It would be desirable to have more accurate measurements of the low-momentum ($\sim$100 Mev/c) Σ^-/Σ^+ production ratio from K^--p reactions, as well as of the K^--d nonabsorptive scattering at higher momenta ($\sim$200 Mev/c). More refined data would require, in turn, a more detailed and careful analysis of the corrections to the impulse approximation for K^--d reactions.

We would like to thank Dr. R. H. Dalitz and Dr. S. F. Tuan for sending us the results of their analysis before publication. We would also like to thank Dr. L. S. Rodberg for several stimulating discussions.

PHYSICAL REVIEW VOLUME 119, NUMBER 3 AUGUST 1, 1960

Gauge Invariance and the Lorentz Pondermotive Force

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(Received March 3, 1958; revised manuscript received January 28, 1959)

In this paper it is shown that if one introduces into the Weyl theory of gauge invariance the two additional conditions that gauge (and therefore length), except for an arbitrary phase factor, be integrable along the path of a particle, and that the change in dimensions of a particle be a minimum, one immediately obtains the Lorentz pondermotive force for a charged particle in an electromagnetic field and the Bohr-Sommerfeld quantum integral.

INTRODUCTION

In a previous paper[1] an attempt was made to obtain the Maxwell-Lorentz equations for an electron by starting out from a generalized Lagrangian which was derived from the contracted Riemann-Christoffel tensor $G_{\mu\nu}$ by imposing the condition that the Lagrangian be gauge invariant in Weyl's sense of the term. This led to the second rank in-tensor as defined by Eddington[2]

$$*G_{\mu\nu}=G_{\mu\nu}-(\kappa_\alpha{}^\alpha-2\kappa_\alpha\kappa^\alpha)g_{\mu\nu}-2\kappa_\mu\kappa_\nu+(\kappa_{\mu,\nu}+\kappa_{\nu,\mu})-2\mathfrak{F}_{\mu\nu}, \quad (1)$$

where the $g_{\mu\nu}$ are the components of the metric tensor, and $\kappa_{\mu,\nu}=\partial\kappa_\mu/\partial X_\nu$, and

$$\mathfrak{F}_{\mu\nu}=\frac{\partial\kappa_\mu}{\partial X_\nu}-\frac{\partial\kappa_\nu}{\partial X_\mu}. \quad (2)$$

We shall see later that the 4-vector κ_μ may be chosen proportional to the electromagnetic 4-vector potential A_μ so that the $\mathfrak{F}_{\mu\nu}$ are essentially the electromagnetic field strengths.

In this note we shall show first that we can derive the Lorentz expression for the pondermotive force on a charge in an electromagnetic field if we impose upon the Weyl theory the condition that the charge must move in such a way that the change in its dimensions along a given path (resulting from the change of gauge along this path) must be a minimum. Secondly, we shall show that the Bohr-Sommerfeld quantum integral follows directly if we impose the condition that the dimensions of a charged particle, except for a possible change of phase, must return to their initial values when the particle moves around a closed orbit. We shall see that this condition will eliminate one of the most serious objections to the Weyl theory arising from the nonintegrability of length that is a consequence of the theory.

LORENTZ PONDERMOTIVE FORCE

The Weyl theory of gauge invariance arose out of the concept that lengths at different places cannot be compared because of the change of gauge that takes place as one moves from point to point in a space-time continuum. Since the gauge was assumed to be determined by a vector field κ_μ, comparison of lengths at different places would be ambiguous because the result of the comparison would depend on the path taken in going from one point to the other. Although this theory introduced a four vector (to be identified with the electromagnetic vector potential) into the description of the world quite naturally, the nonintegrability of length which it brought with it led to apparently insurmountable difficulties concerning the structure of atoms.

Thus the objection was raised that according to the Weyl theory the natural frequency of an atom at a point in space-time should depend on the path the atom took to reach that point. This objection was met by introducing the assumption that although lengths and frequencies depend on the path taken, the effect is much too small to be measurable in actual physical phenomena. This, however, is not a satisfactory way out of the difficulty since the ambiguity is still present in the theory. It is possible to eliminate this ambiguity without destroying the content of the Weyl theory by imposing the condition that the measurable physical dimensions of a particle shall be integrable along the path of its motion. We must note that this is not the same thing as imposing the condition that the gauge

[1] L. Motz, Phys. Rev. **89**, 60 (1953).
[2] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, 1923), p. 204.

be integrable along a path. This latter condition is much too restrictive and would result in the vanishing of the curl of the four-vector κ_μ and hence to the vanishing of the electromagnetic field so that our theory would be empty.

The possibility of imposing integrability on the physically meaningful dimensions and yet not on the gauge arises from the fact that the dimensions must be treated as complex quantities so that they have arbitrary phase factors associated with them. Since these phases need not be integrable, the gauge will not be integrable either, with the result that the content of the theory will remain while the ambiguity is eliminated. We shall come back to this point in our discussion of the Bohr-Sommerfeld quantum condition, but now we shall consider what constraints may be imposed on the motion of a particle without modifying the nonintegrability of gauge at all.

To see what we must do we shall start from Weyl's fundamental assumption that if a length A is displaced from a point x_μ to a nearby point $x_\mu+dx_\mu$, then it suffers a change in length determined by the equation

$$d\ln A=\kappa_\mu dx_\mu, \qquad (3)$$

where κ_μ is a vector field. Let us suppose, now, that we have a particle which moves from some point, P_1, in our space-time continuum to some other point, P_2, along a physically permissible path. What constraint can we impose upon this motion that will be physically significant and yet which will not violate the basic assumptions of the Weyl theory? It is reasonable to assume that a particle will tend to retain its dimensions in so far as possible as it moves along its path. We shall therefore impose the condition that the particle will move along that particular path connecting the end points P_1 and P_2 which results in the smallest change in its dimensions. In other words we shall assume that

$$\int_{P_1}^{P_2} d\ln A=\int_{P_1}^{P_2}\frac{dA}{A}=\int_{P_1}^{P_2}\kappa_\mu dx_\mu=\text{a minimum}. \qquad (4)$$

We have, then, as an additional constraint on the motion of the particle the condition that

$$\delta\int_{P_1}^{P_2}\kappa_\mu dx_\mu=0, \qquad (5)$$

for all variations of the permissible path in which the end points are kept fixed.

If ds is the element of path length along a geodesic between two fixed points, it must satisfy the stationary condition

$$\int_{P_1}^{P_2}\delta(ds)=0. \qquad (6)$$

However, the variation in the integrand of (6) may no longer be taken as arbitrary since only those variations are permissible which are governed by Eq. (5). We may treat this situation in the usual way by varying (5) and incorporating this variation into (6) by means of Lagrangian multipliers.

If we vary (5) we obtain

$$\int_{P_1}^{P_2}\frac{\partial\kappa_\mu}{\partial X_\nu}\frac{dX_\mu}{dS}dS\delta X_\nu+\int_{P_1}^{P_2}\kappa_\mu d(\delta X_\mu)=0. \qquad (7)$$

If we integrate the second term by parts and take account of the fact that the variation must vanish at the end points of the path, we finally obtain

$$\begin{aligned}\int_{P_1}^{P_2}\frac{\partial\kappa_\mu}{\partial X_\nu}\frac{dX_\mu}{dS}\delta X_\nu dS-\int_{P_1}^{P_2}d\kappa_\nu\delta X_\nu &=\int_{P_1}^{P_2}\left[\frac{\partial\kappa_\mu}{\partial X_\nu}-\frac{\partial\kappa_\nu}{\partial X_\mu}\right]\frac{dX_\mu}{dS}dS\delta X_\nu\\ &=\int_{P_1}^{P_2}\mathfrak{F}_{\mu\nu}V_\mu dS\delta X_\nu=0, \qquad (8)\end{aligned}$$

where we have introduced the velocity four-vector V_μ and the antisymmetric tensor $\mathfrak{F}_{\mu\nu}$ defined by (2).

The variation (6) gives rise to the result

$$\int_{P_1}^{P_2}\left[\frac{1}{2}\frac{dX_\mu}{dS}\frac{dX_\sigma}{dS}\left(\frac{\partial g_{\mu\sigma}}{\partial X_\nu}-\frac{\partial g_{\mu\nu}}{\partial X_\sigma}-\frac{\partial g_{\sigma\nu}}{\partial X_\mu}\right)-g_{\epsilon\nu}\frac{d^2X_\epsilon}{dS^2}\right]dS\delta X_\nu=0. \qquad (9)$$

If we now introduce the Lagrangian multiplier, L, we can combine (8) and (9) to give,

$$\int_{P_1}^{P_2}\left[\frac{1}{2}\frac{dX_\mu}{dS}\frac{dX_\sigma}{dS}\left(\frac{\partial g_{\mu\sigma}}{\partial X_\nu}-\frac{\partial g_{\mu\nu}}{\partial X_\sigma}-\frac{\partial g_{\sigma\nu}}{\partial X_\mu}\right)-g_{\epsilon\nu}\frac{d^2X_\epsilon}{dS^2}+L\mathfrak{F}_{\mu\nu}V_\mu\right]dS\delta X_\nu=0. \qquad (10)$$

In this equation the variations are arbitrary so that we obtain the equations of motion of a particle by setting the square bracket in (10) equal to zero:

$$g_{\epsilon\nu}\frac{d^2X_\epsilon}{dS^2}-\frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial X_\nu}-\frac{\partial g_{\mu\nu}}{\partial X_\sigma}-\frac{\partial g_{\sigma\nu}}{\partial X_\mu}\right)\frac{dX_\mu}{dS}\frac{dX_\sigma}{dS}=L\mathfrak{F}_{\mu\nu}V_\mu. \qquad (11)$$

If we multiply through by $g^{\alpha\nu}$, we obtain the Lorentz force equation in the form

$$\frac{d^2X_\alpha}{dS^2}+\Gamma_{\mu\sigma}{}^{\alpha}\frac{dX_\mu}{dS}\frac{dX_\sigma}{dS}=L\mathfrak{F}_\mu{}^{\alpha}V_\mu. \qquad (12)$$

Since κ_μ will later be related to the vector potential A_μ by the equation $\kappa_\mu=(i/\hbar)(e/c)A_\mu$, the Lagrangian

multiplier, L, must be chosen equal to $-i(\hbar/mc)$ in order to make Eq. (12) dimensionally correct. The letters e, $\hbar$, m, and c have their usual meanings.

BOHR-SOMMERFELD QUANTUM INTEGRAL

We have already noted that the nonintegrability of length which follows from the Weyl theory brings certain objectionable features with it which cast doubt on the entire theory. We must therefore try, in so far as is possible, to eliminate these features but not at the expense of the physical content of the theory. We may do this if we note that the quantity $\ln A$ is, in general, not real so that we may write $A=\mathcal{A}e^{i\phi}$, where ϕ is a real number. The arbitrary phase factor will have no effect on the physically meaningful lengths in nature since these are to be obtained from the mathematical quantities A by taking absolute values. We now have from Eq. (3) the result

$$d\ln\mathcal{A}+id\phi=\kappa_\mu dx_\mu,$$

or

$$d\ln\mathcal{A}=\kappa_\mu dx_\mu-id\phi. \tag{13}$$

We shall now impose the condition that $\ln\mathcal{A}$ shall be integrable along any permissible path of a particle but that $\ln A$ need not be. If we now consider a particle moving in a closed orbit in the field κ_μ, we see that we must have

$$\oint d\ln\mathcal{A}=\oint\kappa_\mu dx_\mu-i\oint d\phi=0, \tag{14}$$

or

$$\oint\kappa_\mu dx_\mu=i\oint d\phi.$$

We have complete freedom in terms of our theory as to the change that ϕ must suffer when our particle moves once around in its orbit, but it is most natural to assume the change will be such as to have as small an effect as possible on A, and this will obviously be the case if ϕ changes exactly by an integral multiple of 2π. We therefore have from (14) the additional constraint on the motion of the particle given by

$$\oint\kappa_\mu dx_\mu=2\pi in, \tag{15}$$

where n is any integer.

If we now replace κ_μ by its definition in terms of the vector potential A_μ as given in the last paragraph of the previous section, we have

$$(e/c)\oint A_\mu dx_\mu=nh. \tag{16}$$

We may see what the meaning of this condition is if we consider a charged particle moving in a static central force field. In that case only the fourth component of A_μ will be different from zero, and (16) reduces to

$$(e/c)\oint\Phi cdt=e\oint(\Phi/v)ds=nh, \tag{17}$$

where Φ is the scalar potential of the central field, v is the speed of the particle in its orbit, and ds is an element of path length.

As the simplest case we shall take a Coulomb force field and suppose that our charged particle is moving in a circular orbit with constant speed v. If r is the radius of the orbit, the potential equals e/r, and $ds=rd\theta$, where $d\theta$ is the element of angular displacement. Equation (17) now becomes

$$(e^2/v)\oint d\theta=nh,$$

or

$$v=(2\pi e^2)/(nh). \tag{18}$$

This is just the Bohr condition for the velocity of an electron in a circular orbit and leads to the Bohr energy levels for circular orbits. This means that (15) is equivalent to the Bohr-Sommerfeld quantum integral, and we shall now give a general proof of this.

If we start from (16) and introduce the proper time $d\tau=cdt(1-v^2/c^2)^{\frac{1}{2}}$, we have

$$(e/c)\oint A_\mu V_\mu d\tau=nh, \tag{19}$$

where V_μ is the relativistic four-velocity of the particle in its orbit defined by

$$V_\mu=dx_\mu/d\tau=(p_\mu-eA_\mu/c)/m_0c, \tag{20}$$

where p_μ is the momentum-energy four-vector and m_0 is the rest mass of the particle. We shall now transform (19) by adding and subtracting $p_\mu dx_\mu$ under the integral sign so that we obtain

$$\oint[(e/c)A_\mu-p_\mu]V_\mu d\tau=nh-\oint p_\mu dx_\mu. \tag{21}$$

If we now use the definition (20) and note that

$$-[p_\mu-(e/c)A_\mu]^2=m_0^2c^2,$$

we obtain from (21)

$$\oint m_0cd\tau=nh-\oint p_\mu dx_\mu. \tag{22}$$

We shall split the four-vector p_μ into its space and time parts and transpose the time part, which is just the Hamiltonian of the system, that is the sum of kinetic (including rest mass energy) and potential energies of the particle, $[p_4=(i/c)(T+U)$, where T is the kinetic energy$]$. If we now keep in mind that

$dx_4 = icdt$, we obtain from (22) the equation

$$\oint \{m_0c^2[1-(v/c)^2]^{\frac{1}{2}} - T - U\}dt = nh - \oint \sum_{i=1}^{3} p_i dx_i, \quad (23)$$

where we have replaced $d\tau$ by its definition in terms of dt. Since T is equal to $m_0c^2/[1-(v/c)^2]^{\frac{1}{2}}$, the left-hand side of (23) becomes

$$\oint \{m_0v^2/[1-(v/c)^2]^{\frac{1}{2}} + U\}dt = \oint [\sum_{i=1}^{3} p_i(dx_i/dt) + U]dt. \quad (24)$$

If we now integrate the first term on the right-hand side by parts and use the relationship $dp_i/dt = -\partial U/\partial x_i$, we finally obtain for the left-hand side of (23) the integral

$$\oint \left[\sum_{i=1}^{3} x_i \frac{\partial U}{\partial x_i} + U\right]dt. \quad (25)$$

We see that the integrand will vanish by Euler's theorem on homogeneous functions if U is a homogeneous function of degree -1 in the coordinates x_i. We see, then, that for systems in which the potential energy is a homogeneous function of degree -1, the left-hand side of (23) vanishes and we obtain the Bohr-Sommerfeld quantum condition. It appears from this that the existence of discrete orbits arises from the fact that it is only for such orbits governed by the Bohr-Sommerfeld conditions that the dimensions of a particle are single-valued.

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Electromagnetic Form Factors of the Nucleon*

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The physical interpretation of the electromagnetic form factors is discussed with special reference to the gauge invariance of particular theories. A distinction is made between the condition that the one nucleon matrix element satisfy the equation of continuity ("weak gauge invariance") and the stronger condition imposed by the generalized Ward identity ("strong gauge invariance"). The former is shown to be a consequence of covariance under the improper Lorentz transformations, and hence it has no new content concerning the functional behavior of the form factors. The latter implies restrictions on the current operator which may have an important effect on the results of calculations of form factors.

In connection with the physical interpretation, it is noted that the moments of the charge and current distribution are determined by $F_{\text{ch}} = F_1 - (q^2/2M)F_2$ and $F_{\text{mag}} = (1/2M)F_1 + F_2$. Specifically the second moment of the charge distribution, $-6F_{\text{ch}}'(0)$, is found, in the case of the neutron, to be directly measured by the neutron-electron interaction *without* the intervening subtraction of the Foldy term.

These matters are investigated in detail by means of a specific model of the nucleon which is a covariant generalization of the fixed source static model having the property that it gives results identical with the static model in the limit $M \to \infty$. It is found that strong gauge invariance requires the addition of line currents which make significant contributions to the form factors in general and, in particular, to the proton charge radius even in the static approximation. This suggests that as a consequence of strong gauge invariance, important contributions to the charge radius must arise in any theory from intermediate states of large mass. The model also provides a means of consistently calculating recoil corrections to the static model. They are found to be large.

1. INTRODUCTION

THE electromagnetic interactions of the nucleon provide, in principle, a direct source of information concerning the structure of the nucleon. Although a prodigious amount of experimental information concerning the electromagnetic interactions is available,[1] a satisfactory interpretation of all of the data has not been possible.[2] The failure of any theoretical treatment of the electromagnetic form factors

* Work supported in part by the University of Wisconsni Research Committee with funds granted by the Wisconsin Alumni Research Foundation, in part by the U. S. Atomic Energy Commission, and in part by the U. S. Air Force, through the Research Contracts Division and through the European Office, Air Research and Development Command.

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[1] For a summary of experimental data see R. Hofstadter, F. Bumiller, and M. R. Yearian, Revs. Modern Phys. **30**, 482 (1958).

[2] A summary of the theoretical interpretations predating the use of dispersion relations is provided by D. R. Yennie, M. M. Lévy, and D. G. Ravenhall, Revs. Modern Phys. **29**, 144 (1957).