diffraction minima appear to be suppressed. Their final radial wave equation is similar to the model here, except that they have a variable mass which is enhanced as one goes to the center rather than reduced. With the inclusion of this modification it is possible to fit observed angular distributions whose minima resist fitting with local optical models.

As the experimental results of elastically scattered neutron polarizations are at this time rather sparse, not much effort was directed towards detailed fitting along these lines. However, the characteristic difference of the polarization predictions compared with former work with local models lies in the slightly smaller polarizations predicted by the present model at scattering angles up to 40°. This has been noticed experimentally for intermediate elements at higher scattering energies.[45] Nevertheless, the wide variations in the theoretically predicted polarization (up to $\pm 98\%$ in some cases) for larger angle scatterings are still found—the nonlocality apparently having little effect in smoothing out the polarization patterns (Fig. 6).

Although some of the elastic differential scattering cross-section curves in this nonlocal model are in poor agreement (Fig. 2) with experiment (e.g., Bi at 4.1 Mev), others show structural details observed in experiment, but not predicted on the basis of local models. (See, e.g., Cu and Fe at 14.6 Mev). This structure is generally high, but nevertheless in good shape agreement with experiment. The minimum predicted in Cd at 4.1 Mev between 25 and 50 degrees could well exist and might be looked for in future experiments. The very poor fits of Bi at 4.1 and 7 Mev are probably associated with the poor agreement with experiment of the bound state eigenvalue of that element (see Table III). Finally, it should be mentioned that the spheroidal deformation of some elements could be responsible for some of the discrepancies between our predictions and experiment.

It might be well at this point to call attention to the limitations of this study of the application of nonlocal potentials. This work makes use of the expansion of Frahn and Lemmer in conjunction with a more generalized type of kernel embodying realistic and complex structure functions. Because of the complexity of the finite nucleus problem and the rather detailed involvement of the nuclear surface in the bound-state and scattering data fitted here, it is very difficult to assign the origins of the residual discrepancies between theory and experiment. From crude "wavelength" and nuclear size considerations it would seem that the approximation used should hold good to positive energies of the order of 25 Mev. Hence, it is likely that these discrepancies are associated more with limitations of the particular kernels chosen here. The work of Brueckner et al. and Gomes et al.[34] provides some guidance as to two kernel generalizations of the real portion of the kernel. However, there is as yet little theoretical guidance on the imaginary part. The negative sign of ζ_1, which the fitting of experimental data over a broad energy range has demanded may also be a manifestation of hard core effects in the two-body interaction.

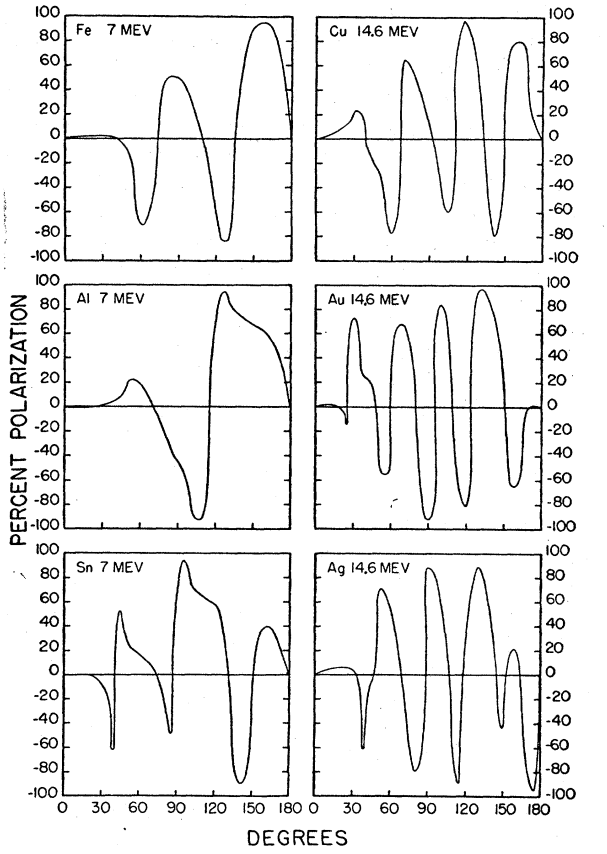

FIG. 6. Representative polarization data based on the present best parameters (Table II). The violent fluctuations present no improvements over the "local" predictions of Bjorklund and Fernbach though the polarization is considerably smaller in the forward direction.

[45] Dr. Jules Levin (private communication).

One must, of course, recognize that the optical model is a highly oversimplified description of the nucleon-nucleus interaction and hence has other limitations. The prescription chosen for separating the observed elastic scattering into shape elastic and compound elastic, has of course considerable bearing upon the values assigned to ζ_0 and ζ_1. The recent work of Alford[4] has particularly suggested that there is a larger contamination of compound elastic in measured cross sections than previously supposed, particularly for even A elements in the mass region around $A=50$. Accordingly, the limits of error assigned to the absorption parameters may be somewhat optimistic.

In conclusion it should be pointed out that the work described herein embodies a particular synthesis of many earlier phenomenological studies of the independent particle model and the nuclear optical model. In view of the many uncertainties and differences of opinion among those pursuing a strictly theoretical approach and in an effort to minimize the number of free parameters, the simplest nonlocal generalizations which might possibly fit a diverse range nucleon-nuclear phenomenon were utilized. It is rather satisfying that the inclusion of nonlocal terms which is so clearly demanded by many theoretical considerations does not occasion any loss from the remarkable accomplishments of local I.P.M. and optical-model description.

Unfortunately, even when constricted by a wide range of phenomenological observations there seems to remain a rather wide choice of parameters. The inclusion of proton scattering data (originally intended as a part of this study) should prove of additional help in narrowing the range of acceptable parameters and hence in sharpening the physical conclusions which might be drawn with such a model.

However, even in its present form and with the parameters given in Table II, the model might be expected to serve as a useful point of departure for exploring many further details of the nucleon-nuclear interaction. From a standpoint of practical applications the model might be expected to be useful in reactor design for interpolation of neutron cross sections over energy gaps or element gaps where experimental data is lacking.

VI. ACKNOWLEDGMENTS

Work on the IBM 704 code for bound states[46] (an extension of an earlier code created for the Oak Ridge Oracle[47]) and the design of the scattering code[48] to accomplish this particular synthesis of nucleon-nuclear phenomena was carried out at Los Alamos Scientific Laboratory. The work continued at Florida State University, using, at times, the IBM 704 Laboratory at Eglin Air Force Base. We wish to acknowledge the kindness of Dr. Carson Mark of LASL and Major Charles Reed of the Air Force Office of Scientific Research who aided in making these arrangements. We wish also to acknowledge the helpful assistance at various points of this study of Dr. R. Lemmer, Dr. Y. C. Tang, Dr. J. Levin, Mr. B. Hill, and Mr. Ulmer Stabler.

[46] J. G. Wills (unpublished).
[47] H. C. Griffith, T. Hildebrandt, and A. E. S. Green (unpublished).
[48] P. J. Wyatt (unpublished).