Electron Scattering in High Magnetic Field

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Electrical conductivity in a strong magnetic field is calculated for the case of scattering by delta-function impurities. The impurity concentration is taken as sufficiently weak that collision broadening may be neglected. The scattering by an individual center is solved exactly rather than by perturbation theory. As a result, transition probabilities for an electron at the bottom of a Landau level vanish, rather than diverge. Expressions are given for the longitudinal and transverse conductivities in the oscillatory range, and in the quantum limit range for degenerate and nondegenerate statistics. The relation of this theory to those employing collision broadening is discussed.

I. INTRODUCTION

The transverse motion of an electron in a magnetic field is greatly restricted; hence, in a qualitative sense, one may view the free motion of the electron as being one-dimensional, along the direction of the field. The density of states is profoundly altered and this manifests itself in the oscillatory variation with field of the diamagnetic susceptibility, and the galvano- and thermomagnetic coefficients.[1] However, we feel that the one-dimensional aspect of the electronic motion has not been given sufficient attention in the study of the scattering of conduction electrons by imperfections. One notes that first-order time-dependent perturbation theory, applied to a one-dimensional elastic scattering problem, predicts infinite transition rates for a very slow incoming electron, in violation of conservation theorems. A correct solution would have the reflection coefficient approach unity.

In recent work in the quantum theory of magnetoconductivity, the scattering rates have been approximated by first order time-dependent perturbation theory.[2–5] This approximation has had the defect in that for electrons with small propagation vectors along the magnetic field, it predicted diverging scattering rates for elastic scattering. This, in turn, gave rise to an infinite transverse conductivity. Cutoff procedures or theories of collision broadening were invoked to obtain meaningful results.[4,6] No quantitative theories of collision broadening have been produced.

In this paper attention will be given to the situation in which the density of scattering centers is so low that collision broadening may be neglected. Finite conductivity is obtained provided that one uses an improvéd treatment of the scattering by an individual center. The model for the scattering potential will be taken as a random array of delta functions. While this is, perhaps, an oversimplified choice, it does lead to solutions in closed form and suggests methods for treating the more difficult cases. This model was used in the analysis of the magnetoresistance of bismuth.[6] Also, it is probable that such quantities as field dependence of magnetoresistance, determined in this way, may be of interest in the analysis of experimental data. The calculations will be performed for an electron with a constant, isotropic effective mass.

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[2] S. Titeica, Ann. Physik **22**, 128 (1935).
[3] P. N. Argyres and E. N. Adams, Phys. Rev. **104**, 900 (1956).
[4] E. N. Adams and T. D. Holstein, J. Phys. Chem. Solids **10**, 254 (1959).
[5] G. E. Zilberman, J. Exptl. Theoret. Phys. (U.S.S.R.) **29**, 762 (1956) [translation: Soviet Phys.-JETP **2**, 650 (1956)].
[6] B. Davydov and I. Pomeranchuk, J. Phys. (U.S.S.R.) **2**, 147 (1940).

II. WAVE FUNCTIONS AND THE SCATTERING PROBLEM

The Hamiltonian for an electron in a magnetic field is[7]

$$\mathcal{H}_0=(1/2m)[\mathbf{p}+(e/c)\mathbf{A}]^2, \tag{2.1}$$

in which m is the effective mass, $\mathbf{p}$ the canonical momentum, e the numerical value of the electronic charge, and $\mathbf{A}$ the vector potential. Electron spin will be neglected in this treatment.[5] The kinetic momentum $m\mathbf{v}$ is related to $\mathbf{p}$ by

$$m\mathbf{v}=\mathbf{p}+(e/c)\mathbf{A}.$$

We choose the gauge $\mathbf{A}=(0,Hx,0)$, for magnetic field H in the z direction. The Hamiltonian is then

$$\mathcal{H}_0=(1/2m)[p_x^2+(p_y+m\omega x)^2+p_z^2], \tag{2.2}$$

where $\omega=eH/mc$. The eigenfunctions and corresponding eigenvalues are

$$\psi_{nk_yk_z}=[1/(L_yL_z)^{\frac{1}{2}}]\phi_n[x+(\hbar k_y/m\omega)]e^{ik_yy}e^{ik_zz}, \tag{2.3a}$$

$$E_{nk_yk_z}=(\hbar^2k_z^2/2m)+[n+\tfrac{1}{2}]\hbar\omega, \tag{2.3b}$$

where k_y and k_z are real propagation constants, n is a positive integer, and ϕ_n is the one-dimensional harmonic oscillator wave function. The x part of the wave function is centered at position $x_0=-\hbar k_y/m\omega$. The eigenfunctions Eq. (2.3a) are normalized in the region $-\infty<x<\infty$, $0\leqslant y\leqslant L_y$, $0\leqslant z\leqslant L_z$. The density of electronic states per unit energy range—unit volume is given by

$$n(E)=\frac{1}{2}\left(\frac{1}{2\pi}\right)^2\hbar\omega\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}\sum_{n=0}^{\max}\frac{1}{[E-(n+\frac{1}{2})\hbar\omega]^{\frac{1}{2}}}, \tag{2.4}$$

[7] Notation is the same as that of reference 1.

where the summation is over all positive integers for which the expression $n(E)$ is real.

We now construct the Green's function, i.e., the solution of the equation

$$(E-\mathcal{H}_0)G_E(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}').$$

In terms of the eigenfunctions (2.3), we have

$$G_E(\mathbf{r},\mathbf{r}')=\lim_{\eta\to 0}\sum_{nk_yk_z}\frac{\psi_{nk_yk_z}{}^*(\mathbf{r}')\psi_{nk_yk_z}(\mathbf{r})}{E-E_{nk_yk_z}+i\eta}. \quad (2.5)$$

The presence of the $i\eta$ in the denominator ensures that the Green's function will contain waves running outward from the source point.[8] Executing the summation over k_z in Eq. (2.5), we obtain

$$G_E(\mathbf{r},\mathbf{r}')=-\frac{im}{\hbar^2}\sum_{nk_y}^{\max}\frac{e^{ik_y(y-y')}}{L_y}\phi_n\left(x'+\frac{\hbar k_y}{m\omega}\right)\phi_n\left(x+\frac{\hbar k_y}{m\omega}\right)\times\frac{e^{ik_n|z-z'|}}{k_n}, \quad (2.6)$$

where we have used the abbreviation

$$k_n=\{(2m/\hbar^2)[E-(n+\tfrac{1}{2})\hbar\omega]\}^{\frac{1}{2}}, \quad (2.7)$$

for the possible magnitudes of k_z at energy E. Further simplification is possible, but not necessary for our purposes. It is to be noted that the part of the Green's function containing $z-z'$, for a fixed n, is of the same form as that for a particle in one-dimensional motion in the absence of a magnetic field.[9] We also observe that $G_E(\mathbf{r},\mathbf{r})=-i\pi n(E)$.

We now compute the scattering by a delta function potential at the coordinate origin, $V(\mathbf{r})=V_0\delta(\mathbf{r})$. Let the incoming wave be of the form $\psi^0=\psi_{nk_yk_z}$, with energy E. Then the wave function for the scattering state will satisfy the equation

$$\psi(\mathbf{r})=\psi^0(\mathbf{r})+\int G_E(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'. \quad (2.8)$$

With the above potential the solution of Eq. (2.8) is

$$\psi(\mathbf{r})=\psi^0(\mathbf{r})+V_0G_E(\mathbf{r},0)\frac{\psi^0(0)}{1-V_0G_E(0,0)}. \quad (2.9)$$

For more complicated potentials variational methods which retain normalization will prove useful. Introducing the magnetic wave functions and the density of states per unit energy range, we find the scattered wave is given by

$$\psi-\psi^0=-\frac{imV_0}{\hbar^2L_y}\frac{\phi_n(\hbar k_y/m\omega)}{1+i\pi V_0n(E)}\times\sum_{n'k_y'}\phi_{n'}\left(\frac{\hbar k_y'}{m\omega}\right)\phi_{n'}\left(x+\frac{\hbar k_y'}{m\omega}\right)\frac{e^{ik_{n'}|z|}}{k_{n'}}. \quad (2.10)$$

From Eq. (2.10) we see that the probabilities of scattering into states $k_z'=\pm k_{n'}$ are equal, i.e., the scattering is symmetric in $\pm k_z$. The probability per unit time of transition from state nk_yk_z to $n'k_y'k_z'$ is given by the current associated with the $n'k_y'k_z'$ component of the scattered wave, Eq. (2.10). We find for this transition probability

$$W_{nk_yk_z\to n'k_y'k_z'}=\frac{mV_0^2}{L_zL_y^2\hbar^3}\frac{1}{|k_{n'}|}\times\frac{|\phi_n(\hbar k_y/m\omega)|^2|\phi_{n'}(\hbar k_y'/m\omega)|^2}{1+\pi^2V_0^2n(E)^2}\times\delta_{k_z',k_{n'}}. \quad (2.11)$$

Had we used the Born approximation, i.e., the first iteration of Eq. (2.8), the term $\pi^2V_0^2n(E)^2$ in the denominator of Eq. (2.11) would have been absent. It is this term which prevents W from becoming infinite if $k_{n'}\to 0$. A divergence of transition probabilities for vanishingly small wave vector is characteristic of the results obtained when one uses the Born approximation in a one-dimensional problem. This divergence is in no way dependent on the form of the potential. We also note that for large V_0 the transition probability no longer depends on V_0.

In the applications of the following sections we shall be interested in the scattering by a random distribution of impurity centers. It is assumed that for weak impurity concentration we may neglect interference between scattering by different centers. Making this assumption the transition probability per unit time, summed and averaged over the distribution of impurities of concentration n_I becomes

$$W_{nk_yk_z\to n'k_y',\pm k_{n'}}=\frac{n_IV_0^2m}{\hbar^3L_y}\frac{1}{|k_{n'}|}\frac{1}{1+\pi^2V_0^2n(E)^2}\times\int_{-\infty}^{\infty}dX\left|\phi_n\left(X+\frac{\hbar k_y}{m\omega}\right)\right|^2\left|\phi_{n'}\left(X+\frac{\hbar k_y'}{m\omega}\right)\right|^2. \quad (2.12)$$

III. LONGITUDINAL CONDUCTIVITY

The relaxation time for longitudinal conductivity τ_L has been derived by Argyres.[10] It is just the time for the decay of momentum along the direction of the

[8] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Publishing Company, New York, 1953), see Chap. 7.

[9] See reference 8, p. 1071.

[10] P. N. Argyres, J. Phys. Chem. Solids 4, 19 (1958).

magnetic field and is given by

$$\frac{1}{\tau_{nk}}=\sum_{n'k_y'k_z'}\left(1-\frac{k_z'}{k_z}\right)W_{nk_yk_z\to n'k_y'k_z'}. \tag{3.1}$$

This formula was derived on the basis of time-dependent perturbation theory. We shall assume that it is correct when the more exact values of the previous section are used for W. To evaluate τ_L we first note that W vanishes unless $k_z'=\pm k_{n'}$. Then for the scattering by delta functions, we perform the sum on k_z' and obtain

$$\frac{1}{\tau_{nk}}=2\sum_{n'k_y'}W_{nk_yk_z\to n'k_y'k_{n'}}, \tag{3.2}$$

where $k_{n'}$ takes positive values only. Applying Eq. (2.12), we obtain

$$\frac{1}{\tau_L}=n_IV_0^2\frac{2\pi}{\hbar}\frac{n(E)}{1+\pi^2V_0^2n(E)^2}. \tag{3.3}$$

The longitudinal relaxation time is thus found to depend only on the energy. We note that when the energy is very close to $(n+\frac{1}{2})\hbar\omega$, i.e., close to the bottom of a Landau level, $\tau\to\infty$. This follows from the discontinuous behavior of $n(E)$ as may be seen from Eq. (2.4). This result differs significantly from those predicted by the Born approximation, which yields a vanishing τ for such energies.

For the longitudinal conductivity σ_L, one may use a distribution function analysis.[11] The Boltzmann-type equation is

$$-\frac{e}{\hbar}E_z\frac{\partial f}{\partial k_z}=-\frac{f-f_0}{\tau_L(E)}, \tag{3.4}$$

where f is the distribution function over the magnetic states, f_0 its equilibrium value, E_z the applied longitudinal field, and τ_L the relaxation time calculated above. We solve to first order in the applied electric field and calculate the current. The result is[10]

$$\begin{aligned}\sigma_L&=J_z/E_z=(2/L_xL_y)\sum_{nk_yk_z}f(nk_yk_z)(-e\hbar k_z/m)\\&=-\frac{2e^2}{m}\frac{1}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\hbar\omega\sum_n\int_{(n+\frac{1}{2})\hbar\omega}^{\infty}\\&\qquad\times[E-(n+\tfrac{1}{2})\hbar\omega]^{\frac{1}{2}}\tau(E)\frac{\partial f_0}{\partial E}dE.\end{aligned} \tag{3.5}$$

The first factor of 2 in Eq. (3.5) takes account of the electron spin.

The conductivity can be evaluated for several ranges of interest which are specified by the relative magnitudes of the Fermi energy, E_F, $\hbar\omega$, and kT.

(a) Oscillatory range, degenerate statistics, ($E_F\gg kT$, $E_F\gg\hbar\omega$): The conductivity is expanded in powers of $\hbar\omega/E_F$ with the aid of Poisson's summation formula.[12] The first two terms of this expansion yield

$$\begin{aligned}\sigma_L=&\frac{2}{3}\frac{e^2}{\pi m}\frac{\hbar E_F}{V_0^2n_I}\left[1+\frac{m^3E_FV_0^2}{2\pi^2\hbar^6}\right]\\&\times\left[1-\frac{\sqrt{2}\pi^2kT}{\hbar\omega}\left(\frac{\hbar\omega}{E_F}\right)^{\frac{1}{2}}\sum_{r=1}^{\infty}(-1)^r(r)^{\frac{1}{2}}\right.\\&\qquad\left.\times\frac{\cos[(2\pi rE/\hbar\omega)-(\pi/4)]}{\sinh(2\pi^2rkT/\hbar\omega)}\right].\end{aligned} \tag{3.6}$$

This result agrees with those which would be determined by perturbation theory, but for the factor $(1+m^3E_FV_0^2/2\pi^2\hbar^6)$.

(b) Quantum limit, degenerate statistics ($E_F<\frac{3}{2}\hbar\omega$, $E_F\gg kT$): In this domain, the only terms contributing to Eq. (3.5) and $n(E)$ are those for which $n=0$. The result is

$$\sigma_L=\frac{8}{9}\frac{e^2E_F{}^03}{\pi mn_IV_0^2\hbar\omega^2}+\frac{e^2m^2\omega^2}{4\pi^3\hbar^3n_I}, \tag{3.7}$$

where $E_F{}^0$ is the Fermi energy for $H=0$, $T=0$. The second term represents a deviation from H^2 dependence of the resistivity. It is probably not important, as at extremely high fields the electron gas becomes nondegenerate.

(c) Quantum limit, nondegenerate statistics ($E_F\ll\frac{3}{2}\hbar\omega$, $E_F\ll kT$): The result is

$$\sigma_L=\frac{2e^2n\hbar^3}{m^3n_IV_0^2\omega}(2\pi mkT)^{\frac{1}{2}}+\frac{e^2m^2\omega n}{\pi\hbar n_I(2\pi mkT)^{\frac{1}{2}}} \tag{3.8}$$

The resistance in this range is linear, but departs from this behavior at very high fields, low temperature and strong scattering potential.

IV. TRANSVERSE CONDUCTIVITY

For the transverse case, we shall begin with the expressions for conductivity given by Argyres,[13] Adams and Holstein,[3] and Kubo *et al.*[14] Their results for strong magnetic fields are:

$$\sigma_{xx}=-\frac{e^2}{\Omega}\sum_{\substack{nk_yk_z\\n'k_y'k_z'}}\left(\frac{\partial f_0}{\partial E}\right)_{nk}\left(\frac{\hbar k_y}{m\omega}-\frac{\hbar k_y'}{m\omega}\right)^2\times W_{nk_yk_z\to n'k_y'k_z'}, \tag{4.1}$$

$$\sigma_{xy}=(ne^2/m\omega). \tag{4.2}$$

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[11] P. N. Argyres, Phys. Rev. **109**, 1115 (1958).

[12] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Julius Springer Verlag, Berlin, 1931), Vol. I, see pp. 63–65.

[13] P. N. Argyres and L. M. Roth, J. Phys. Chem. Solids (to be published).

[14] R. Kubo, H. Hasegawa, and N. Hashitsume, J. Phys. Soc. Japan **14**, 56 (1959).

These formulas relate the diffusion of carriers under scattering to the conductivity. The transverse resistivity is given by

$$\rho_T=\sigma_{xx}/(\sigma_{xx}^2+\sigma_{xy}^2). \tag{4.3}$$

Though the results, Eqs. (4.1) and (4.2), were obtained by perturbation theory, we shall carry them over to our analysis by using our more accurate value of W. Substituting Eqs. (2.12), (4.1), and (4.2) into Eq. (4.3) and simplifying, we obtain

$$\rho_T=-\frac{n_I m^4 V_0^2\omega^3}{4\pi^3 n^2 e^2\hbar^4}\sum_{n,n'}\times\int\frac{(n+n'+1)}{[E-(n+\frac{1}{2})\hbar\omega]^{\frac{1}{2}}[E-(n'+\frac{1}{2})\hbar\omega]^{\frac{1}{2}}}\times\frac{1}{1+V_0^2\pi^2 n(E)^2}\frac{\partial f_0}{\partial E}dE. \tag{4.4}$$

If the factor $[1+\pi^2V_0^2n(E)^2]$ were absent, the integral would diverge for $T\neq 0$. We evaluate this expression for several regions of interest.

(a) Oscillatory range, degenerate statistics ($E_F\gg kT$, $E_F\gg\hbar\omega$): The conductivity again is expanded in powers of $\hbar\omega/E_F$. The result is

$$\sigma_{xx}=\frac{4}{3\pi^3}\frac{e^2 n_I m^2 V_0^2}{\hbar^5}\left(\frac{E_F}{\hbar\omega}\right)^2\times\left\{1+\frac{3}{4}\frac{\hbar\omega}{E_F}+\frac{11}{\sqrt{2}}\pi^2\frac{kT}{\hbar\omega}\left(\frac{\hbar\omega}{E_F}\right)^{\frac{1}{2}}\sum_\nu(-1)^\nu(\nu)^{\frac{1}{2}}\times\frac{\cos[(2\pi E_F/\hbar\omega)-(\pi/4)]}{\sinh(2\pi^2\nu kT/\hbar\omega)}\right\}. \tag{4.5}$$

In this last result we have used the limit of small V_0 after performing the integration of Eq. (4.4). This result is in agreement with that of Zilberman.[5]

(b) Quantum limit, degenerate statistics ($E_F<\frac{3}{2}\hbar\omega$, $E_F\gg kT$): The only relevant terms in Eq. (4.4) are those for which $n=n'=0$. On the assumption that $\sigma_{xy}\gg\sigma_{xx}$, we obtain a transverse resistivity

$$\rho_T=\frac{9n_I m^4V_0^2\omega^5}{16n^2\hbar^2\pi^3}\frac{1}{(E_F^0)^3+(9/32)(\omega^4V_0^2m^3/\pi^2\hbar^2)}. \tag{4.6}$$

Thus we expect the quantum limit, degenerate, magnetoresistance for point defects to vary first as H^5 and then linearly with H. At sufficiently high fields the electron gas will become nondegenerate.

(c) Quantum limit, nondegenerate statistics ($E_F\ll kT\ll\hbar\omega$): On the assumption that $\sigma_{xy}\gg\sigma_{xx}$, we obtain

$$\rho_T=-\frac{n_IV_0^2m^4\omega^2}{4ne^2\hbar^2(2\pi mkT)^{\frac{1}{2}}}\exp\left[\frac{V_0^2(\hbar\omega)^2}{2^7\pi^2kT}\left(\frac{2m}{\hbar^2}\right)^3\right]\times Ei\left[-\frac{V_0^2(\hbar\omega)^2}{2^7\pi^2kT}\left(\frac{2m}{\hbar^2}\right)^3\right]. \tag{4.7}$$

At very high fields the asymptotic form, $-e^xEi(-x)\to 1/x$ may be used to give the result

$$\rho_T\xrightarrow[H\to\infty]{}\frac{2\pi n_I\hbar^2}{ne^2}\frac{1}{(2\pi mkT)^{\frac{1}{2}}}; \tag{4.8}$$

i.e., the resistivity saturates.

V. CONSIDERATIONS OF COLLISION BROADENING

Previous theories have obtained finite transverse conductivities by invoking collision broadening.[4] This takes account of multiple scattering, which is not treated in this paper. Collision broadening is introduced by "folding" into the density of states a factor $(\hbar/\pi\tau)\times[E^2+(\hbar/\tau)^2]^{-1}$, where τ is the decay time for energy relaxation. This process removes the infinity in the density of states associated with the bottom of each Landau level. In the present theory the removal of the infinity in transverse conductivity is accomplished by the factor $[1+\pi^2V_0^2n(E)^2]^{-1}$ of Eq. (3.3). The results of this paper are valid for cases in which this last factor dominates in removing the infinity. This domain is defined by the following conditions:

(a) $\pi^2V_0^2n(E)^2\gg 1$, implying that first order perturbation theory is not valid. In this range we have

$$1/\tau_L=2n_I/\pi\hbar n(E). \tag{5.1}$$

(b) $E\gg\hbar/\tau$; i.e., that collision broadening may be neglected. The relaxation times τ and τ_L will be of the same order of magnitude.

The most striking anomalies occur in the quantum limit range. Combining the above conditions, we find for degenerate statistics in the quantum limit, that the present approach is pertinent, provided

$$n_I\lesssim(\pi/8)n, \tag{5.2}$$

a condition which can be met in semimetals.

For extrinsic semiconductors the condition of Eq. (5.2) is not met and a theory of collision broadening must be used. However, we feel that a valid theory for this case should contain a proper treatment of the scattering by one center from the start. We are led to this view by the following estimate. If we use the delta-function impurity as a model for n-type, degenerate InSb, we may choose V_0 to fit the conductivity in zero magnetic field. Using this value of V_0 in the presence of the magnetic field, we find that $\pi^2V_0^2n(E)^2\gg 1$ for typical quantum limit conditions.

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