$dx_4 = icdt$, we obtain from (22) the equation

$$\oint \{m_0c^2[1-(v/c)^2]^{\frac{1}{2}} - T - U\}dt = nh - \oint \sum_{i=1}^{3} p_i dx_i, \quad (23)$$

where we have replaced $d\tau$ by its definition in terms of dt. Since T is equal to $m_0c^2/[1-(v/c)^2]^{\frac{1}{2}}$, the left-hand side of (23) becomes

$$\oint \{m_0v^2/[1-(v/c)^2]^{\frac{1}{2}} + U\}dt = \oint [\sum_{i=1}^{3} p_i(dx_i/dt) + U]dt. \quad (24)$$

If we now integrate the first term on the right-hand side by parts and use the relationship $dp_i/dt = -\partial U/\partial x_i$, we finally obtain for the left-hand side of (23) the integral

$$\oint \left[\sum_{i=1}^{3} x_i \frac{\partial U}{\partial x_i} + U\right]dt. \quad (25)$$

We see that the integrand will vanish by Euler's theorem on homogeneous functions if U is a homogeneous function of degree -1 in the coordinates x_i. We see, then, that for systems in which the potential energy is a homogeneous function of degree -1, the left-hand side of (23) vanishes and we obtain the Bohr-Sommerfeld quantum condition. It appears from this that the existence of discrete orbits arises from the fact that it is only for such orbits governed by the Bohr-Sommerfeld conditions that the dimensions of a particle are single-valued.

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Electromagnetic Form Factors of the Nucleon*

F. J. Ernst
University of Wisconsin, Madison, Wisconsin and Columbia University, New York, New York

R. G. Sachs†
University of Wisconsin, Madison, Wisconsin and University of Paris, Paris, France

AND

K. C. Wali
University of Wisconsin, Madison, Wisconsin

(Received March 18, 1960)

The physical interpretation of the electromagnetic form factors is discussed with special reference to the gauge invariance of particular theories. A distinction is made between the condition that the one nucleon matrix element satisfy the equation of continuity ("weak gauge invariance") and the stronger condition imposed by the generalized Ward identity ("strong gauge invariance"). The former is shown to be a consequence of covariance under the improper Lorentz transformations, and hence it has no new content concerning the functional behavior of the form factors. The latter implies restrictions on the current operator which may have an important effect on the results of calculations of form factors.

In connection with the physical interpretation, it is noted that the moments of the charge and current distribution are determined by $F_{\mathrm{ch}} = F_1 - (q^2/2M)F_2$ and $F_{\mathrm{mag}} = (1/2M)F_1 + F_2$. Specifically the second moment of the charge distribution, $-6F_{\mathrm{ch}}'(0)$, is found, in the case of the neutron, to be directly measured by the neutron-electron interaction *without* the intervening subtraction of the Foldy term.

These matters are investigated in detail by means of a specific model of the nucleon which is a covariant generalization of the fixed source static model having the property that it gives results identical with the static model in the limit $M \to \infty$. It is found that strong gauge invariance requires the addition of line currents which make significant contributions to the form factors in general and, in particular, to the proton charge radius even in the static approximation. This suggests that as a consequence of strong gauge invariance, important contributions to the charge radius must arise in any theory from intermediate states of large mass. The model also provides a means of consistently calculating recoil corrections to the static model. They are found to be large.

1. INTRODUCTION

THE electromagnetic interactions of the nucleon provide, in principle, a direct source of information concerning the structure of the nucleon. Although a prodigious amount of experimental information concerning the electromagnetic interactions is available,[1] a satisfactory interpretation of all of the data has not been possible.[2] The failure of any theoretical treatment of the electromagnetic form factors

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† Guggenheim Fellow, 1959–60.

[1] For a summary of experimental data see R. Hofstadter, F. Bumiller, and M. R. Yearian, Revs. Modern Phys. **30**, 482 (1958).

[2] A summary of the theoretical interpretations predating the use of dispersion relations is provided by D. R. Yennie, M. M. Lévy, and D. G. Ravenhall, Revs. Modern Phys. **29**, 144 (1957).

to give quantitative agreement with the data has been interpreted as an indication of a serious difficulty with the theory. However, there are several general questions concerning the nature of the approximations used and their interpretation which seem to warrant careful investigation before drawing any such conclusion. The purpose of this work is to study in some detail two such questions, one being the problem of gauge invariance and the other the physical interpretation of the form factors.

The requirement of gauge invariance is usually limited to the condition that the one-nucleon matrix element of the current should satisfy the equation of continuity. However, it is found that this condition is automatically satisfied for any matrix element which is covariant under the improper Lorentz transformations. Hence, this condition, which will be referred to as "weak gauge invariance," does not, in fact, guarantee the conservation of charge. We suggest here that the minimum requirement is the generalized Ward identity, which is a statement of the equation of continuity for the photon vertex off the nucleon mass shell. This will be called the condition of "strong gauge invariance." It will be shown, by making use of a specific model, that strong gauge invariance requires additions to the current operators which make important contributions to the form factors.

The model used here is a covariant generalization of the fixed source static model, designed in such a way as to give the same results as the static model in the limit of large nucleon mass. Strong gauge invariance is established by introducing "line currents" which can be used to estimate the effect of charge conservation on the form factors. A particularly surprising result is that there is a large line-current contribution to the isoscalar second moment of the charge distribution even in the static limit. This is of particular interest since the large value of the experimental isoscalar second moment has been a major source of difficulty in earlier work.[2]

The model also provides a consistent procedure for calculating recoil corrections to the static model. The fact that recoil effects are included means that another question of some interest can be investigated. In 1951, Foldy[3] showed that the neutron-electron interaction could be described as the sum of two terms, one of them being given exactly in terms of the known anomalous magnetic moment of the nucleon. He interpreted the other term as the second moment of the pion charge cloud distribution. The contribution of the magnetic moment, called the "Foldy term" is inversely proportional to the nucleon mass. Therefore it was argued, in particular by Salzman,[4] that it is not taken into account by the static model, which corresponds to the limit $M\rightarrow\infty$. Hence, the neutron-electron interaction as calculated by the static model has usually been compared to the difference between the observed interaction and the Foldy term. The disagreement measured in that way is enormous.

The interpretation of Foldy's result is re-examined below, by making use of the fact that the limit $M\rightarrow\infty$ can be handled systematically. We find that in a consistent treatment of the static model, the result of the static model is to be compared directly with the observed neutron-electron interaction, without making the subtraction described above. Therefore the discrepancy between the static model and observation is not nearly so great as had been thought, nor is the experimental result as mysterious as it seemed.

As an illustration of the possibility of calculating recoil corrections, we shall obtain the terms of order M^{-1} in the magnetic moments. These will turn out to be large, of the same order as the static contributions, so that any success of the static model is probably fortuitous. Of particular interest is the isoscalar magnetic moment, which is strictly a recoil effect. Its evaluation to order M^{-1} by means of the present model gives an expression in close agreement with that obtained from the static model on the basis of the mirror theorem.[5] Perhaps this agreement is not so surprising since the early result is just a consequence of the conservation of angular momentum. As is well known, the result disagrees seriously with the observed value of the isoscalar magnetic moment.

2. ELECTROMAGNETIC FORM FACTORS

The interactions under discussion involve the emission and absorption of a single photon and may be described in terms of the matrix element $\langle P',s'|j_\mu(0)|P,s\rangle$ of the current density operator between nucleon states of 4-momentum P and P' and spin s and s'. The operator is evaluated at the space-time point $x_\mu=0$. It is well known[4] that this matrix element may be written in the form

$$\langle P',s'|j_\mu(0)|P,s\rangle=\frac{ie}{(2\pi)^3}\frac{M}{(P_0P_0')^{\frac{1}{2}}}\bar{u}_{s'}(\mathbf{P}')\times[F_1(q^2)\gamma_\mu-F_2(q^2)\sigma_{\mu\nu}q_\nu]u_s(\mathbf{P}),\quad(1)$$

where $q_\mu=P_\mu'-P_\mu$, $\sigma_{\mu\nu}=(1/2i)[\gamma_\mu,\gamma_\nu]$, and $u_s(\mathbf{P})$ is the positive energy Dirac spinor of a free nucleon of mass M, momentum $\mathbf{P}$ and spin s.

The matrix element is restricted to the form Eq. (1) by the requirement of covariance under the improper Lorentz group. In particular, it is shown in the Appendix that the additional term of the form $F_3(q^2)q_\mu$ is excluded by the requirements imposed on the current density under the time-reversal transformation. The condition imposed on the matrix element by the equation of

[3] L. L. Foldy, Phys. Rev. **83**, 688 (1951); **87**, 688 (1952); **87**, 693 (1952).
[4] G. Salzman, Phys. Rev. **99**, 973 (1955).
[5] R. G. Sachs, Phys. Rev. **87**, 1100 (1952).

continuity, namely,

$$q_\mu\langle P',s'|j_\mu(0)|P,s\rangle=0 \tag{2}$$

is satisfied by Eq. (1). Thus the condition of weak gauge invariance, Eq. (2), is a direct consequence of the covariance[6] and does not, in fact, contain any information concerning the consequences of charge conservation. Therein lies its weakness. It is satisfied for any choice of the functions $F_1(q^2)$ and $F_2(q^2)$. On the other hand, it is clear that the form of the functions, as calculated in any given model, is affected by the condition of charge conservation. This results from the restriction on the form of the current operator imposed by the equation of continuity,

$$\partial j_\mu(x)/\partial x_\mu=0. \tag{3}$$

Not only does Eq. (3) imply the condition Eq. (2) on the one-nucleon matrix element, but it also implies an extension of the condition off the nucleon mass shell, namely, the generalized Ward identity.[7] This condition of strong gauge invariance would appear to be a minimum requirement on any calculation of electromagnetic properties of the nucleon.

It is of interest to discuss the physical meaning of the form factors $F_1(q^2)$ and $F_2(q^2)$ which describe, in some sense, the distribution of charge and magnetization in the nucleon. The various moments of these distributions are often used to give an insight into the form of the distribution; examples are the magnetic moment, second moment of the charge distribution, and so on. The simplest moment is the zero-order moment of F_1, namely,

$$F_1(0)=\tfrac{1}{2}(1+\tau_3) \tag{4}$$

is just the charge of the nucleon in units of e.

To obtain the magnetic moment, for example, we wish to calculate the expectation value of the operator $\frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{j}(x)$ in a state in which the nucleon is at rest. This expectation value is most easily determined for a wave packet $\int d^3P\,a(\mathbf{P})|P,s\rangle$. Since

$$\langle P'|j_\mu(x)|P\rangle=e^{-i(P'-P)\cdot x}\langle P'|j_\mu(0)|P\rangle, \tag{5}$$

we find for the magnetic moment

$$\begin{aligned}\langle\mathfrak{M}\rangle=&\frac{1}{2}\int d^3P'\,a^*(P')\int d^3P\,a(P)\int d^3r\,e^{-i(P'-P)\cdot x}\mathbf{r}\\&\times\langle P',s'|\mathbf{j}(0)|P,s\rangle\\=&-(2\pi)^3i\frac{1}{2}\int d^3P'\,a^*(P')\int d^3P\,a(P)\boldsymbol{\nabla}_P\delta(\mathbf{P}-\mathbf{P}')\\&\times\langle P',s'|\mathbf{j}(0)|P,s\rangle e^{i(P_0'-P_0)t}.\end{aligned}$$

We now substitute for the matrix element from Eq. (1), integrate by parts, and finally take as the wave packet

$$|a(\mathbf{P})|^2=\delta(\mathbf{P}),$$

with the result

$$\langle\mathfrak{M}\rangle=e[F_1(0)/2M+F_2(0)]\bar{u}_{s'}(0)\boldsymbol{\sigma}u_s(0). \tag{6}$$

Since $eF_1(0)/2M$ is, according to Eq. (4), the Dirac moment, we see that $eF_2(0)$ is the anomalous moment of the nucleon.

The second moment of the charge distribution may be calculated in a similar manner; the expectation value of the quantity $\int d^3r\,r^2j_4(x)$ is required. Proceeding in just the manner described above, we find for the second moment

$$\langle\mathfrak{R}^2\rangle=-6\left[F_1'(0)-\frac{F_2(0)}{2M}\right]+F_1(0)\left[-\frac{3}{4M^2}+\int d^3P\frac{P_0}{M}\nabla_Pa(\mathbf{P})\cdot\nabla_Pa(\mathbf{P})\right], \tag{7}$$

where the prime denotes differentiation with respect to q^2. The ambiguous term proportional to $F_1(0)$, which vanishes for the neutron, is just the uninteresting effect of distributing the charge of the proton in a wave packet. It will be dropped henceforth.[8]

The results Eq. (6) and Eq. (7) suggest that we introduce new form factors, the charge form factor

$$F_{\mathrm{ch}}(q^2)=F_1(q^2)-(q^2/2M)F_2(q^2), \tag{8}$$

and the magnetic form factor

$$F_{\mathrm{mag}}(q^2)=(1/2M)F_1(q^2)+F_2(q^2). \tag{9}$$

Note that this usage differs from the usual, but the difference has an important physical significance. F_{ch} measures the charge distribution, not F_1 as is so often assumed. Also, F_{mag} rather than F_2 measures the distribution of magnetization. This point was surmised by Yennie, Lévy, and Ravenhall,[2] and a clear demonstration that it is a reasonable interpretation has been given by Walecka,[9] who shows that F_{ch} and F_{mag} measure the interaction with static electric and magnetic fields, respectively.

We now have from Eqs. (7) and (8)

$$\langle\mathfrak{R}^2\rangle=-6F_{\mathrm{ch}}'(0), \tag{10}$$

after dropping the uninteresting term, and from Eqs. (6) and (9),

$$\mathfrak{M}=eF_{\mathrm{mag}}(0)\boldsymbol{\sigma}. \tag{11}$$

We are now in a position to comment on the Foldy term. Foldy[3] showed that the neutron-electron interaction is proportional to $F_1'(0)-F_2(0)/2M$, which we now see is just $F_{\mathrm{ch}}'(0)$. Thus a measurement of the

[6] Note that this result would not obtain if the initial and final states described particles of different mass.

[7] Y. Takahashi, Nuovo cimento 6, 370 (1957).

[8] Yennie, Lévy, and Ravenhall, reference 2, include part of this term in their expression for the second moment but its contribution is too small to have a significant effect on their discussion.

[9] J. P. Walecka, Nuovo cimento 11, 821 (1959).

neutron-electron interaction is simply a measurement of $F_{\text{ch}}'(0)$, which is also seen to be directly proportional to the second moment of the total charge distribution. A calculation of the second moment should then be directly comparable to the measurement, without subtracting the Foldy term. In particular, it will be shown in Sec. 7 that the limit as $M\rightarrow\infty$ of $-6F_{\text{ch}}'(0)$ is identical with the second moment of the pion charge distribution as calculated in the static model. Hence, a consistent interpretation of the static model would associate the entire neutron-electron interaction with the second moment of the pion charge distribution.

3. DESCRIPTION OF THE MODEL

The model to be considered is a covariant generalization of the fixed-source static model in its most primitive form. No effort will be made here to determine rescattering corrections. Thus our starting point is the lowest-order covariant perturbation theory. The modification is simply to insert at the pion-nucleon vertices a general vertex function, as indicated in Fig. 1, and to calculate the form factors in terms of the vertex function. The vertex function will be found to play the same role as the source function in the static model.

There are several terms in the general vertex function corresponding to the several possible vertex form factors. For the sake of simplicity, only the pseudovector term is included; it is the one having the closest correspondence to the static model. This has the form $\gamma_5(\boldsymbol{P}_2-\boldsymbol{P}_1)$ multiplied by a form factor, where $\boldsymbol{P}\equiv\gamma_\mu P_\mu$. The vertex form factor is, in general, an invariant function of P_1 and P_2 (see Fig. 1), thus any three combinations of the three invariant variables P_1^2, P_2^2, $(P_1\cdot P_2)$ may appear in it. On the other hand, in the static model the source function depends on only one variable. In order to study the approach to the static limit, the vertex function is also taken to be a function of a single invariant variable $\xi(P_1,P_2)$ the variable being chosen to be such a combination of P_1^2, P_2^2 and $P_1\cdot P_2$ as to make the correspondence with the static model as close as possible. This choice will be specified below. The vertex part, Fig. 1 is therefore taken to be of the form

$$i(8\pi)^{\frac{1}{2}}fv(\xi)\gamma_5(\boldsymbol{P}_2-\boldsymbol{P}_1), \tag{12a}$$

for a charged pion and

$$i(4\pi)^{\frac{1}{2}}fv(\xi)\gamma_5(\boldsymbol{P}_2-\boldsymbol{P}_1) \tag{12b}$$

for a neutral pion. We shall normalize v in such a way

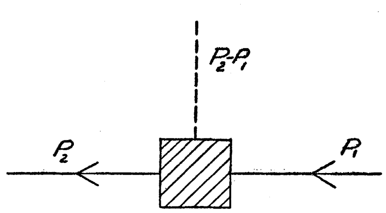

FIG. 1. Diagram of the pion-nucleon vertex.

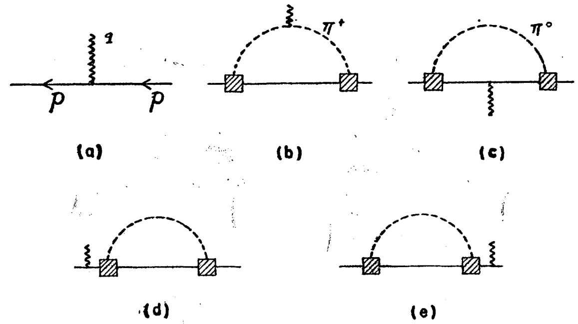

FIG. 2. Diagrams representing the electromagnetic interaction of the proton.

that f corresponds to the usual renormalized, unrationalized coupling constant, $f^2\approx 0.08$.

The calculation of the electromagnetic form factor consists of substituting the vertex part Eq. (12a) or Eq. (12b) in the matrix element obtained from the diagrams Fig. 2 for the proton and Fig. 3 for the neutron. For a given function $v(\xi)$, these diagrams could be evaluated directly without further assumptions or approximations. However, in order to avoid introducing a specific function and to study the expansion in powers of M^{-1}, we proceed in the following manner. Consider for example the term corresponding to Fig. 2(c): The contribution to $\langle P',s'|j_\mu(0)|P,s\rangle$ is

$$\frac{4\pi ef^2}{(2\pi)^7}\frac{M}{(P_0P_0')^{\frac{1}{2}}}\bar{u}_{s'}(\mathbf{P}')\int d^4k\,\gamma_5\boldsymbol{k}v(\xi_1)S_F(P'-k)$$
$$\times\gamma_\mu S_F(P-k)\gamma_5\boldsymbol{k}v(\xi_2)\Delta_F(k)u_s(\mathbf{P}),$$

where

$$\xi_1=\xi(P',P'-k),\quad \xi_2=\xi(P-k,P),$$
$$S_F(p)=(i\boldsymbol{p}-M)/(p^2+M^2-i\epsilon),$$

and

$$\Delta_F(k)=(k^2+\mu^2-i\epsilon)^{-1}.$$

Our purpose is to reduce this to an integral in the 3-dimensional $\mathbf{k}$ space, similar to the type of integral occurring in the static model. Therefore we assume that $v(\xi_1)$ and $v(\xi_2)$ are analytic in the upper half of the k_0 plane, vanishing on the infinite semicircle, and integrate k_0 over a contour closed in the upper half plane.[10] Contributions from the three poles on the negative k_0 axis are thereby obtained.

When the variable $\xi(P_1,P_2)$ is chosen to be

$$\xi(P_1,P_2)=\left[\frac{4M^2+\mu^2+(P_2+P_1)^2}{4M\mu}\right]^2+\frac{\mu^2+(P_2-P_1)^2}{\mu^2}, \tag{13}$$

very simple results are obtained for the contributions of these three poles and the poles appearing from the other diagrams. In fact, this choice leads directly to the static model in the no-recoil limit.

[10] It is not implied that this simple behavior is to be expected for the vertex function arising from any realistic, relativistic theory. These assumptions are introduced only as a prescription for reproducing the results of the static model.

An expansion in powers of $|\mathbf{k}|/M$ and $|\mathbf{q}|/M$ is now possible if it is assumed that at each pole

$$v(\xi)\approx 0, \tag{14}$$

except for

$$|k|^2/M^2 \ll 1.$$

This rather unrealistic assumption is in the spirit of the static model and will be the basis for our discussion. Evaluation of the diagrams of Figs. 2 and 3 to any order in M^{-1} is now a straightforward but tedious matter. The results to lowest order are presented in Sec. 7 and some of the first order corrections in Sec. 8.

4. CONSEQUENCES OF GAUGE INVARIANCE

It is immediately recognized that the diagrams in Figs. 2 and 3 do not present a gauge invariant result. As long as the black boxes represent anything other than a local pseudoscalar interaction, diagrams of the type shown in Fig. 4 are expected to contribute appreciably to the electromagnetic interactions. For example, if the black box represents vertex parts contributed by higher order diagrams, each of the internal lines carrying charge must, in turn, be permitted to produce the photon.

Nevertheless, the gauge invariance condition Eq. (2) *is* satisfied by the matrix elements obtained from Figs. 2 and 3. Indeed, it is easily shown that *each of the terms* indicated by Figs. 2(a), 2(b), 2(c) and 3 satisfies this condition, as does the sum of diagrams Figs. 2(d) and 2(e), as long as the function $v(\xi)$ is symmetric under interchange of P_1 and P_2. Note that for a local pseudovector vertex, not even the usual catastrophic terms arising from the momentum dependence are required to satisfy the condition Eq. (2). The reason for this is simply that Eq. (2) follows from covariance under a Lorentz transformation, as remarked in Sec. 2, and each of these terms satisfies separately the covariance condition. These remarks serve to illustrate the weakness of Eq. (2).

To satisfy the requirements of strong gauge invariance, the behavior of the photon vertex off the nucleon mass shell must be considered. Then it is found that the diagrams indicated in Fig. 4 are required. We shall give a prescription for writing down terms of the type indicated in Fig. 4 for an arbitrary vertex function $v(\xi)$. However, there is no unique prescription for determining these terms, there are always many ways to satisfy the condition of gauge invariance. That given

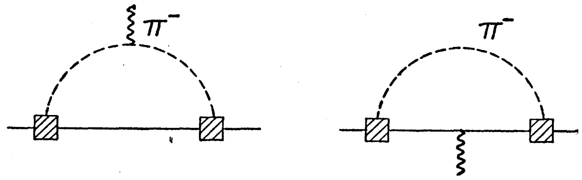

FIG. 3. Diagrams representing the electromagnetic interaction of the neutron.

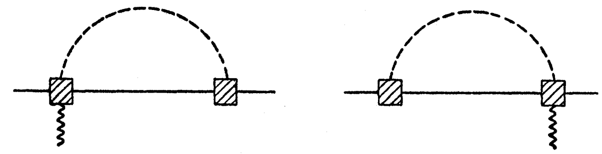

FIG. 4. Vertex current contributions to the nucleon form factors.

here seems to be the simplest, it makes use of line currents which are a natural generalization[11] of the usual substitution of $P_\mu-(e/c)A_\mu$ for P_μ.

The currents to be introduced may be described in configuration space as currents flowing along a line connecting two points in space-time which are coupled by the nonlocal pion-nucleon vertex function. In this way it is seen that they serve to transfer the electric charge across the nonlocal vertex. We shall not go into this representation in configuration space here, but merely remark that the form of the line is arbitrary. It will be taken to be a straight line for the sake of simplicity. The corresponding matrix element is given below in momentum representation and we shall simply demonstrate that the term used does indeed satisfy the condition of strong gauge invariance.

If we write

$$\langle P',s'|j_\mu(0)|P,s\rangle=\frac{4\pi ief^2}{(2\pi)^7}\frac{M}{(P_0P_0')^{\frac{1}{2}}}\bar{u}_{s'}(\mathbf{P}')J_\mu u_s(\mathbf{P}), \tag{15}$$

the contributions to J_μ from the diagrams in Fig. 4 are as follows: For the proton with intermediate π^0,

$$\begin{aligned}J_\mu{}^L(\pi^0)=\int d^4k\Big\{\gamma_5\Big[\int_0^1 d\alpha\Big(\frac{\partial}{\partial P_\mu'}+\frac{\partial}{\partial P_\mu}\Big)\\ \times g(P'-\alpha q,\, P-k+(1-\alpha)q)\Big]\\ \times S_F(P-k)\gamma_5 g(P-k,\, P)\\ +\gamma_5 g(P',\, P'-k)S_F(P'-k)\gamma_5\\ \times\Big[\int_0^1 d\alpha\Big(\frac{\partial}{\partial P_\mu'}+\frac{\partial}{\partial P_\mu}\Big)\\ \times g(P'-k-\alpha q,\, P+(1-\alpha)q)\Big]\Big\}\Delta_F(k);\end{aligned} \tag{16a}$$

for the proton with intermediate π^+,

$$\begin{aligned}J_\mu{}^L(\pi^+)=2i\int d^4k\Big\{\gamma_5\Big[\int_0^1 d\alpha\,\frac{\partial}{\partial P_\mu'}g(P'-\alpha q,\, P-k)\Big]\\ \times S_F(P-k)\gamma_5 g(P-k,\, P)\\ +\gamma_5 g(P',\, P'-k)S_F(P'-k)\gamma_5\\ \times\Big[\int_0^1 d\alpha\,\frac{\partial}{\partial P_\mu}g(P'-k,\, P+(1-\alpha)q)\Big]\Big\}\Delta_F(k);\end{aligned} \tag{16b}$$

[11] R. G. Sachs, Phys. Rev. 74, 433 (1948); 75, 1605 (1949).

and, finally, for the neutron with intermediate π^-,

$$J_\mu{}^L(\pi^-)=2\int d^4k\bigg\{\gamma_5\bigg[\int_0^1 d\alpha\frac{\partial}{\partial P_\mu}$$
$$\times g(P',P-k+(1-\alpha)q)\bigg]S_F(P-k)$$
$$\times\gamma_5 g(P-k,P)+\gamma_5 g(P',P'-k)S_F(P'-k)\gamma_5$$
$$\times\bigg[\int_0^1 d\alpha\frac{\partial}{\partial P_\mu{}'}g(P'-k-\alpha q,P)\bigg]\bigg\}\Delta_F(k). \quad (16c)$$

The definition of the function g is

$$g(p_2,p_1)=(p_2-p_1)v(\xi), \quad (17)$$

so that the indicated differentiations of the factor (p_2-p_1) lead to the usual "catastrophic" terms associated with the pseudovector coupling, while the derivatives of $v(\xi)$ provide the pure line currents.

The test of strong gauge invariance may be taken to be

$$q_\mu J_\mu=0, \quad (18)$$

where J_μ includes all contributions other than that of Fig. 2(a). The point is that Eq. (18) applies off the mass shell for the nucleons. It will be found to be equivalent to the generalized Ward identity. Before applying this test, it is helpful to take account of mass renormalization, which also has an effect on the electromagnetic interactions.

5. MASS RENORMALIZATION

Since our procedure is analogous to the lowest order perturbation diagram in meson theory, the mass renormalization consists merely of the subtraction of a term having the form suggested by the diagram Fig. 5. We may make the usual Dyson expansion of this diagram

$$\Sigma(p)=A+(ip+M)B+(ip+M)^2\Sigma_f(p), \quad (19)$$

where A and B are constants. Insertion of $\Sigma(p)$ between external lines leaves only the constant A to be subtracted. However, we wish to perform the renormalization without reference to the external lines, that is, in J_μ. Hence, we introduce the renormalization diagram Fig. 6, where the $\times$ denotes the momentum-dependent mass correction

$$\delta M=-[A+(ip+M)B]. \quad (20)$$

The introduction of the electromagnetic interaction now must be made in such a way as to maintain the gauge invariance of this momentum-dependent mass correction. Since δM is in effect a nonlocal operator, this can again be accomplished by means of line currents. Hence, in the one-photon matrix element for the proton, we must introduce terms corresponding to the diagrams of Fig. 7, where the contribution to J_μ of Fig. 7(c) is just

$$J_\mu{}^{\delta M}=-\frac{\partial}{\partial P_\mu{}'}\int_0^1 d\alpha\,\Sigma(P'-\alpha q), \quad (21)$$

and

$$\Sigma(p)=3\int d^4k\,\gamma_5 g(P,P-k)S_F(P-k)$$
$$\times\gamma_5 g(P-k,P)\Delta_F(k) \quad (22)$$

may be expanded in the form of Eq. (19). The term in Σ_f does not contribute to the matrix element, Eq. (15), but it is required in Eq. (21) for strong gauge invariance. There is no contribution of the type $J_\mu{}^{\delta M}$ for the neutron.

It is immediately evident that diagrams Fig. 7(a) and Fig. 7(b) cancel those of Figs. 2(d) and 2(e), respectively. Hence, the sum of all proton diagrams except Fig. 2(a) is given by inserting into Eq. (15), the

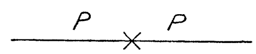

FIG. 6. Mass correction.

expression

$$J_\mu(\text{proton})=J_\mu{}^N(\pi^0)+J_\mu{}^\pi(\pi^+)+J_\mu{}^L(\pi^0)$$
$$+J_\mu{}^L(\pi^+)+J_\mu{}^{\delta M}, \quad (23a)$$

where the last three terms are given by Eqs. (16) and (21) while the nucleonic contribution [Fig. 2(c)] is

$$J_\mu{}^N(\pi^0)=i\int d^4k\,\gamma_5 g(P',P'-k)S_F(P'-k)$$
$$\times\gamma_\mu S_F(P-k)\gamma_5 g(P-k,P)\Delta_F(k), \quad (24a)$$

and the pionic contribution [Fig. 2(b)] is

$$J_\mu{}^\pi(\pi^+)=-2\int d^4p\,\gamma_5 g(P',p)S_F(p)\gamma_5 g(p,P)$$
$$\times\Delta_F(P'-p)(P_\mu{}'+P_\mu-2p_\mu)\Delta_F(P-p). \quad (24b)$$

For the neutron, the sum of the diagrams in Figs. 3 and 4 is

$$J_\mu(\text{neutron})=J_\mu{}^N(\pi^-)+J_\mu{}^\pi(\pi^-)+J_\mu{}^L(\pi^-), \quad (23b)$$

where $J_\mu{}^L(\pi^-)$ is given by Eq. (16c) and

$$J_\mu{}^N(\pi^-)=2J_\mu{}^N(\pi^+), \quad (24c)$$

$$J_\mu{}^\pi(\pi^-)=-J_\mu{}^\pi(\pi^+). \quad (24d)$$

The condition of strong gauge invariance, Eq. (18)

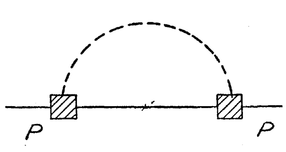

FIG. 5. Self-energy contribution to the nucleon mass.

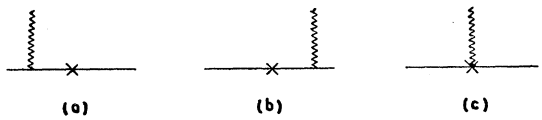

FIG. 7. Contributions of the mass renormalization to the electromagnetic interaction of the proton.

can now be easily verified by noting that

$$(i\not{p}+M)S_F(p)=-1,$$

whence

$$S_F(P'-k)i\mathbf{q}S_F(P-k)=S_F(P'-k)-S_F(P-k).$$

Similarly

$$q_\mu\Delta_F(P'-p)(P_\mu'+P_\mu-2p_\mu)\Delta_F(P-p) =\Delta_F(P-p)-\Delta_F(P'-p).$$

The following relationships are also used:

$$\int_0^1 d\alpha\, q_\mu\left(\frac{\partial}{\partial P_\mu'}+\frac{\partial}{\partial P_\mu}\right)\Phi(P'-\alpha q,\, P+(1-\alpha)q) =-\int_0^1 d\alpha\,\frac{d\Phi}{d\alpha}=\Phi(P',P')-\Phi(P,P),$$

$$\int_0^1 d\alpha\, q_\mu\frac{\partial}{\partial P_\mu'}\Phi(P'-\alpha q,\, P)=\Phi(P',P)-\Phi(P,P),$$

and

$$\int_0^1 d\alpha\, q_\mu\frac{\partial}{\partial P_\mu}\Phi(P',\, P+(1-\alpha)q)=\Phi(P',P')-\Phi(P',P),$$

where $\Phi(x,y)$ is an arbitrary function of the two 4-dimensional variables x, y.

Since

$$q_\mu J_\mu^{\delta M}=-q_\mu\frac{\partial}{\partial P_\mu'}\int_0^1 d\alpha\,\Sigma(P'-\alpha q)=\Sigma(P)-\Sigma(P'),$$

we see that strong gauge invariance for the proton is equivalent to the generalized Ward identity

$$(P_\mu'-P_\mu)\Lambda_\mu(P',P)=\Sigma(P')-\Sigma(P), \tag{25}$$

where $\Lambda_\mu(P',P)$ is the photon vertex function

$$\Lambda_\mu(P',P)=J_\mu{}^N(\pi^0)+J_\mu{}^\pi(\pi^+)+J_\mu{}^L(\pi^0)+J_\mu{}^L(\pi^+). \tag{26}$$

According to Eqs. (21) and (19)

$$\bar{u}_{s'}(\mathbf{P}')J_\mu{}^{\delta M}u_s(\mathbf{P})=-iB\bar{u}_{s'}(\mathbf{P}')\gamma_\mu u_s(\mathbf{P}). \tag{27}$$

Comparison of Eq. (15) with Eq. (1) therefore yields

$$\bar{u}_{s'}(\mathbf{P}')[F_1(q^2)\gamma_\mu-F_2(q^2)\sigma_{\mu\nu}q_\nu]u_s(\mathbf{P}) =4\pi f^2\bar{u}_{s'}(\mathbf{P}')\{\gamma_\mu[1-(2\pi)^{-4}iB] +(2\pi)^{-4}\Lambda_\mu(P',P)\}u_s(\mathbf{P}). \tag{28}$$

But the Ward identity

$$\Lambda_\mu(P,P)=\partial\Sigma/\partial P_\mu$$

combined with Eq. (19) shows that

$$\bar{u}_{s'}(\mathbf{P}')\Lambda_\mu(P,P)u_s(\mathbf{P})=iB\bar{u}_{s'}(\mathbf{P}')\gamma_\mu u_s(\mathbf{P}),$$

whence it follows from Eq. (28) that $F_1(0)=1$, in accordance with the required normalization for the proton, Eq. (4). To circumvent the calculation of B, we may first obtain an unrenormalized ${}^uF_1(q^2)$ defined by

$$\bar{u}_{s'}(\mathbf{P}')[{}^uF_1(q^2)\gamma_\mu-\sigma_{\mu\nu}q_\nu\, F_2(q^2)]u_s(\mathbf{P}) =4\pi^2(2\pi)^{-4}f^2\bar{u}_{s'}(\mathbf{P}')\Lambda_\mu(P',P)u_s(\mathbf{P}), \tag{29}$$

and then determine $F_1(q^2)$ from

$$F_1(q^2)={}^uF_1(q^2)-{}^uF_1(0)+1. \tag{30}$$

For the neutron, the condition of strong gauge invariance is satisfied by Eq. (23b) without the introduction of a mass normalization term. Hence, Eq. (25) is replaced by

$$(P_\mu'-P_\mu)\Lambda_\mu(P',P)=0, \tag{31}$$

and Eq. (4) follows directly by noting that Eq. (31) is equivalent to setting $B=0$, although B does not vanish, of course, for the mass renormalization of the neutron.

6. CHOICE OF REFERENCE FRAME

It is convenient to perform the computations in the Breit frame defined by the condition

$$\mathbf{P}'=-\mathbf{P}. \tag{32}$$

In this case the left-hand side of Eq. (29) can be simplified by making use of the Dirac equation to give

$$i\,{}^uF_{\mathrm{ch}}(q^2)=4\pi(2\pi)^{-4}f^2\bar{u}(-\mathbf{P})\Lambda_4(P',P)u(\mathbf{P}), \tag{33a}$$

and

$$i(\boldsymbol{\sigma}\times\mathbf{q})\;{}^uF_{\mathrm{mag}}(q^2) =4\pi(2\pi)^{-4}f^2\bar{u}(-\mathbf{P})\boldsymbol{\Lambda}(P',P)u(\mathbf{P}), \tag{33b}$$

where a 2×2 spin matrix notation has been substituted for the labeling of the spin states. Here

$${}^uF_{\mathrm{ch}}(q^2)={}^uF_1(q^2)-(q^2/2M)F_2(q^2),$$

and

$${}^uF_{\mathrm{mag}}(q^2)=(1/2M)\,{}^uF_1(q^2)+F_2(q^2)$$

are the unrenormalized charge and magnetic form factors defined in accordance with Eqs. (8) and (9). The renormalized form factors are then found from Eq. (30) to be

$$F_{\mathrm{ch}}(q^2)={}^uF_{\mathrm{ch}}(q^2)-{}^uF_{\mathrm{ch}}(0)+1, \tag{34a}$$

and

$$F_{\mathrm{mag}}(q^2)={}^uF_{\mathrm{mag}}(q^2)+(1/2M)[1-{}^uF_{\mathrm{ch}}(0)]. \tag{34b}$$

7. THE STATIC LIMIT

The calculation of $F_{\mathrm{ch}}(q^2)$ and $F_{\mathrm{mag}}(q^2)$ on the basis of Eqs. (33), (26), (24), and (16) may now be carried out by the method outlined in Sec. 3, which produces an expansion in powers of M^{-1}. In this section, attention

is limited to the lowest order, or static terms. The results are as follows (we set $\mu=1$ as well as $\hbar=c=1$):

For the neutron,

$$[F_{\mathrm{ch}}^{(0)}(q^2)]_{\mathrm{neutron}}=-\frac{f^2}{\pi^2}\int d^3k\bigg[\bigg(\mathbf{k}^2-\frac{\mathbf{q}^2}{4}\bigg)\times\frac{v_+v_-}{\omega_+\omega_-(\omega_++\omega_-)}-\frac{1}{2}\frac{\mathbf{k}^2v_0^2}{\omega^3}\bigg],\quad(35\mathrm{a})$$

where

$$\omega_\pm=[1+|\mathbf{k}\pm\tfrac{1}{2}\mathbf{q}|^2]^{\frac{1}{2}},\qquad \omega=[1+\mathbf{k}^2]^{\frac{1}{2}}.\quad(36)$$

$$v_\pm=v(\omega_\pm^2);\quad v_0=v(\omega^2).\quad(37)$$

Note that $q^2=\mathbf{q}^2$ in the Breit frame. It is found immediately that for $q^2=0$

$$[F_{\mathrm{ch}}^0(0)]_{\mathrm{neutron}}=0,$$

in accordance with Eq. (4) [note that Eq. (8) gives $F_{\mathrm{ch}}(0)=F_1(0)$, in general]. Also

$$[F_{\mathrm{mag}}^{(0)}(q^2)]_{\mathrm{neutron}}=-\frac{f^2}{2\pi^2}\int d^3k\bigg\{\bigg[\mathbf{k}^2-\frac{(\mathbf{q}\cdot\mathbf{k})^2}{q^2}\bigg]\times\frac{v_+v_-}{\omega_+^2\omega_-^2}\bigg\}.\quad(38\mathrm{a})$$

For the proton,

$$[{}^uF_{\mathrm{ch}}^{(0)}(q^2)]_{\mathrm{proton}}=\frac{f^2}{\pi^2}\int d^3k\bigg((\mathbf{k}^2-q^2/4)\times\frac{v_+v_-}{\omega_+\omega_-(\omega_++\omega_-)}+\frac{1}{4}\frac{\mathbf{k}^2}{\omega^3}v_0^2-4\int_0^1 d\alpha\frac{v_0}{\omega}\{\tfrac{1}{2}\mathbf{k}^2v_0'+[\mathbf{k}^2+\alpha(\mathbf{q}\cdot\mathbf{k})]v_\alpha'\}\bigg),\quad(35\mathrm{b})$$

where

$$v_\alpha=v(1+[\mathbf{k}+\alpha\mathbf{q}]^2),\qquad v_\alpha'=v'(1+[\mathbf{k}+\alpha\mathbf{q}]^2),\quad(39)$$

and

$$v'(\xi)=dv/d\xi.$$

The quantity required by Eqs. (34) for renormalization is

$$[{}^uF_{\mathrm{ch}}^{(0)}(0)]_{\mathrm{proton}}=\frac{3}{4}\frac{f^2}{\pi^2}\int\frac{d^3k}{\omega}\bigg[\frac{\mathbf{k}^2}{\omega^2}v_0^2-8\mathbf{k}^2v_0v_0'\bigg].\quad(40)$$

The second term can be integrated by parts to yield

$$[{}^uF_{\mathrm{ch}}^{(0)}(0)]_{\mathrm{proton}}=\frac{3}{4}\frac{f^2}{\pi^2}\int d^3k\frac{v_0^2}{\omega}\bigg[6-\frac{\mathbf{k}^2}{\omega^2}\bigg].\quad(41)$$

It can be seen from Eq. (34b) that the renormalization of F_{mag} is of order M^{-1}, hence it is not to be included in the static limit. Then, for the proton,

$$[F_{\mathrm{mag}}^{(0)}(q^2)]_{\mathrm{proton}}=-[F_{\mathrm{mag}}^{(0)}(q^2)]_{\mathrm{neutron}},\quad(38\mathrm{b})$$

where the expression on the right-hand side may be obtained from Eq. (38a).

In Eqs. (35) and (38), the terms proportional to v_+v_- are the contributions of the pion current J_μ^π. They are identical with the Born terms obtained by Walecka[9] in his calculation of the form factors in the static model. The terms proportional to v_0^2 [in Eqs. (35) only] arise from the nucleon current J_μ^N and they are independent of $\mathbf{q}$. Hence, their role is just to maintain charge conservation. The line-current contributions are easily recognized by their α dependence. The only such term in the static approximation occurs for $F_{\mathrm{ch}}^{(0)}$ of the proton.[12]

From $F_{\mathrm{ch}}^{(0)}$ we may now calculate the second moment of the charge distribution $(\mathfrak{R}^2)^{(0)}=-6(dF_{\mathrm{ch}}^{(0)}/dq^2)_{q^2=0}$ in the static limit. Since the form factor is the same, the result for the neutron is identical with that obtained in the static model by Salzman,[4] for example:

$$(\langle\mathfrak{R}^2\rangle^{(0)})_{\mathrm{neutron}}=-\frac{f^2}{\pi}\int_0^\infty dk\frac{\mathbf{k}^4}{\omega^7}\bigg[2\omega^4\bigg(\frac{dv_0}{dk}\bigg)^2+5(3\omega^2-2k^2)v_0^2\bigg].\quad(42\mathrm{a})$$

This is the anticipated result which shows that the second moment of the pion charge distribution calculated in the static model is to be compared to the complete neutron-electron interaction without subtracting the Foldy turn. The static model therefore gives much closer agreement with experiment than had previously been thought. For the usual choice of a cutoff close to the nucleon mass, the calculated value is about twice the experimental value.[13] However, it is to be noted that, with this high a cutoff, the neglect of recoil terms is clearly not justified for our procedure.

The second moment of the proton charge distribution is made up of two terms, one is the pionic contribution which is the negative of the neutron second moment and the other is the line-current contribution:

$$(\langle\mathfrak{R}^2\rangle^{(0)})_{\mathrm{proton}}=-(\langle\mathfrak{R}^2\rangle^{(0)})_{\mathrm{neutron}}+4\frac{f^2}{\pi}\int_0^\infty dk\frac{\mathbf{k}^2}{\omega^7}\times\bigg[5v_0^2+\omega^4(1+\tfrac{2}{3}\omega^2)\bigg(\frac{dv_0}{dk}\bigg)^2\bigg].\quad(42\mathrm{b})$$

It is clear that the second term, due entirely to the line currents, is somewhat larger than the first.

It has been remarked by Yennie, Lévy, and Ravenhall[2] that application of the mirror theorem[5] to $\langle\mathfrak{R}^2\rangle$ indicates that the isoscalar quantity $\langle\mathfrak{R}^2\rangle_{\mathrm{proton}}$

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[12] To demonstrate that the line-current contributions to $F_{\mathrm{mag}}^{(0)}$ vanish, an integration by parts in $\cos\theta=(\mathbf{k}\cdot\mathbf{q})/|\mathbf{k}||\mathbf{q}|$ has been performed. The fact that these terms vanish is probably related to the fact that the line currents flow radially in the static model and therefore do not give rise to any magnetic multipole moments. See R. H. Capps and W. G. Holladay, Phys. Rev. **99**, 931 (1955).

[13] S. Treiman and R. G. Sachs, Phys. Rev. **103**, 435 (1956).

$+\langle \mathcal{R}^2\rangle_{\text{neutron}}$ depends only on contributions from the nucleon (core) charge distribution. Hence, the line-current contribution is presumably nucleonic in origin. For a usual value of the cutoff, its order of magnitude agrees with the experimental value[2]

$$[\langle \mathcal{R}^2\rangle_{\text{proton}}+\langle \mathcal{R}^2\rangle_{\text{neutron}}]^{\frac{1}{2}}\approx 0.75\times 10^{-13}\text{ cm},$$

which is, of course, a surprisingly large number. Evidently, in the process of introducing a cutoff, the requirements of charge conservation are such as to cause a large shift in the electric charge, even if the cutoff is limited to a small spatial region. In this connection, it should be remarked that the line-current contribution is obtained in the form given in Eq. (42b) only after several integrations by parts. As can be seen from Eq. (35b), the contribution originally involves at least one derivative of the source function, hence for a constant v, the line-current term would vanish.

From Eq. (38b) it is evident that only pionic contributions to the magnetic form factor occur in the static limit. The neutron and proton magnetic moments are oppositely equal in this approximation and they agree exactly with the values obtained in the usual static model. Hence, the magnetic moments are identical with those obtained from the static model. As Miyazawa has shown,[14] these may be brought into reasonable agreement with the isotopic vector part of the observed nucleon moment by including rescattering corrections.

8. RECOIL CORRECTIONS TO THE MAGNETIC MOMENT

Although it is actually a recoil effect, it is customary in applications of the static model to include a nucleon (core) spin contribution to the magnetic moment. This has been a constant source of failure for the static model since it leads to much too large a value of the isoscalar moment.[14] However, it has often been suggested that a consistent treatment of recoil effects would overcome this difficulty. We have now made a consistent calculation on the basis of the procedure outlined above with the following results:

For the neutron, the contribution of the pion current is

$$[\Delta_\pi F_{\text{mag}}^{(1)}(0)]_{\text{neutron}}=\frac{1}{6M}\frac{f^2}{\pi^2}\int d^3k\frac{\mathbf{k}^2}{\omega}\times\left[\frac{3}{2}\frac{\mathbf{k}^2}{\omega^4}v_0{}^2-2v_0v_0'\right],\quad (43\text{a})$$

that of the nucleon (core) current is

$$[\Delta_N F_{\text{mag}}^{(1)}(0)]_{\text{neutron}}=-\frac{1}{12M}\frac{f^2}{\pi^2}\int d^3k\frac{\mathbf{k}^2}{\omega^3}v_0{}^2,\quad (44\text{a})$$

and that of the line current is

$$[\Delta_L F_{\text{mag}}^{(1)}(0)]_{\text{neutron}}=-\frac{1}{6M}\frac{f^2}{\pi^2}\int\frac{d^3k}{\omega}\times\left[\left(3+\frac{\mathbf{k}^2}{\omega^2}\right)v_0{}^2+2\mathbf{k}^2v_0v_0'\right].\quad (45\text{a})$$

The total $1/M$ effect on the neutron moment is therefore

$$[F_{\text{mag}}^{(1)}(0)]_{\text{neutron}}=-\frac{1}{2M}\frac{f^2}{\pi^2}\int\frac{d^3k}{\omega}\times\left[\left(1+\frac{1}{2}\frac{\mathbf{k}^2}{\omega^4}\right)v_0{}^2+\tfrac{4}{3}\mathbf{k}^2v_0v_0'\right].\quad (46\text{a})$$

For the proton, the pion contribution and the nucleon contributions to the unrenormalized form factors are related to those of the neutron as follows:

$$[\Delta_\pi\, {}^uF_{\text{mag}}(0)]_{\text{proton}}=-[\Delta_\pi\, {}^uF_{\text{mag}}^{(1)}(0)]_{\text{neutron}},\quad (43\text{b})$$

and

$$[\Delta_N\, {}^uF_{\text{mag}}^{(1)}(0)]_{\text{proton}}=\tfrac{1}{2}[\Delta_N{}^uF_{\text{mag}}^{(1)}(0)]_{\text{neutron}}.\quad (44\text{b})$$

The line-current contribution is

$$[\Delta_L\, {}^uF_{\text{mag}}^{(1)}(0)]_{\text{proton}}=\frac{1}{6M}\frac{f^2}{\pi^2}\int\frac{d^3k}{\omega}\times\left[\left(3+\frac{\mathbf{k}^2}{\omega^2}\right)v_0{}^2+8\mathbf{k}^2v_0v_0'\right].\quad (45\text{b})$$

Also for the proton, there are the additional terms produced by renormalization in accordance with Eq. (34b):

$$F_{\text{mag}}^{(1)}(0)={}^uF_{\text{mag}}^{(1)}(0)+\frac{1}{2M}[1-{}^uF_{\text{ch}}^{(0)}(0)],\quad (47)$$

where ${}^uF_{\text{ch}}^{(0)}(0)$ is given by Eq. (40) or Eq. (41). In this way, we find for the total recoil correction to the magnetic moment of the proton:

$$[F_{\text{mag}}^{(1)}(0)]_{\text{proton}}=\frac{1}{2M}+\frac{1}{2M}\frac{f^2}{\pi^2}\int\frac{d^3k}{\omega}\times\left[\left(\frac{1}{2}\frac{\mathbf{k}^2}{\omega^4}+1-\frac{1}{2}\frac{\mathbf{k}^2}{\omega^2}\right)v_0{}^2+\frac{14}{3}\mathbf{k}^2v_0v_0'\right].\quad (46\text{b})$$

The order of magnitude of the recoil correction to the neutron or proton moment is a sensitive (quadratic) function of the cutoff in v_0. For the usual choice of a cutoff close to the nucleon mass, the correction is very large, of the order of several nuclear magnetons. This is not very surprising since the expansion in powers of $1/M$ is based on the assumption that the cutoff is much smaller than M. Hence we come to the same

[14] H. Miyazawa, Phys. Rev. **101**, 1564 (1956).

conclusion as Walecka[9] and others, that the static model is a very poor approximation in view of the probable importance of the recoil corrections. The apparent agreement with experiment (of the isovector magnetic moment, for example) is evidently fortuitous.

Despite this conclusion, it is of some interest to consider the sum of neutron and proton moments. From Eqs. (38b), (46a), and (46b), we find to order $1/M$,

$$[F_{\rm mag}(0)]_{\rm proton}+[F_{\rm mag}(0)]_{\rm neutron} = (1/2M)[1-\tfrac{4}{3}\,{}^{u}F_{\rm ch}(0)], \quad (48)$$

where ${}^{u}F_{\rm ch}(0)$ is given by Eq. (40). This is evidently the same result obtained from the mirror theorem in the static model,[5] since ${}^{u}F_{\rm ch}(0)$ may be interpreted as the probability for finding the nucleon accompanied by a pion field. The success of the static model in this respect is not very surprising since the derivation of the result relied mainly on the conservation of angular momentum.

Because of its general nature, Eq. (48) poses a serious problem for any theory of the electromagnetic form factors. From the observed magnetic moments it is found by means of Eq. (48) that

$$[{}^{u}F_{\rm ch}(0)]\approx 0.09. \quad (49)$$

If one calculates ${}^{u}F_{\rm ch}{}^{(0)}$ from Eq. (41) using the accepted value of $f^2\approx 0.08$ and a cutoff corresponding to the nucleon mass, a much larger value than that given by Eq. (49) is obtained. Hence, this is a clear point of failure of the first-order recoil corrections to the static model.

9. CONCLUSION

The importance of taking into account the condition of strong gauge invariance in this work suggests that it must be kept in mind in any attempt to calculate electromagnetic form factors. We have found, for example, that the separation of the contribution to the matrix element of the pion current from that of the nucleon current is not really gauge invariant although it satisfies the condition of weak gauge invariance. The latter condition was noted by Federbush, Goldberger, and Treiman[15] in connection with the corresponding separation of the integral equations arising from dispersion relations. Therefore, the gauge invariance of their approximate form of the integral equations is subject to serious doubt.

Furthermore, any other mutilation of the exact equations is subject to doubt on the same grounds. Such approximations provide integral equations involving strong (pion-nucleon, etc.) vertex functions and we have seen how the condition of charge conservation requires that a means must be provided for transferring the charge across such a nonlocal vertex. The importance of this consideration is illustrated by the result that the charge radius of the proton turns out to be very large through the influence of line currents. The line currents are just a manifestation of the condition of charge conservation. They represent the electromagnetic effects of higher order diagrams. In this connection, it is interesting that even for a cutoff near the nucleon mass, the line-current contribution to the charge form factor of the proton is most significant. This indicates that the electromagnetic contributions of intermediate particles of large mass may, through conditions imposed by gauge invariance, be comparable to the pion contribution, in spite of the large denominators appearing with such terms.

[15] P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958).

APPENDIX

We wish to demonstrate that the time reversal properties of $j_\mu(x)$ are such that the matrix element $\langle P',s'|j_\mu(0)|P,s\rangle$ cannot contain a term of the form $\bar{u}_{s'}(\mathbf{P}')q_\mu F_3(q^2)u_s(\mathbf{P})$. Under the Wigner time inversion operator K, the transformation of the current density is

$$\begin{aligned} K\mathbf{j}(0)K^{-1}&=-\mathbf{j}(0),\\ K\rho(0)K^{-1}&=\rho(0), \end{aligned} \quad (\text{A-1})$$

where $\rho(x)=-ij_4(x)$. Phases of the state vectors are chosen so that

$$K|P,s\rangle=|P_-,-s\rangle, \quad (\text{A-2})$$

where $P_-=(-\mathbf{P},P_0)$ if $P=(\mathbf{P},P_0)$. Since K is an antiunitary operator, and $\mathbf{j}(0)$, $\rho(0)$ are Hermitian operators, we may write

$$\begin{aligned} \langle K(P',s')|K\mathbf{j}(0)|P,s\rangle&=\langle P,s|\mathbf{j}(0)|P',s'\rangle,\\ \langle K(P',s')|K\rho(0)|P,s\rangle&=\langle P,s|\rho(0)|P',s'\rangle. \end{aligned} \quad (\text{A-3})$$

But also

$$\begin{aligned} \langle K(P',s')|K\mathbf{j}(0)|P,s\rangle&=\langle K(P',s')|K\mathbf{j}(0)K^{-1}K|P,s\rangle\\ &=-\langle K(P',s')|\mathbf{j}(0)K|P,s\rangle, \end{aligned}$$

by Eq. (A-1). Then from Eqs. (A-2) and (A-3) we have

$$\langle P,s|\mathbf{j}(0)|P',s'\rangle=-\langle P_-',-s'|\mathbf{j}(0)P_-,-s\rangle. \quad (\text{A-4})$$

The corresponding result for $\rho(0)$ is

$$\langle P,s|\rho(0)|P',s'\rangle=\langle P_-',-s'|\rho(0)|P_-,-s\rangle. \quad (\text{A-5})$$

It is now easily seen by means of a straightforward substitution that $\bar{u}_{s'}(\mathbf{P}')q_\mu u_s(\mathbf{P})$ does not satisfy the conditions (A-4) and (A-5) while Eq. (1) does do so.