energy has its maximum value for a system in equilibrium at a negative temperature would remain valid.

If d_iS be written formally as

$$d_iS \equiv dQ'/T,$$

thereby introducing the "uncompensated heat," dQ', of Clausius[4] this heat must be *negative* or zero for systems at negative absolute temperatures.

We may consider all the other usual thermodynamic potentials in the same way. The heat content or enthalpy H is defined as

$$H = E + PV. \tag{6}$$

Substitution into (5) gives

$$dH = TdS - Td_iS + VdP. \tag{7}$$

For a change at constant S and P at positive temperatures

$$-dH \geq 0,$$

while for negative temperatures

$$+dH \geq 0.$$

[4] See, for example, I. Prigogine and R. Defay, *Chemical Thermodynamics* (Longman's Green and Company, New York, 1954), Chap. III.

The Helmholtz free energy, F, is defined as

$$F = E - TS. \tag{8}$$

Substitution into (5) gives

$$dF = -SdT - Td_iS - PdV. \tag{9}$$

F is the appropriate potential for constant T and V and at positive temperatures

$$-dF \geq 0,$$

while for negative temperatures

$$+dF \geq 0.$$

The Gibbs free energy, G, is defined as

$$G = H - TS = E + PV - TS. \tag{10}$$

Substitution into (5) gives

$$dG = -SdT - Td_iS + VdP. \tag{11}$$

G is the appropriate potential for constant T and P and at positive temperatures

$$-dG \geq 0$$

while for negative temperatures

$$+dG \geq 0.$$

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Effects of Collisions on the Cyclotron Radiation from Relativistic Particles*

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The effects of collisions on the cyclotron radiation from relativistic particles are investigated, supposing a uniform velocity for the particles. For the collision process it is assumed that the particles start to radiate at a certain time and are abruptly stopped by the collision, after which they start to radiate again with random phase. The probabilities for the occurrence of a certain time interval between two collisions are then assumed to be distributed according to a statistical law.

The field equations are derived from the familiar Liénard-Wiechert potentials and Fourier-analyzed. Simple integral representations are found for the combined spectral and angular distribution of the radiation. Under the basic assumption that the particles make many revolutions between two collisions, it is shown that the radiation pattern is that of a series of spectral lines, each having a dispersion profile. The relative intensities of the lines are given by the formula well known from previous work neglecting collisions.

For the "slightly relativistic" energy range, $\beta_0 = 0.5$–0.9, of interest to thermofusion experiments, graphs are given for the actual intensity distribution of the various harmonics as a function of the angle between the observer and the orbital plane.

1. INTRODUCTION

THERE are three major processes which are of interest for the spectral and angular distribution of radiation from any assembly of particles: Doppler effects due to collective or individual motions of the particles with respect to the observer, energy losses due to the emission of radiation, and disturbances of the wave trains due to collisions of the radiating particles.

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There is, of course, no general rule to describe the various effects, since their consequences depend largely on the type of radiating particle and its quantum state. While usually the radiation characteristics can be said to be "continuous" or to form a spectral "line," according as the emission is from a free or a bound particle, the cyclotron or synchrotron[1] radiation

[1] The terms cyclotron radiation and synchrotron radiation are sometimes restricted in the literature to the radiation from either nonrelativistic or relativistic particles, respectively. We do not think that this semantic distinction is of great importance.

has the pecularity of being emitted by a free particle, but in the form of spectral lines. In the general case, the maximum intensities of these lines occur at integer multiples of the gyrofrequency

$$\omega_0 = |e|H/mc, \tag{1}$$

their relative intensities being a function of the particle energy. In Eq. (1) e stands for the electron charge, H for the magnetic field strength, m for the mass of the particle, and c for the velocity of light.

One expects from these facts that the actual spectral and angular behavior of the cyclotron lines will closely correspond to the behavior of spectral lines in ordinary optics. It has been shown[2] in the nonrelativistic case that this is true in all major respects.

The question now arises as to how these results can be generalized for the case of relativistic particle energies. The difficulties lie mainly in the problem of how to account for the collisions, since they affect the emission mechanism itself, while Doppler motions can be considered after the emission behavior has been completely described.[3,4] For this reason, we focus our attention on the collisions which will be described in such a fashion that it can be easily combined with Doppler motions for velocity distributions of interest. Energy losses by the radiation itself are practically always negligible, while other randomizing effects, such as field fluctuations, etc., have to be treated individually for each plasma under investigation. A study of the radiation from some typical magnetoplasmas met in nature and laboratory, with the inclusion of Doppler effects, will be published elsewhere.

Incidentally, the mathematical formulation to be presented provides us with a comparatively simple method of evaluating the spectral and angular behavior of the higher harmonics.

2. MODEL ASSUMPTIONS

In order to single out the collision effects on the radiation it is necessary to make certain assumptions regarding the velocity distribution of the radiating electrons. As long as one deals with nonrelativistic particle energies, the velocity distribution commonly met under practical conditions is a Maxwell-Boltzmann distribution. The spectral distribution within the (single) line then can be found by well-known methods.[2]

If, on the other hand, the particle energies are relativistic, the assumption of a Maxwell-Boltzmann distribution[3] becomes somewhat dubious and, of course, conceals some of the typical collision effects. Thus, we assume our particles to have a uniform energy which is maintained throughout the emission of radiation and, also, during collisions. The latter assumption is no physical restriction on the problem, since the actual velocity changes can be combined with our collision model by subsequently taking into account a velocity distribution of the radiating particles.

As is well known, the velocity components perpendicular to the magnetic field do *not* give rise to Doppler effects, but to the cyclotron radiation itself, while the components parallel to the magnetic field are unchanged, the particle moving on a helical path along the magnetic lines of force. This simple form of motion is, of course, restricted to homogeneous magnetic fields. Our model assumption thus means that all particles are moving with a constant velocity v_0 on circular orbits perpendicular to the magnetic field. The field itself is assumed to be homogeneous over distances of the order of the Larmor radius.

The formulas to be derived are written for an observer who sees no motions of the particles along the magnetic field lines. However, any such motion can be included by a contraction of the frequency scale according to the familiar expressions for the Doppler effects, taking due account of the angle between the line of sight and the magnetic field.

Next, we have to select a model for the collision event. As long as we investigate only the effects of the mere existence of collisions on the radiation pattern, very little sophistication is necessary, or indeed possible, since for the nonrelativistic case a rigorous treatment of the collision event is still missing. We follow a procedure which led to good results in spectroscopy where Lorentz[5] was able to explain the major characteristics of the collision broadening of spectral lines, while the actual processes occurring during such a collision were still unknown.

Thus, we assume that the particles radiate during a certain time Δt, beginning with a random phase and being stopped abruptly at the end of the interval Δt. The particles start to radiate again after the interruption. The probability of occurrence of a time interval Δt during which the particle emits an undisturbed wave train is

$$\exp[-\Delta t/T_0]d(\Delta t)/T_0, \tag{2}$$

where T_0 designates an average time between collisions.

This model describes the collision effects essentially through the mean time T_0 which may be velocity dependent. Thus, we succeed in eliminating from our discussion the major uncertainty, namely, the nature of the interaction, i.e., whether the collision takes place between radiating electrons themselves or between electrons and ions, neutral particles, etc. In our formulation this is a secondary problem which,

[2] L. Oster, Phys. Rev. **116**, 474 (1959).

[3] B. A. Trubnikov, Doklady Akad. Nauk S.S.S.R. **118**, 913 (1958) [translation: Soviet Phys.-Doklady **3**, 136 (1958)]. D. B. Beard, Phys. Fluids **2**, 379 (1959).

[4] Earlier references are given in the article by A. Sokolov, Nuovo cimento Suppl. **3**, 743 (1956), which also contains a short review of the calculations by D. Ivanenko and A. Sokolov, published in their book *Classical Theory of Fields*, (Moscow-Leningrad, 1951). I would like to thank Professor Ivanenko, who kindly informed me of this work, for his comments.

[5] H. A. Lorentz, Verslag Amsterdam Acad. **14**, 518 (1905).

when solved, would specify T_0 in terms of other parameters such as temperature, density, type of scatterer, etc. For a very first approximation one may tentatively derive T_0 from scattering cross sections.[6]

We do not want to conceal, however, the major weakness of this model, which is the assumption of an abrupt stop and start of the radiation process. Since the most important interaction under practical conditions is the Coulomb interaction which takes a comparatively long time and often results in multiple collisions, our approximation seems to be rather poor. However, matters are not so bad as long as the particle is allowed to make many revolutions between two collisions, i.e., as long as the ratio between cyclotron frequency and collision frequency is large:

$$(\Delta t)\omega_0 \gg 1. \tag{3}$$

Equation (3) also justifies to some extent the conventional assumption that one may combine the emitted cyclotron radiation with the bremsstrahlung which is a *direct* consequence of Coulomb interactions without mutual interference, i.e., add their intensities linearly. Equation (3) will be used throughout our treatment as a vital assumption in order to simplify the mathematical work.

3. FIELD EQUATIONS

The easiest way to derive expressions for the spectral and angular distribution is to start from the equations for the electric and magnetic field, Fourier-analyze the components, and form Poynting's vector. This then gives the emitted radiation as a function of the angle between the observer and the magnetic field and of the frequency.

The familiar Liénard-Wiechert potentials for an accelerated point charge yield for the electrical and magnetic field vectors $\mathbf{E}$ and $\mathbf{H}$ the following expressions[7]:

$$\mathbf{E}=\mathbf{n}\times(\mathbf{n}^*\times\mathbf{a}_0)es^{-3}c^{-2}, \tag{4}$$

$$\mathbf{H}=\mathbf{n}\times\mathbf{E}. \tag{5}$$

Here, $\mathbf{n}$ is a unit vector in an arbitrary direction; $\mathbf{n}^*$ and s are defined by the relations

$$\mathbf{n}^*=\mathbf{n}-v_0/c, \tag{6}$$

$$s=1-\mathbf{n}\cdot\mathbf{v}_0/c. \tag{7}$$

$\mathbf{v}_0$ is the velocity of the particle, $\mathbf{a}_0$ the acceleration; $\mathbf{v}_0$ and $\mathbf{a}_0$ are assumed to be of constant amplitude, as discussed above: This corresponds to a neglect of energy losses due to the emitted radiation. Equations (4) and (5) are written for a unit distance from the source, thus all quantities derived with the help of Eqs. (4) and (5) are also normalized to unit distance. This allows us to omit a factor specifying the source-observer distance without a physical restriction: The only restriction to Eqs. (4) and (5) is that their derivation from the Liénard-Wiechert potentials is valid only for distances large compared with the Larmor radius, i.e., in a distance where the arriving waves can be treated as plane waves. Also, the wavelengths have to be small compared with the source-observer distance, for the same reason. All our equations are fully valid under relativistic conditions.

The next step is to define a suitable reference frame. We use the same system as in our investigation on nonrelativistic particle energies,[2] i.e., the reference frame is chosen in such a way that the angle between the observer (direction given by the vector $\mathbf{n}$) and the orbital plane described by the polar angle θ does not change in time, while the phase of the orbital motion is described by the other polar angle, ϕ. See Fig. 1. This definition, again, is only possible when the observer is so far away from the source that the orbit of the particle can be considered as coincident with the origin.

The quantities $\mathbf{v}_0$, $\mathbf{a}_0$, $\mathbf{n}$, $\mathbf{n}^*$, and s then have the following components in the. rectangular coordinate system (x,y,z) of Fig. 1:

$$\mathbf{v}_0=(v_0\cos\phi, v_0\sin\phi, 0), \tag{8}$$

$$\mathbf{a}_0=d\mathbf{v}_0/dt=(-v_0\omega_0\sin\phi, v_0\omega_0\cos\phi, 0), \tag{9}$$

$$\mathbf{n}=(\cos\theta, 0, \sin\theta), \tag{10}$$

$$\mathbf{n}^*=(\cos\theta-\beta_0\cos\phi, -\beta_0\sin\phi, \sin\theta), \tag{11}$$

$$s=1-\beta_0\cos\phi\cos\theta, \tag{12}$$

where we used the fact that the phase angle ϕ and the time are connected by the relation

$$\phi=\omega_0 t, \tag{13}$$

while

$$\beta_0=v_0/c. \tag{14}$$

The cyclotron frequency ω_0 is given again by Eq. (1). In terms of the rest mass m_0 of the electron, however,

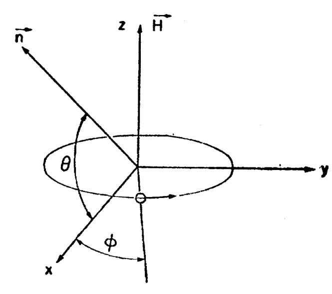

FIG. 1. The polar system of reference.

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[6] A general discussion of the derivation of scattering cross sections and their application to plasma problems is given by W. P. Allis, *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 21, Secs. 48 and 49. The use of scattering cross sections in radiation problems is discussed in reference 2.

[7] See, for instance, W. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955).

we have

$$\omega_0 = eH[1-\beta_0^2]^{\frac{1}{2}}/m_0c. \quad (15)$$

The electric and magnetic field vectors become

$$\mathbf{E} = \frac{ev_0\omega_0}{c^2(1-B_0\cos\phi)^3} \times(\sin\phi\sin^2\theta,\ B_0-\cos\phi,\ -\sin\phi\sin\theta\cos\theta), \quad (16)$$

$$\mathbf{H} = \frac{ev_0\omega_0}{c^2(1-B_0\cos\phi)^3} \times[-\sin\theta(B_0-\cos\phi),\ \sin\phi\sin\theta,\ \cos\theta(B_0-\cos\phi)], \quad (17)$$

where

$$B_0 \equiv \beta_0\cos\theta. \quad (18)$$

We note here that

$$E^2 = H^2, \quad (19)$$

and that Poynting's vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} = \frac{e^2v_0^2\omega_0^2}{4\pi c^3} \times\frac{(B_0\cos\theta-\cos\phi)^2+\sin^2\theta\sin^2\phi}{(1-B_0\cos\phi)^6}\mathbf{n} \quad (20)$$

can be written

$$\mathbf{S} = (c/4\pi)E^2\mathbf{n} = (c/4\pi)H^2\mathbf{n}. \quad (21)$$

4. FOURIER ANALYSIS

Equations (16) and (17) already contain all information necessary to compute the angular distribution of radiation. In order to derive the spectral behavior we have to Fourier-analyze each of the components of the electrical field vector **E** or the magnetic field vector **H**. The result will be the same, since we are interested only in the squares of the vector components and their Fourier transforms, Eq. (19) being equally valid for the vector components and their Fourier transforms.[8] We chose the electric vector **E**.

We see from Eq. (16) that both the x and z component have the same dependance on ϕ and thus on the time. We carry out the Fourier analysis for the x and y components, the treatment of the z component being exactly the same as for the x component.

In the relativistic case, where we have to distinguish carefully between the actual time of the observer's scale t^*, in which the Fourier integration is carried out, and the retarded time, in which all quantities referring to the orbital motion are measured.

Finally, it has to be mentioned that the functions E_x and E_y, for example, are not simply odd and even when an arbitrary phase constant is added.

[8] W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954), Chap. I, 4.

The complete expressions for the two field components thus are

$$E_x(\omega) = \frac{ev_0\omega_0}{\pi c^2}\sin^2\theta\int_{-\Delta t^*/2}^{+\Delta t^*/2} \times\frac{\sin(\omega_0t+\delta)[\sin\omega t^*+\cos\omega t^*]}{[1-B_0\cos(\omega_0t+\delta)]^3}dt^*, \quad (22)$$

$$E_y(\omega) = \frac{ev_0\omega_0}{\pi c^2}\int_{-\Delta t^*/2}^{+\Delta t^*/2} \times\frac{[B_0-\cos(\omega_0t+\delta)][\sin\omega t^*+\cos\omega t^*]}{[1-B_0\cos(\omega_0t+\delta)]^3}dt^*. \quad (23)$$

As noted in Sec. 2, we have to introduce phase constants δ into the expression for the orbital motion. δ is a measure of the angle around the orbit, counted from a fixed direction, at which the particle undergoes a collision. Assuming that these phase angles are distributed at random, we thus have to average over δ later in this section.

This picture implies mathematically that the orbital function $\propto \exp(i\omega_0t)$ and the Fourier function $\propto \exp(i\omega t^*)$ are out of phase. The customary description is to fix the latter function symmetrically in a certain time or azimuthal scale. Then the orbital function contains an arbitrary phase constant which varies at random between 0 and 2π.

Actual and retarded time are connected by the relation

$$dt^* = [1-B_0\cos(\omega_0t+\delta)]dt, \quad (24)$$

which leads to

$$t^* = t - B_0\sin(\omega_0t+\delta)/\omega_0. \quad (25)$$

Equation (25) does not contain an additional phase constant, since in nonrelativistic limit ($B_0\to 0$) the time scales must be equal.

With the help of Eqs. (24) and (25) we find for the field components

$$E_x(\omega) = \frac{ev_0\omega_0}{\pi c^2}\sin^2\theta\int_{-\Delta t/2}^{+\Delta t/2}\sin(\omega_0t+\delta) \times[1-B_0\cos(\omega_0t+\delta)]^{-2} \times\{\sin\omega[t-B_0\sin(\omega_0t+\delta)/\omega_0] +\cos\omega[t-B_0\sin(\omega_0t+\delta)/\omega_0]\}dt, \quad (26)$$

$$E_y(\omega) = \frac{ev_0\omega_0}{\pi c^2}\int_{-\Delta t/2}^{+\Delta t/2}[B_0-\cos(\omega_0t+\delta)] \times[1-B_0\cos(\omega_0t+\delta)]^{-2} \times\{\sin\omega[t-B_0\sin(\omega_0t+\delta)/\omega_0] +\cos\omega[t-B_0\sin(\omega_0t+\delta)/\omega_0]\}dt. \quad (27)$$

The correct procedure would now be to evaluate the integrals, Eqs. (26) and (27), for all parameters B_0, ω, etc., of interest and all phase angles δ between 0 and 2π, then to square the components and average over δ. As we shall see, the main complication in this program is the appearance of the phase angle δ. Fortunately, the inclusion of δ is not vital for the accuracy of the results as we shall show in the next sections. Thus, subject to further justification, we put

$$\delta \equiv 0, \tag{28}$$

and use the complete equations (26) and (27) only as a reference when we discuss the consequences of restriction (28). Anticipating the results, Eq. (28) is not a stronger restriction than Eq. (3) and may be used whenever the collision frequency is small compared with the gyrofrequency.

Introducing again the azimuthal angle ϕ as variable instead of the time, using the abbreviations

$$2\phi_0 \equiv \omega_0 \Delta t, \quad \Omega \equiv \omega/\omega_0, \tag{29}$$

and *omitting* the phase constant, we find

$$E_x = 2\frac{ev_0}{\pi c^2}\sin^2\theta \int_0^{\phi_0} \sin\phi[1-B_0\cos\phi]^{-2} \times \sin[\Omega(\phi - B_0\sin\phi)]d\phi, \tag{30}$$

$$E_y = 2\frac{ev_0}{\pi c^2} \int_0^{\phi_0} [B_0 - \cos\phi][1-B_0\cos\phi]^{-2} \times \cos[\Omega(\phi - B_0\sin\phi)]d\phi, \tag{31}$$

$$E_z = -2\frac{ev_0}{\pi c^2}\sin\theta\cos\theta \int_0^{\phi_0} \sin\phi[1-B_0\cos\phi]^{-2} \times \sin[\Omega(\phi - B_0\sin\phi)]d\phi. \tag{32}$$

The integrals

$$I_1(\phi_0, B_0, \Omega) \equiv \int_0^{\phi_0} \sin\phi[1-B_0\cos\phi]^{-2} \times \sin[\Omega(\phi - B_0\sin\phi)]d\phi, \tag{33}$$

and

$$I_2(\phi_0, B_0, \Omega) \equiv \int_0^{\phi_0} [B_0 - \cos\phi][1-B_0\cos\phi]^{-2} \times \cos[\Omega(\phi - B_0\sin\phi)]d\phi \tag{34}$$

will be discussed in Sec. 11. From integrations by parts, we readily obtain the following alternative forms:

$$I_1 = -\frac{\sin[\Omega(\phi_0 - B_0\sin\phi_0)]}{B_0(1-B_0\cos\phi_0)} + \frac{\Omega}{B_0}\int_0^{\phi_0} \cos[\Omega(\phi - B_0\sin\phi)]d\phi, \tag{35}$$

or

$$I_1 = -\frac{\cos\phi_0 \sin[\Omega(\phi_0 - B_0\sin\phi_0)]}{1-B_0\cos\phi_0} + \Omega\int_0^{\phi_0} \cos\phi\cos[\Omega(\phi - B_0\sin\phi)]d\phi, \tag{36}$$

which can be used even in the limit $B_0 \to 0$. We find a similar expression for I_2:

$$I_2 = -\frac{\sin\phi_0 \cos[\Omega(\phi_0 - B_0\sin\phi_0)]}{1-B_0\cos\phi_0} - \Omega\int_0^{\phi_0} \sin\phi\sin[\Omega(\phi - B_0\sin\phi)]d\phi. \tag{37}$$

5. POYNTING'S VECTOR AND AVERAGE RADIATION

Equation (21) transforms into the following expression for the emission per unit frequency:

$$\mathbf{S}_{\Delta t}(\omega) = \frac{c}{4\pi}E^2\mathbf{n} = \frac{e^2v_0^2}{\pi^3c^3}[(\sin^2\theta)I_1^2 + I_2^2]\mathbf{n}. \tag{38}$$

The frequency distribution is connected with the total emission through Parseval's theorem[9]:

$$\pi\int_0^{\infty} \mathbf{S}_{\Delta t}(\omega)d\omega = \int_{-\Delta t/2}^{+\Delta t/2} \mathbf{S}dt. \tag{39}$$

However, we are not interested in the radiation of particles which are all radiating during the same time interval Δt, but in radiation times which are distributed at random. Introducing instead of the mean collision time T_0 the corresponding mean angle[10] by

$$2\bar{\phi} \equiv \omega_0 T_0, \tag{40}$$

we can write the statistical law (2) as

$$\exp[-\phi_0/\bar{\phi}]d\phi_0/\bar{\phi}. \tag{41}$$

The average radiation is then found from

$$\bar{S}(\omega) = \frac{e^2v_0^2}{\pi^3c^3}\left(\sin^2\theta\int_0^{\infty} I_1^2(\phi_0)\exp(-\phi_0/\bar{\phi})d\phi_0/\bar{\phi} + \int_0^{\infty} I_2^2(\phi_0)\exp(-\phi_0/\bar{\phi})d\phi_0/\bar{\phi}\right), \tag{42}$$

with the integral relation

$$\pi\int_0^{\infty} \bar{S}(\omega)d\omega = \int_0^{T_0} S(t)dt, \tag{43}$$

corresponding to Eq. (39).

6. PERIODICITY RELATIONS

Although we need not consider, in general, the integrals of Eq. (42) directly, we would like to collect here a few relations which will be useful in treating the

[9] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge University Press, New York, 1952), p. 182.

[10] The factor 2 is introduced to maintain the similarity with the definition of ϕ_0; see Eq. (29).

limiting case of zero damping. The basis of these considerations is that the functions I_1 and I_2 are integrals over periodic functions.

Let us first note that both integrals are strictly periodic for all noninteger values of Ω and that they alternate between positive and negative values. The period of the *squares* is

$$p=\pi/b, \tag{44}$$

with b related to the frequency by the definition

$$\Omega\equiv a/b. \tag{45}$$

a and b are integers, so that a/b is an irreducible fraction.

We then have

$$[I(p+\phi_0)]^2=[I(\phi_0)]^2, \quad \Omega\neq\text{integer}. \tag{46}$$

The symbol I in this and the following equations stands for both I_1 and I_2.

If, on the other hand, Ω is an integer, the period is always 2π and the relation corresponding to Eq. (46) is

$$[I(2n\pi+\phi_0)]^2=[nI(2\pi)+I(\phi_0)]^2; \quad n=1,2,3,\cdots. \tag{47}$$

With the help of Eqs. (46) and (47), the integrals in Eq. (42) can be simplified by changing the infinite interval of integration into a sum of integrals over one period. One finds after some manipulation

$$\begin{aligned}\int_0^\infty I^2(\bar{\phi}x)dx=&e^{-x_0}(1+e^{-x_0})(1-e^{-x_0})^{-2}I^2(\bar{\phi}x_0)\\&+2e^{-x_0}(1-e^{-x_0})^{-2}\int_0^{x_0}I(\bar{\phi}x)e^{-x}dx\\&+(1-e^{-x_0})^{-1}\int_0^{x_0}I^2(\bar{\phi}x)e^{-x}dx,\end{aligned} \tag{48}$$

where the following abbreviations have been used:

$$x=\phi_0/\bar{\phi}, \quad x_0=2\pi/\bar{\phi}. \tag{49}$$

Finally, if $\Omega\neq$integer,

$$x_0=p/\bar{\phi} \tag{50}$$

with p defined by Eq. (45), and

$$I(\bar{\phi}x_0)\equiv 0. \tag{51}$$

Then, Eq. (48) is simply

$$\int_0^\infty I^2(\bar{\phi}x)e^{-x}dx=(1-e^{-x_0})^{-1}\int_0^{x_0}I^2(\bar{\phi}x)e^{-x}dx. \tag{52}$$

7. THE NONRELATIVISTIC LIMIT

The next few sections will be devoted to the discussion of the asymptotic behavior of our basic Eqs. (30)–(32). We consider first the nonrelativistic limit, because it provides us with familiar expressions which we then use in an expansion technique to handle the basic equations without numerical calculations. It also helps us to settle the question of phase constants.

If the particle velocity is small compared with the velocity of light,

$$\beta_0\ll 1, \quad B_0\ll 1, \tag{53}$$

and the fields are described by the relations

$$\mathbf{E}=(ev_0\omega_0/c^2)(\sin^2\theta\sin\phi,\ -\cos\phi,\ -\sin\theta\cos\theta\sin\phi), \tag{54}$$

$$\mathbf{H}=(ev_0\omega_0/c^2)(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ -\cos\theta\cos\phi) \tag{55}$$

instead of Eqs. (16) and (17), while Eq. (18), of course, retains its form. Poynting's vector is now written

$$\mathbf{S}=(e^2v_0\omega_0^2/4\pi c^3)(1-\cos^2\theta\sin^2\phi)\mathbf{n}. \tag{56}$$

In carrying out the Fourier analysis, it is not necessary to specify the local and retarded time and we find for the field components from our previous Eqs. (26) and (27)

$$\begin{aligned}E_x(\omega)=\frac{ev_0\omega_0}{\pi c^2}\sin^2\theta\int_{-\Delta t/2}^{+\Delta t/2}&\sin(\omega_0t+\delta)\\&\times[\sin(\omega t)+\cos(\omega t)]dt,\end{aligned} \tag{57}$$

$$\begin{aligned}E_y(\omega)=-\frac{ev_0\omega_0}{\pi c^2}\int_{-\Delta t/2}^{+\Delta t/2}&\cos(\omega_0t+\delta)\\&\times[\sin(\omega t)+\cos(\omega t)].\end{aligned} \tag{58}$$

From here we obtain with the help of the addition theorems and the fact that symmetrical integrals over products $\sin x\cdot\cos x$ vanish:

$$\begin{aligned}E_x(\omega)=\frac{ev_0\omega_0}{\pi c^2}\sin^2\theta\Big[\cos\delta\int&\sin(\omega_0t)\sin(\omega t)dt\\&+\sin\delta\int\cos(\omega_0t)\cos(\omega t)dt\Big],\end{aligned} \tag{59}$$

and

$$\begin{aligned}E_y(\omega)=-\frac{ev_0\omega_0}{\pi c^2}\Big[\cos\delta\int&\cos(\omega_0t)\cos(\omega t)dt\\&-\sin\delta\int\sin(\omega_0t)\sin(\omega t)dt\Big].\end{aligned} \tag{60}$$

We now define the following auxiliary functions:

$$\Delta_m(\omega)\equiv\frac{\sin[(\omega+m\omega_0)\Delta t/2]}{\omega+m\omega_0}-\frac{\sin[(\omega-m\omega_0)\Delta t/2]}{\omega-m\omega_0}, \tag{61}$$

$$\Sigma_m(\omega)\equiv\frac{\sin[(\omega+m\omega_0)\Delta t/2]}{\omega+m\omega_0}+\frac{\sin[(\omega-m\omega_0)\Delta t/2]}{\omega-m\omega_0}, \tag{62}$$

and obtain with their help for the field components

$$\begin{aligned}E_x(\omega)=-(ev_0\omega_0/\pi c^2)\sin^2\theta[&\Delta_1(\omega)\cos\delta\\&-\Sigma_1(\omega)\sin\delta],\end{aligned} \tag{63}$$

$$E_y(\omega)=-(ev_0\omega_0/\pi c^2)[\Sigma_1(\omega)\cos\delta+\Delta_1(\omega)\sin\delta], \tag{64}$$

$$\begin{aligned}E_z(\omega)=+(ev_0\omega_0/\pi c^2)\sin\theta\cos\theta[&\Delta_1(\omega)\cos\delta\\&-\Sigma_1(\omega)\sin\delta].\end{aligned} \tag{65}$$

The frequency distribution of the radiation, given by Poynting's vector, becomes

$$S_{\Delta t}(\omega,\delta)=(c/4\pi)E^2(\omega)=(e^2v_0^2\omega_0^2/4\pi^3c^3)\times\{\sin^2\theta[\Delta_1(\omega)\cos\delta-\Sigma_1(\omega)\sin\delta]^2+[\Sigma_1(\omega)\cos\delta+\Delta_1(\omega)\sin\delta]^2\}. \quad (66)$$

Now we average over the phase constants δ, i.e., we perform the integration

$$\frac{1}{2\pi}\oint S_{\Delta t}(\omega,\delta)d\delta\equiv S_{\Delta t}(\omega), \quad (67)$$

and find from

$$\langle\sin^2\delta\rangle_\delta=\langle\cos^2\delta\rangle_\delta=\tfrac{1}{2},\quad \langle\sin\delta\cos\delta\rangle_\delta=0 \quad (68)$$

the desired result

$$S_{\Delta t}(\omega)=(e^2v_0^2\omega_0^2/4\pi^3c^3)\tfrac{1}{2}(1+\sin^2\theta)\times[\Delta_1^2(\omega)+\Sigma_1^2(\omega)]. \quad (69)$$

Note that the cross terms $\Sigma\Delta$ disappeared quite naturally from the final result.

The last step is to evaluate the average with respect to Δt. With the help of the general expressions

$$\langle\Delta_m^2(\omega)\rangle_{\Delta t}=\tfrac{1}{2}[(\omega+m\omega_0)^2+T_0^{-2}]^{-1}+\tfrac{1}{2}[(\omega-m\omega_0)^2+T_0^{-2}]^{-1}-T_0^{-2}[(m\omega_0)^2+T_0^{-2}][\omega^2+T_0^{-2}], \quad (70)$$

$$\langle\Sigma_m^2(\omega)\rangle_{\Delta t}=\tfrac{1}{2}[(\omega+m\omega_0)^2+T_0^{-2}]^{-1}+\tfrac{1}{2}[(\omega-m\omega_0)^2+T_0^{-2}]^{-1}+T_0^{-2}[(m\omega_0)^2+T_0^{-2}][\omega^2+T_0^{-2}], \quad (71)$$

we find for the average of Poynting's vector

$$\langle S_{\Delta t}(\omega)\rangle_{\Delta t}=(e^2v_0^2\omega_0^2/4\pi^3c^3)\tfrac{1}{2}(1+\sin^2\theta)\times\{[(\omega+\omega_0)^2+T_0^{-2}]^{-1}+[(\omega-\omega_0)^2+T_0^{-2}]^{-1}\}. \quad (72)$$

8. INTEGRAL RELATIONS

Let us now consider some relations holding for the time and space integrals over the emission which are of interest and, applied to the results of the expansion analysis, will enable us to make a direct comparison with previously published results.

Our starting points are Eqs. (69) and (72). We begin with the total radiation emitted into the solid angle 4π. Because of the normalization of the field vectors to unit distance, we find the total emission by an integration over the unit sphere. The radiation described by Eqs. (69) and (72) is already an average over the azimuthal angle ϕ, since it is a result of a time integration. Thus, we merely multiply through by 2π. The integration over θ then goes from $-\pi/2$ to $+\pi/2$ with $\cos\theta$ as weight function.

We obtain from $S_{\Delta t}(\omega)$

$$\oint S_{\Delta t}(\omega)do=(e^2v_0^2\omega_0^2/2\pi^2c^3)\tfrac{4}{3}[\Delta_1^2(\omega)+\Sigma_1^2(\omega)], \quad (73)$$

with a corresponding expression for $\langle S_{\Delta t}(\omega)\rangle_{\Delta t}$.

If finally we also integrate over all frequencies, we find after some algebra and with the help of the integral

$$\int_{-\infty}^{+\infty} x^{-2}[\sin^2(x\Delta t/2)]dx=\pi\Delta t/2 \quad (74)$$

from Eq. (73):

$$\pi\oint do\int_0^\infty d\omega S_{\Delta t}(\omega)=\frac{2}{3}\frac{e^2}{c^3}v_0^2\omega_0^2\Delta t. \quad (75)$$

Equation (75) gives the total intensity emitted into all solid angles and in all frequencies during a time Δt between collisions.

In the same way, we find with the help of the integral

$$\int_{-\infty}^{+\infty} d\omega/(\omega^2+T_0^{-2})=\pi T_0 \quad (76)$$

from Eq. (72):

$$\pi\oint do\int_0^\infty d\omega\langle S_{\Delta t}(\omega)\rangle=\frac{2}{3}\frac{e^2}{c^3}v_0^2\omega_0^2T_0. \quad (77)$$

The result is as expected: Eq. (77) gives the radiation emitted during the *average* collision time T_0, in agreement with the fact that Eq. (77) could also be obtained directly from Eq. (75) with the help of the trivial relation

$$\langle\Delta t\rangle=T_0. \quad (78)$$

9. THE NEGLECT OF PHASE CONSTANTS IN THE NONRELATIVISTIC CASE

In the last section we derived the radiation laws in the nonrelativistic limit with the full use of phase constants. Now we determine what alterations appear when these phase constants are neglected. The answer will provide a clearer picture of the physical restrictions under which this neglect is justified. As we shall see, the same restrictions apply to the general relativistic case.

If we put

$$\delta\equiv 0,\quad \sin\delta=0,\quad \cos\delta=1, \quad (79)$$

the field components, Eqs. (57) and (58), are purely odd or even functions of the time. Poynting's vector, Eq. (66), becomes

$$S_{\Delta t}(\omega,\,\delta\equiv 0)=(e^2v_0^2\omega_0^2/4\pi^3c^3)\times[\Delta_1^2(\omega)\sin^2\theta+\Sigma_1^2(\omega)]. \quad (80)$$

This does not agree with the result of the averaging over the phase constants, Eq. (69).

If we carry out the averaging over collision times which previously resulted in Eq. (72), we find

$$\langle S_{\Delta t}(\omega,\delta\equiv 0)\rangle_{\Delta t}=(e^2v_0^2\omega_0^2/4\pi^3c^3)\times(\tfrac{1}{2}(1+\sin^2\theta)\{[(\omega+\omega_0)^2+T_0^{-2}]^{-1}+[(\omega-\omega_0)^2+T_0^{-2}]^{-1}\}+(1-\sin^2\theta)T_0^{-2}[\omega_0^2+T_0^{-2}]^{-1}\times[\omega^2+T_0^{-2}]^{-1}). \quad (81)$$

Here, the term proportional to T_0^{-2} marks the difference and appears in addition to the correct terms in Eq. (63).

To see the cause of this disagreement we carry out the integrations over the solid angles and frequencies. Making use of the definitions of Δ_1 and Σ_1, we find from Eq. (80)

$$\oint doS_{\Delta t}(\omega,\delta\equiv 0)=(e^2v_0^2\omega_0^2/2\pi^2c^3)\tfrac{4}{3}\{\Delta_1^2(\omega)+\Sigma_1^2(\omega)+\sin[(\omega+\omega_0)\Delta t/2]\times\sin[(\omega-\omega_0)\Delta t/2]/(\omega^2-\omega_0^2)\}. \quad (82)$$

The integration over the frequencies yields

$$\oint do\int_0^\infty d\omega S_{\Delta t}(\omega,\delta\equiv 0)=\frac{2}{3}\frac{e^2}{c^3}v_0^2\omega_0^2[\Delta t+\sin(\omega_0\Delta t)/2\omega_0] \quad (83)$$

instead of Eq. (75).

The corresponding expression for the average radiation is found from Eq. (81) to be

$$\oint do\int_0^\infty d\omega\langle S_{\Delta t}(\omega,\delta\equiv 0)\rangle_{\Delta t}=\frac{2}{3}\frac{e^2}{c^3}v_0^2\omega_0^2[T_0+\tfrac{1}{2}T_0/(1+\omega_0^2T_0^2)], \quad (84)$$

which may also be obtained from Eq. (83).

Equations (83) and (84) contain the information we were looking for. The neglect of the random phase constants affects the expression for the total emission between collisions as well as the spectral and angular distribution and does not, of course, average out even with a random distribution of collision times.

However, the additional terms entering Eqs. (83) and (84) compared with Eqs. (75) and (77) are of no importance, as long as $T_0\gg\omega_0^{-1}$, a condition we upheld from the beginning for physical reasons. As a matter of fact, the error introduced by the neglect of the phase constants is unimportant when the particle on the average completes at least one or two full revolutions between collisions.

This restriction to "weak damping" has another interesting consequence. As one finds from Eq. (81), the line shape becomes steeper with increasing time intervals between collisions. At the same time, the only important term is the resonance term, containing the difference $(\omega-\omega_0)^2$. Thus, we may approximate our auxiliary functions by the following expression.

$$\Sigma_m(\omega)\approx-\Delta_m(\omega)\approx\sin[(\omega-m\omega_0)\Delta t/2]/(\omega-m\omega_0), \quad (85)$$

which is called "dispersion distribution" in optics and was used in our previous paper.[2]

We see now that the simplified analysis describes the actual conditions very well, as long as we deal only with weak damping. Then, the important contributions to the total amount of emitted radiation come from a narrow frequency range around the resonance, and, in order to keep the approximations consistent, we may use Eq. (85) to describe the actual line shape.

In the line "wings," i.e., where the intensity is only a small fraction of the central intensity, the approximation of Eq. (85) becomes poor. This, too, does not affect our treatment of cyclotron radiation, since we are not interested in the depressions between the resonance peaks, where the energy distribution is determined by bremsstrahlung anyway.

Going back now to the general relativistic case, we first show that the restriction to weak damping is again the justification of the neglect of phase constants. Then, we expand the basic integrals in terms of functions Δ_m and Σ_m whose general behavior was outlined and consider the restrictions which arise under the condition of weak damping.

10. INCLUSION OF PHASE CONSTANTS IN THE RELATIVISTIC CASE

We start from Eqs. (26) and (27) and use the following abbreviations:

$$\omega_0t+\delta\equiv\phi+\delta\equiv\chi;\quad \phi_0+\delta\equiv\chi_0, \quad (86)$$

$$\Omega B_0\sin(\phi+\delta)\equiv\Lambda(\chi). \quad (87)$$

Instead of Eqs. (33) and (34) for the integrals I_1 and I_2, we have the expressions

$$2I_1=\int_{-\chi_0+\delta}^{+\chi_0+\delta}(1-B_0\cos\chi)^{-2}\sin\chi\times[\sin(\Omega\chi-\Omega\delta-\Lambda)+\cos(\Omega\chi-\Omega\delta-\Lambda)]d\chi, \quad (88)$$

$$2I_2=\int_{-\chi_0+\delta}^{+\chi_0+\delta}(1-B_0\cos\chi)^{-2}(B_0-\cos\phi)\times[\sin(\Omega\chi-\Omega\delta-\Lambda)+\cos(\Omega\chi-\Omega\delta-\Lambda)]d\chi \quad (89)$$

for which we write

$$2I_1=\int X(\chi)Z(\Omega,\chi,\delta)d\chi, \quad (90)$$

and

$$2I_2=\int Y(\chi)Z(\Omega,\chi,\delta)d\chi. \quad (91)$$

Note that $X(\chi)$ is a purely odd, $Y(\chi)$ a purely even function of χ.

We now split $Z(\chi)$ into single functions of χ, Ω, and δ. This leads after some manipulations to the following expression

$$Z(\Omega,\chi,\delta)=[\cos(\Omega\delta)+\sin(\Omega\delta)]\sin(\Omega\chi-\Lambda)+[\cos(\Omega\delta)-\sin(\Omega\delta)]\cos(\Omega\chi-\Lambda). \quad (92)$$

The main idea, in order to discard the phase constant, is that the integrals (90) and (91) do not change appreciably by replacing the limits $-\chi_0+\delta$ and $+\chi_0+\delta$ by $-\chi_0$ and $+\chi_0$, when the integral is extended over many periods of the functions X, Y, and Z, i.e., again, as long as Eq. (3) and the condition of weak damping are valid. Then, making use of the form (92) for the function $Z(\chi)$, the integrals (90) and (91) can be reduced to

$$2I_1=[\cos(\Omega\delta)+\sin(\Omega\delta)]\int_{-\chi_0}^{+\chi_0}X(\chi)\times\sin[\Omega\chi-\Lambda(\chi)]d\chi, \quad (93)$$

$$2I_2=[\cos(\Omega\delta)-\sin(\Omega\delta)]\int_{-\chi_0}^{+\chi_0}Y(\chi)\times\cos[\Omega\chi-\Lambda(\chi)]d\chi. \quad (94)$$

In averaging over δ, we have to square $2I_1$ and $2I_2$ and integrate over δ from 0 to 2π. This results exactly in Eqs. (33) and (34). Thus, we have found that even in the strictly relativistic case, the phase constants may be neglected if one restricts considerations to the case of weak damping.

11. EXPANSION ANALYSIS

We are now ready to study the basic Eqs. (33) to (37) in detail and derive from them the line contour of the higher harmonics and their relative intensities. The program is to expand the second trigonometric function under the integrals in Eqs. (36) and (37) in a power series in (ΩB_0) and then single out the contributions to the various harmonics.

Let us disregard the constant terms on the right side of Eqs. (36) and (37) for the time being and denote the remaining integrals by I_1^* and I_2^*. We find

$$I_1^*=\Omega\int_0^{\phi_0}\cos\phi\cos(\Omega\phi)\cos(\Omega B_0\sin\phi)d\phi+\Omega\int_0^{\phi_0}\cos\phi\sin(\Omega\phi)\sin(\Omega B_0\sin\phi)d\phi, \quad (95)$$

$$I_2^*=-\Omega\int_0^{\phi_0}\sin\phi\sin(\Omega\phi)\cos(\Omega B_0\sin\phi)d\phi+\int_0^{\phi_0}\sin\phi\cos(\Omega\phi)\sin(\Omega B_0\sin\phi)d\phi. \quad (96)$$

If we now introduce for $\sin(\Omega B_0\sin\phi)$ and $\cos(\Omega B_0\sin\phi)$ in Eqs. (95) and (96) the well-known power series expansions, we find

$$I_1^*=\Omega\sum_{s=0}^{\infty}(-1)^{-s}\frac{(B_0\Omega)^{2s}}{(2s)!}\int_0^{\phi_0}\cos(\Omega\phi)\cos\phi\sin^{2s}\phi d\phi+\Omega\sum_{s=0}^{\infty}(-1)^{s}\frac{(B_0\Omega)^{2s+1}}{(2s+1)!}\int_0^{\phi_0}\sin(\Omega\phi)\cos\phi\times\sin^{2s+1}\phi d\phi, \quad (97)$$

$$I_2^*=-\Omega\sum_{s=0}^{\infty}(-1)^{s}\frac{(B_0\Omega)^{2s}}{(2s)!}\int_0^{\phi_0}\sin(\Omega\phi)\sin^{2s+1}\phi d\phi+\Omega\sum_{s=0}^{\infty}(-1)^{s}\frac{(B_0\Omega)^{2s+1}}{(2s+1)!}\int_0^{\phi_0}\cos(\Omega\phi)\times\sin^{2s+2}\phi d\phi. \quad (98)$$

The powers of the trigonometric functions can be expressed once again as sums of functions[11] involving multiples of ϕ. Inserting these sums into Eqs. (97) and (98) leads to the familiar integrals

$$\Delta_m(\Omega)=-\int_{-\phi_0}^{+\phi_0}\sin(\Omega\phi)\sin(m\phi)d\phi, \quad (99)$$

and

$$\Sigma_m(\Omega)=+\int_{-\phi_0}^{+\phi_0}\cos(\Omega\phi)\cos(m\phi)d\phi. \quad (100)$$

The functions Δ_m and Σ_m are defined in the same way as previously in Eqs. (61) and (62) but with Ω instead of ω as variable.

However, we have learned meanwhile that our basic assumptions restrict the validity of our treatment to frequencies near the resonance. Under the condition

$$|\Omega-m|\ll m, \quad (101)$$

the first terms in Eqs. (61) and (62) are small compared with the second. We then write

$$\Sigma_m\approx-\Delta_m\approx\sin[(\Omega-m)\phi_0]/(\Omega-m). \quad (102)$$

At the same time, we may neglect the constant terms already omitted in writing down Eqs. (95) and (96), since they are also small compared with the resonance terms (102). We consequently drop the asterisk. By the same argument, all terms containing Δ_0 and Σ_0 are omitted.

After some rearranging we obtain finally for the integrals I_1 and I_2

[11] See for instance P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, New York, 1953), Vol. II, p. 1320.

$$-2I_1=\Omega\sum_{s=0}^{\infty}(-1)^s\alpha^{2s}\Bigg(\sum_{m=0}^{s}\frac{(-1)^m}{(s-m)!(s+m)!}\times(\Delta_{2m+1}+\Delta_{2m-1})-\frac{1}{s!s!}\Delta_1\Bigg)+\Omega\sum_{s=0}^{\infty}(-1)^s\frac{\alpha^{2s+1}}{2s+1}\times\Bigg(\sum_{m=0}^{s}\frac{(-1)^m}{(s-m)!(s+m)!}(\Delta_{2m+2}-\Delta_{2m-2}{}^*)-\frac{1}{s!s!}\Delta_2\Bigg), \quad (103)$$

$$2I_2=\Omega\sum_{s=0}^{\infty}(-1)^s\alpha^{2s}\Bigg(\sum_{m=0}^{s}\frac{(-1)^m}{(s-m)!(s+m)!}\times(\Delta_{2m+1}-\Delta_{2m-1}{}^*)-\frac{1}{s!s!}\Delta_1\Bigg)-\Omega\sum_{s=0}^{\infty}(-1)^s\alpha^{2s+1}(2s+2)\sum_{m=0}^{s+1}\times\frac{(-1)^m}{(s-m-1)!(s+m+1)!}\Delta_{2m}. \quad (104)$$

The quantity α is defined by

$$\alpha=\Omega B_0/2. \quad (105)$$

The factor 2 on the left side comes from the fact that the functions Δ_m are defined for an interval of integration between $-\phi_0$ and $+\phi_0$, while I_1 and I_2 are defined for half this interval. Some of the functions Δ can take on a negative index for $m=0$. Here, we have to make the distinction between Δ_m and $\Delta_m{}^*$:

$$\Delta_{-m}{}^*=-\Delta_{+m}{}^*=-\Delta_{+m}=-\Delta_{-m}. \quad (106)$$

The reason for this behavior can be readily found by comparing Eqs. (61) and (62). Again, Δ_0 should be discarded, since it does not contain a real resonance.

In Eqs. (103) and (104), the running index m determines the harmonic. There is an infinite set of terms associated with Δ_m which represent a power series in α^2, the lowest order term being always proportional to α^{2m-1}.

We now single out the contributions to a given harmonic and then determine the relative intensities.

In the case of I_1, we find for a given harmonic of order $2k+1$

$$-2I_1(2k+1)=\Omega\sum_{s=k}^{\infty}(-1)^{s+k}\alpha^{2s}\frac{\Delta_{2k+1}}{(s-k)!(s+k)!}-\Omega\sum_{s=k+1}^{\infty}(-1)^{s+k}\alpha^{2s}\times\frac{\Delta_{2k+1}}{(s-k-1)!(s+k+1)!}=\Omega\sum_{s=k}^{\infty}(-1)^{s+k}\alpha^{2s}\frac{2k+1}{(s-k)!(s+k+1)!}\times\Delta_{2k+1}. \quad (107)$$

Equation (107) is valid for frequencies near the resonance $\Omega=2k+1$. Thus, we may replace $2k+1$ by Ω in the coefficients and find after some rearranging

$$-2I_1(\Omega)=\frac{\Omega^2}{\alpha}\Delta_\Omega\sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k+\Omega}}{k!(k+\Omega)!}. \quad (108)$$

It is easy to show from equations similar to Eq. (107) that the same expression (108) holds for the even harmonics as well. Also included in the range of validity of Eq. (108) are the first two harmonics for which the representation Eq. (103) contains extra terms, as one may find from a straightforward calculation.

The application of the same principles to the second integral leads to the following expression:

$$2I_2(2k+1)=\Omega\sum_{s=k}^{\infty}(-1)^{s+k}\alpha^{2s}\frac{\Delta_{2k+1}}{(s-k)!(s+k)!}+\Omega\sum_{s=k+1}^{\infty}(-1)^{s+k}\alpha^{2s}\frac{\Delta_{2k+1}}{(s-k-1)!(s+k+1)!}=\Omega\sum_{s=k}^{\infty}(-1)^{s+k}\alpha^{2s}\frac{2s+1}{(s-k)!(s+k+1)!}\times\Delta_{2k+1} \quad (109)$$

from which we derive

$$2I_2(\Omega)=\frac{\Omega}{\alpha}\Delta_\Omega\sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k+\Omega}}{k!(k+\Omega)!}(2k+\Omega). \quad (110)$$

Again, Eq. (110) is valid for even and odd harmonics.

Equations (108) and (110) contain already a most valuable result: In the approximation made throughout this investigation, a given harmonic can be represented by a dispersion function,[12] while the central intensity is given by an expression which depends essentially on B_0. Thus, the frequency conditions and other restrictions we derived in detail in Secs. 8 and 9 govern the various harmonics in the relativistic case as well.

The sums in Eqs. (108) and (110) can be expressed in terms of Bessel functions of the order Ω and the argument 2α and their derivatives, since

$$\sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k+\Omega}}{k!(k+\Omega)!}=J_\Omega(\Omega B_0), \quad (111)$$

and

$$\sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k+\Omega}}{k!(k+\Omega)!}(2k+\Omega)=\Omega B_0\cdot J_\Omega'(\Omega B_0). \quad (112)$$

Thus, the final result may be written in the form (n being the order of the harmonic)

$$I_1(n)=-(\Omega/B_0)\Delta_nJ_n(nB_0), \quad (113)$$

[12] This has been postulated previously by J. L. Hirshfield and D. E. Baldwin, Bull. Am. Phys. Soc. **5**, 321 (1960). See also reference 4.

and

$$I_2(n)=\Omega\Delta_n J_n'(nB_0). \quad (114)$$

I_1 and I_2 must be squared before applying the averaging procedure. Under the aforementioned conditions

$$|\Delta_m\Delta_n|\ll\Delta_m^2 \quad \text{or} \quad \Delta_n^2; \quad m\neq n. \quad (115)$$

Then

$$I_1^2=n^2B_0^{-2}\Delta_n^2[J_n(nB_0)]^2, \quad (116)$$

and

$$I_2^2=n^2\Delta_n^2[J_n'(nB_0)]^2. \quad (117)$$

Since Δ_n is the only term which depends on ϕ_0, we find, with the help of the integral

$$\langle\Delta_n\rangle_{\phi_0}\equiv\tfrac{1}{2}[(\Omega-n)^2+(2\bar{\phi})^{-2}]^{-1} =\tfrac{1}{2}\omega_0^2[(\omega-n\omega_0)^2+T_0^{-2}]^{-1}, \quad (118)$$

for Poynting's vector [see Eq. (42)] the formula

$$\bar{S}_n(\omega)d\omega=\frac{e^2v_0^2\omega_0^2}{2\pi^3c^3}n^2\left\{\sin^2\theta\frac{[J_n(n\beta_0\cos\theta)]^2}{\beta_0^2\cos^2\theta} +[J_n'(n\beta_0\cos\theta)]^2\right\}[(\omega-n\omega_0)^2 +T_0^{-2}]^{-1}d\omega. \quad (119)$$

$\bar{S}_n(\omega)$ gives the total radiation in the frequency interval $d\omega$ centered at $\omega=n\omega_0$, into a unit solid angle in a direction making an angle θ with the orbital plane, and during a time T_0/π [see Eq. (43)].

In writing Eq. (119), we have exchanged the integers n and Ω. The reason is, of course, that under the conditions Eqs. (3) and (101), Ω^2 and, especially, the Bessel functions and their derivatives are slowly varying functions of Ω.

Equation (119) is the important result of this investigation: It says that the cyclotron radiation under relativistic conditions is a sum of individual, equidistant spectral lines, each of them having a dispersion profile determined by the collision probability. Equation (119) also contains the relative intensities of these lines in terms of the particle velocity and the angle between the observer and the orbital plane.

Let us state once more the limits of validity:

$$\bar{\phi}\gg2\pi \quad \text{or} \quad T_0^{-1}\ll\omega_0, \quad (120)$$

and

$$|\Omega-m|\ll m \quad \text{or} \quad |\omega-m\omega_0|\ll m\omega_0 \quad (121)$$

for the mth harmonic.

It might also be repeated that conditions (120) and (121) hardly affect the practical applicability of Eq. (119). This is easily seen by calculating the (half) half-width $\Delta\omega_H$ of the distribution function $[(\omega-n\omega_0)^{-2}+T_0^{-2}]^{-1}$. One finds

$$\Delta\omega_H=T_0^{-1} \quad (122)$$

independant of the order of the harmonic. $\Delta\omega_H$ is small compared with the gyrofrequency ω_0 according to Eq. (120) and, thus, *a fortiori*, compared with the resonance frequency $m\omega_0$. Hence, the dispersion profile ought to be a good approximation over the whole range of frequencies of interest.

The total emission in a given harmonic n can be found by integrating over the line contour, considering n^2 and the Bessel functions as slowly varying, as we did before. With the help of the integral[13] Eq. (76) we obtain

$$\pi\bar{S}_n=\frac{e^2v_0^2\omega_0^2}{2\pi c^3}n^2\left(\frac{\sin^2\theta}{\beta_0^2\cos^2\theta}J_n^2+J_n'^2\right)T_0. \quad (123)$$

According to Eq. (43), $\pi\bar{S}_n$ is the total radiation in the nth harmonic during the time T_c. This is a well known result, given for example by Schwinger,[14] Eq. (III.28), for the case of vanishing damping. Within the limits of our approximation, the radiation per unit time does not depend on the number of collisions, as it is well known from optical spectroscopy.

12. THE LIMIT OF VANISHING DAMPING

Equations (113) and (114) are valid independant of the number of collisions occurring. We would like to present, however, a very simple and elementary derivation of these expressions in the usually treated case of vanishing damping, i.e., when $T_0,\phi\rightarrow\infty$.

We first remark that in this limit the quantities defined in Eq. (49) reduce as follows:

$$x\equiv\phi_0/\bar{\phi}\rightarrow0, \quad x_0\equiv2\pi/\bar{\phi}\rightarrow0(\bar{\phi}\rightarrow\infty), \quad (124)$$

while

$$e^{-x_0}\rightarrow1 \quad \text{and} \quad (1-e^{-x_0})\approx2\pi/\bar{\phi}=\sigma(\bar{\phi}^{-1}). \quad (125)$$

The integrals in Eq. (48) vanish, making the second and the third term small compared with the first one. Thus we have

$$\int_0^\infty I^2(\bar{\phi}x)e^{-x}dx\approx\frac{\bar{\phi}^2}{2\pi^2}I^2(\phi_0=2\pi). \quad (126)$$

In Eq. (126), I stands for the integrals, Eqs. (35) and (37).

Since

$$I_1(2\pi)=2I_1(\pi) \quad \text{and} \quad I_2(2\pi)=2I_2(\pi), \quad (127)$$

we can express the special values of the integrals (35) and (37) which enter Eq. (126) at once in terms of Bessel functions:

$$I_1(\pi)=\frac{\Omega}{B_0}\pi J_\Omega(B_0\Omega), \quad I_2(\pi)=-\Omega\pi\frac{d}{dz}J_\Omega(z)|_{z=\Omega B_0}. \quad (128)$$

[13] The lower limit $-\infty$ corresponds to the summation over resonance *and* antiresonance term in Eqs. (70) and (71). However, we know from Secs. 8 and 9 that neglect or inclusion of the antiresonance term does not matter within the limits of our approximation.

[14] J. Schwinger, Phys. Rev. **75**, 1912 (1949).

13. ANGULAR DISTRIBUTION

The relative intensities and the angular distribution of the various harmonics have been investigated previously with the main accent on very high frequencies and ultrarelativistic particle energies.[15] This case is interesting in the study of accelerator devices and some astrophysical problems. However, if one has thermofusion plasmas in mind, it might be worthwhile to collect some simple results of slightly relativistic energies, let us say, for $0.5 \leqslant \beta_0 \leqslant 0.9$.

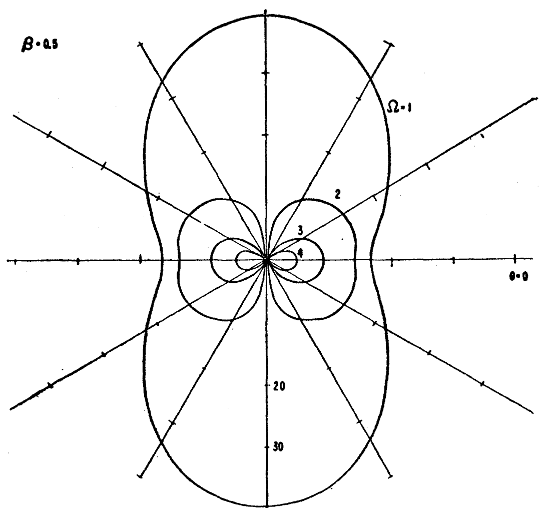

FIG. 2. Intensity distribution in the center of the various harmonics as function of the angle θ between observer and orbital plane. The particle's energy corresponds to $\beta_0=0.5$. The numerical values represent $\sin^2\theta\langle I_1^2\rangle+\langle I_2^2\rangle$ for the model case $\phi=2\pi$.

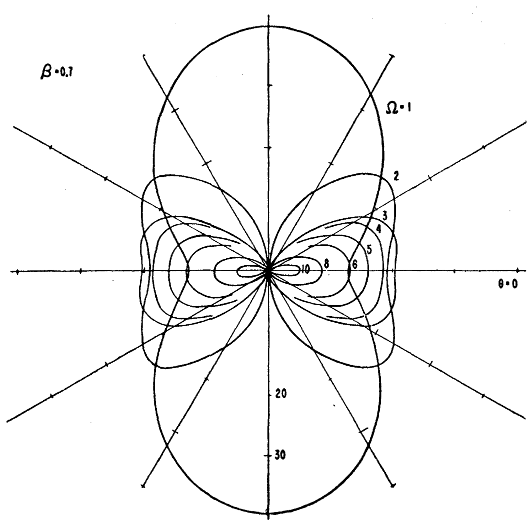

FIG. 3. Same as Fig. 2. Particle's energy corresponds to $\beta_0=0.7$.

[15] See for instance the graphs given by H. Rosner, Republic Aviation Corporation, Missile Systems Division, Technical Report No. 206-950-3, ASTIA AD 208 852 (unpublished).

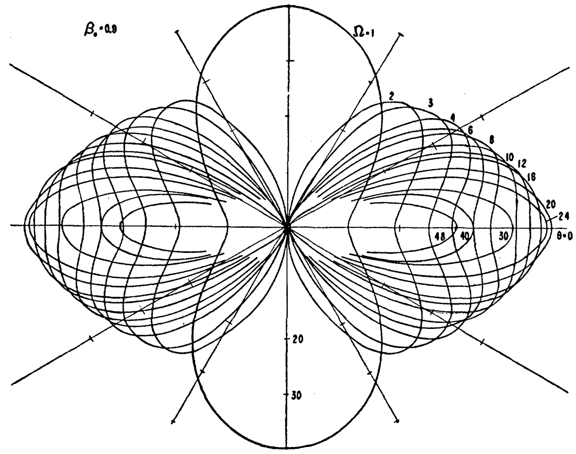

FIG. 4. Same as Fig. 2. Particle's energy corresponds to $\beta_0=0.9$.

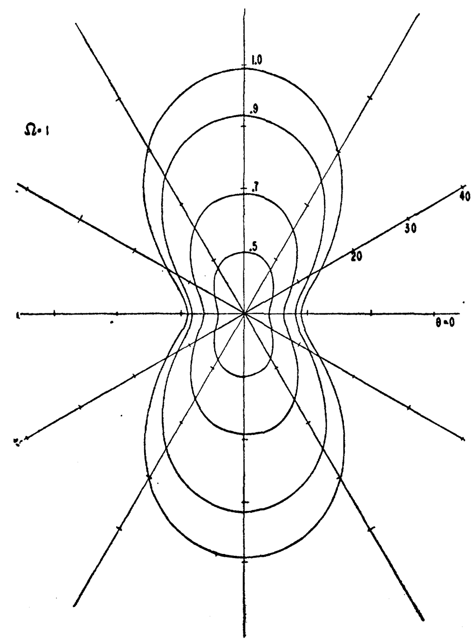

FIG. 5. Angular distribution of the first harmonic for different particle energies, represented by β_0. The numerical values correspond to the expression plotted in Figs. 2–5 multiplied by β_0^2.

In Figs. 2–4, the intensity distribution at the center of the various harmonics has been plotted for fixed particle energies (represented by β_0) and as a function of the angle θ from the orbital plane. The scale is somewhat arbitrary; the plotted numerical values correspond to

$$\sin^2\theta\langle I_1^2\rangle+\langle I_2^2\rangle \qquad (129)$$

for the model case $\bar{\phi}=2\pi$. Expression (129) is directly proportional to $\langle S_n\rangle$.

The figures illustrate clearly the fact that the higher frequencies are emitted in a very narrow angular range around the orbital plane, defined by

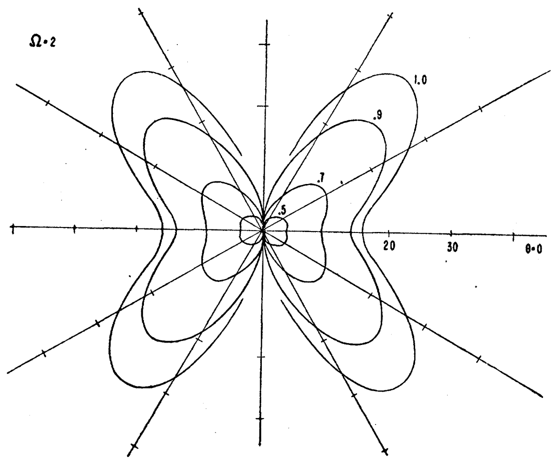

FIG. 6. Angular distribution of the second harmonic for different particle energies. Scales as in Fig. 5.

$\theta=0$. Perpendicular to the orbit only the first harmonic is observed, regardless of the particle's energy.

In Figs. 5 and 6, the angular distributions of the first and second harmonic, chosen as the two typical examples, are plotted for different particle energies. For easier comparison, the numerical values of the expression (129) have been multiplied[16] by β_0^2.

Note the marked angular dependence of the first harmonic in the case of high particle energy compared with the case of lower energy. Also remarkable is the increase in maximum angle in the case of the second harmonic and, of course, in the same manner of all higher harmonics. This again is a straightforward consequence of the basic dependence on $\beta_0 \cos\theta$.

The included limits $B_0=1$ need some comment, although they should not be taken literally. Obviously, the intensities of all harmonics of *finite* order stay finite; see Eq. (119). As one may easily prove by first integrating over the dispersion function, summing over all harmonics n by expanding the Bessel functions, and then by integrating over the sphere, Eq. (119) leads to the correct total energy

$$\tfrac{2}{3}(e^2/c^3)v_0^2\omega_0^2\Delta t/(1-\beta_0^2)^2. \tag{130}$$

[16] The dependance of the intensity on β_0 through ω_0 is omitted in this representation. This defect is irrelevant for our purpose of pointing out the angular dependance.

However, the *complete* expression for the even component, I_2, does diverge, as is obvious from Eq. (34); the odd terms, Eq. (33), stay finite, except in the limit of infinite frequency. On the other hand, the complete expression Eq. (34) leads *exactly* to the representation by Bessel functions in the limit $\bar{\phi}\to\infty$, as has been shown in Sec. 12.

The reason for this can be understood from Eq. (37). Here, the integral remains finite, but the "constant" term has a singularity whenever

$$\phi_0=m\pi, \quad m=1,2,3,\cdots. \tag{131}$$

These terms were neglected in the general derivation of the line contour, because they are small compared with the resonance contained in the integrals. This statement is apparently not true in the limit $B_0=1$, unless $\bar{\phi}\to\infty$ at the same time. As a matter of fact, the first term in Eq. (37) vanishes when averaged over ϕ_0 for $\bar{\phi}\to\infty$. We conclude that Eq. (119) holds for very high energies only for correspondingly small damping.

These details are, actually, without importance even in the case of ultrarelativistic particles, since their radiation characteristics are "abnormal" in this sense only in a very narrow angle around the orbit.

14. CONCLUSIONS

It has been shown that under the condition of weak damping, i.e., when the particle performs many revolutions in the magnetic field between two collisions, the relativistic cyclotron radiation can be represented by a sum of single spectral lines of which the contours are given by the same dispersion profile. This is similar to the behavior of lines in optical spectroscopy. At the same time, the relative intensities of the various lines are functions of the particle energy. This likewise is similar to optical spectroscopy. There is, however, the strong angular dependence which is characteristic of cyclotron radiation.

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