Energy Dependence of the Nucleon-Nucleon Phase Shifts*

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Starting from the analytic structure of partial wave amplitudes predicted by the Mandelstam representation, relativistic formulas are derived for the energy dependence of the phase shifts for nucleon-nucleon scattering, neglecting inelastic processes. These formulas depend on integrals over functions defined by a (numerically) soluble integral equation whose kernel is determined from the absorptive part of the amplitude in the nonphysical region. The contribution to this kernel from single-pion exchange is explicitly exhibited and the contribution from two-pion exchange is calculable. A generalization of the formulas to include phenomenological constants representing the unknown contribution of multimeson and other particle exchanges is given. The dependence of the phase shifts on these parameters is sufficiently simple to allow the formulas to be used for the least-squares fitting of empirical data. Further, these constants can be varied independently, and as much empirical information as is desired can be incorporated into the formulas without destroying this independence. In the case of coupled states, the phenomenological formulas satisfy unitarity only approximately; this approximation can be removed by a subsidiary calculation, which destroys the independence of the parameters for these states. Because of the neglect of inelastic processes, the range of validity of the formulas is expected to be from 0 to approximately 400 Mev.

I. INTRODUCTION

IN order to make use of nucleon-nucleon scattering data at more than one energy to cut down on the multiplicity of phase shift solutions often obtained when analyzing data at a single energy, and to make use of the charge independence assumption to assist in the analysis of *n-p* experiments for the isotopic singlet phase shifts, it is necessary to make some assumption about the energy dependence of the phase shifts. It is clearly desirable that any formula used for this purpose should: (1) allow any phenomenological parameters introduced to be given a theoretical interpretation; (2) allow these parameters to be varied independently; (3) contain all that is currently calculable from meson theory of the two-nueleon interaction; (4) be flexible enough to incorporate new theoretical and experimental information as it becomes available without requiring recoding of the machine calculation used to determine the phenomenological parameters from experiment. We believe that the formulas developed below meet all these criteria in a straightforward way. Although the practical necessity which led to their development was connected with the problem of using large electronic computers to analyze scattering data, the formulas exhibit in a very simple way the structure of partial wave amplitudes predicted by the Mandelstam representation. We will see that, although we deal with a velocity-dependent interaction, its structure is expressible in terms of a simple function which in various regions corresponds to the exchange of systems of different mass, and which therefore can be roughly pictured as giving both the strength and the localization of the interaction energy in configuration space. We therefore believe that this simplified discussion may be of interest to those concerned with the two-nucleon problem, even though some may not be directly interested in the problem of data analysis.

II. THEORY

It has been conjectured by Mandelstam[1] that two-particle scattering amplitudes can always be written in terms of a double spectral representation whose density functions are nonzero only in certain regions which can be determined from the mass spectrum of strongly interacting particles. Although this representation has yet to be derived from the postulates of local field theory, it is valid to at least sixth order in perturbation theory[2] and many of the necessary conditions for its validity have been proved to higher orders.[3] It is rigorously valid for the Schrödinger equation with any potential which can be represented by a superposition of Yukawa potentials.[4] If one projects out a particular partial wave amplitude from a system of two equal mass particles, this representation predicts[5] that this amplitude is a real analytic function of q^2, the center-of-mass momentum, except for two cuts on the real axis. For $q^2>0$ the discontinuity across the cut (twice the imaginary part of the amplitude) is known from unitarity up to the lowest threshold for the production of an additional particle. Above this threshold the unitarity condition will couple this amplitude to amplitudes for multiparticle systems which we are at present unable to calculate. The fundamental approximation made in this approach is to ignore this coupling. Fortunately we know from

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] R. J. Eden, C. Enz, and J. Lascoux, Bull. Am. Phys. Soc. **5**, 284 (1960).
[3] R. J. Eden (private communication).
[4] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Trieman (to be published).
[5] See G. F. Chew, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, California, 1959), Vol. 9, p. 29 for a general discussion and for references.

Phillips[6] that for the two-nucleon system the experimentally determined inelastic amplitudes are small enough so that the elastic amplitude may be accurately described by real phase shifts up to about 400 Mev, which gives us a useful range of applicability. It should also be noted that if inelastic processes are consistently ignored, the Mandelstam representation is rigorously true to all orders in perturbation theory.[7]

For $q^2<0$, the Mandelstam prescription tells us that the cut will start at $q^2=-E^2/4$, where E is the lowest energy state of the particle-antiparticle system with the same quantum numbers as the scattering state we are considering. If, as in the case we consider, this system is a single particle, the discontinuity is just twice the imaginary part of the relativistic Born approximation with the coupling constant interpreted as the renormalized coupling constant. Additional cuts will start according to the same prescription whenever more massive states can be reached from the particle-antiparticle system. In our case, the lowest state is the pion; the next is the two-pion system, which can also be treated by means of the Mandelstam representation. Explicit formulas for the two-pion discontinuity have been derived by Goldberger, MacDowell, Grisaru, and Wong.[8] The three-pion state which comes next is a function of five relativistic invariants and is beyond the reach of present techniques unless we approximate it by the bound state suggested by Chew[9] or the neutral vector boson proposed by Sakurai.[10] We conclude that for the present the portion of the cut beyond $q^2=-9\mu^2/4$ will have to be treated phenomenologically. This has the comforting aspect that it is only this region of the complex plane for which reasonable doubts still remain as to the validity of the Mandelstam representation. Further, for fixed angular momentum states and equal masses, Bjorkèn[11] derived the analytic structure just described to all orders in perturbation theory, without restriction to elastic processes. Consequently, we consider the present application to be largely independent of the ultimate status of the more general theory. We can also hope that this region may be represented by a reasonably small number of phenomenological constants for the calculation of scattering amplitudes over an interesting range of energies—but this remains to be seen.

In order to apply our knowledge of the analytic structure just described to the calculation of partial wave amplitudes, we need to know precisely which partial wave amplitudes possess only the Mandelstam singularities. This has been discussed in detail by Goldberger, MacDowell, Grisaru, and Wong.[8] They conclude that it is true for $[(q^2+M^2)/q^2]^{\frac{1}{2}}e^{i\delta}\sin\delta=h(q^2)$. Since, according to our fundamental approximation, the phase shift is real for $q^2>0$, we have immediately that $\mathrm{Im}h^{-1}=-q/(q^2+M^2)^{\frac{1}{2}}$ for this cut. We also know that $\mathrm{Im}h=\pi r(q^2)$ for $q^2<-\frac{1}{4}\mu^2$, where r is some function to be computed from meson theory. Since Mandelstam has shown[12] that for a function with this structure (i.e., real phase on the right and a left cut), we can always make the decomposition $h(q^2)=N(q^2)/D(q^2)$, where N has only the left cut and D the right cut, we see that

$$\mathrm{Im}D(q^2)=-\frac{qN(q^2)}{(q^2+M^2)^{\frac{1}{2}}},\quad q^2>0;$$
$$\mathrm{Im}N(q^2)=\pi r(q^2)D(q^2),\quad q^2<-\tfrac{1}{4}\mu^2. \tag{1}$$

If we can assume that h vanishes at infinity (see below), we can immediately write down the dispersion relations[13]

$$N(q^2)=\int_{-\infty}^{-\frac{1}{4}\mu^2}dp^2\frac{r(p^2)D(p^2)}{p^2-q^2}, \tag{2}$$

and

$$D(q^2)=1-\frac{q^2}{\pi}\int_0^\infty dp^2\frac{N(p^2)}{p(p^2+M^2)^{\frac{1}{2}}(p^2-q^2)}, \tag{3}$$

where we have made use of the arbitrary constant given us by taking a ratio to make a subtraction in the D equation. Substituting the expression for N into the equation for D, we obtain[14]

$$D(q^2)=1+q^2\int_{-\infty}^{-\frac{1}{4}\mu^2}dp^2K(p^2,q^2)r(p^2)D(p^2), \tag{4}$$

where

$$\begin{aligned}K(p^2,q^2)&=\frac{1}{\pi}\int_0^\infty ds^2\frac{1}{s(s^2+M^2)^{\frac{1}{2}}(s^2-q^2)(s^2-p^2)}\\&=\frac{1}{p^2-q^2}[T(p^2)-T(q^2)],\end{aligned} \tag{5}$$

and

$$T(p^2)=\frac{2\tan^{-1}[(p^2+M^2)/(-p^2)]^{\frac{1}{2}}}{\pi[-p^2(p^2+M^2)]^{\frac{1}{2}}}. \tag{6}$$

Although at first sight there might be singularity at $q^2=-M^2$, the kernel and its derivatives are in fact continuous at this point, and h has only the required Mandelstam singularities. Making the useful change of variable $q^2=-\mu^2/4y$, $D(q^2)=f(y)$, and $r(q^2)$

[6] R. N. J. Phillips (private communication).
[7] S. Mandelstam, Phys. Rev. **115**, 1752 (1959).
[8] M. L. Goldberger, S. W. MacDowell, M. Grisaru, and D. Wong (to be published).
[9] G. F. Chew, Phys. Rev. Letter 4, 142 (1960).
[10] J. J. Sakurai, Ann. Phys. (to be published).
[11] J. Bjorkèn, Bull. Am. Phys. Soc. **4**, 448 (1959).
[12] S. Mandelstam (private communication).
[13] H. P. Noyes and D. Y. Wong, Phys. Rev. Letters **3**, 191 (1959).
[14] This kernel differs from the similar expression given by G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); and University of California Radiation Laboratory Report UCRL-8728 (unpublished), in their discussion of the pion-pion problem because the scattering amplitude is normalized at a different point.

$=-\frac{1}{4}G^2yR(y)$, we obtain

$$f(y)=1+\frac{G^2\mu}{4M}\int_0^1 dz\, R(z)f(z)\frac{t(z)-t(y)}{z-y},$$
$$t(z)=2z(M/\mu)\{\tan^{-1}[4(M/\mu)^2z-1]^{\frac{1}{2}}\}/\pi[4(M/\mu)^2z-1]^{\frac{1}{2}}. \tag{7}$$

Here G^2 is the renormalized pion-nucleon coupling constant, M the nucleon mass, and μ the pion mass. Since for $q^2>0$, $q^2T(q^2)=iq/(q^2+M^2)^{\frac{1}{2}}+L(q^2)$

$$L(q^2)=\frac{2q\ln\{[(M^2+q^2)^{\frac{1}{2}}+q]/M\}}{\pi(q^2+M^2)^{\frac{1}{2}}}, \tag{8}$$

we can write down the phase shift in the physical region in terms of the solution of the integral equation for $f(y)$ as

$$\frac{(q^2+M^2)^{\frac{1}{2}}}{q}\tan\delta=\frac{\dfrac{G^2}{4}\displaystyle\int_0^1\frac{R(y)f(y)dy}{1+4yq^2/\mu^2}}{1+\dfrac{G^2q^2}{M\mu^2}\displaystyle\int_0^1\frac{R(y)f(y)t(y)}{1+4yq^2/\mu^2}-\frac{1}{4}L(q^2)G^2\int_0^1\frac{R(y)f(y)dy}{1+4yq^2/\mu^2}}\equiv\frac{N(q^2)}{d(q^2)}. \tag{9}$$

Our theoretical problem is therefore reduced to computing as much as we can of the function $r(q^2)=-\frac{1}{4}yG^2R(y)$, specifying the rest phenomenologically, and solving Eq. (7). Experience shows that this equation is readily soluble numerically, so that in the future the specification of a meson theory for nuclear forces or of a phenomenological model could be considered to consist of the specification of $R(y)$ for each angular momentum state. From the above discussion, we see that the ranges of the variable y can be directly related to the mass-energy of the quanta being exchanged; therefore, by the uncertainty principle, they can be roughly considered as specifying the range of the interaction, while the magnitude and sign of this function tells us the effective strength of the interaction at that range and whether it is attractive or repulsive. Thus, although we have abandoned the static potential picture of the interaction, we still have a model with intuitively descriptive physical properties.

So far we have not discussed the convergence of the integrals in our dispersion relations. Since we know that δ_l must go to zero as q^{2l+1}, we can always make l subtractions to guarantee this property; as one subtraction insures convergence, even if the phase shift goes to a constant value at infinity, the problem arises only for S waves. The paper already referred to[8] shows that the full triplet amplitude requires no subtraction for convergence, and, if known, determines the value of the singlet amplitude at $q^2=-M^2$. Therefore we can make a subtraction at this point and determine the singlet amplitude with no arbitrary parameters. This clears up a puzzle which has existed about the relativistic Born approximation. Since the one pion interaction function for S waves is given by[13] $R(y)=1$, the Born approximation to Eq. (9) [obtained by letting $f(y)=1$ and $d=1$] gives

$$\frac{(q^2+M^2)^{\frac{1}{2}}}{q}\tan\delta_0{}^B=\frac{1}{4}G^2\frac{\mu^2}{4q^2}\ln\left(1+\frac{4q^2}{\mu^2}\right), \tag{10}$$

rather than the relativistic Born approximation[15]

$$\frac{(q^2+M^2)^{\frac{1}{2}}}{q}\tan\delta_0{}^B=\frac{1}{4}G^2\left[\frac{\mu^2}{4q^2}\ln\left(1+\frac{4q^2}{\mu^2}\right)-1\right]. \tag{11}$$

If we make the subtraction at $q^2=-M^2$, using the Born approximation for the value of the triplet amplitude at this point, we obtain Eq. (11). Whereas Eq. (10) gives an S phase which goes to zero as q (and in fact is just the Born approximation for a static Yukawa potential), Eq. (11) has the unreasonable threshold dependence q^3. From the above discussion, however, we see that this is due to taking an unrealistic value for the amplitude at $q^2=-M^2$. In practice, since the point $q^2=-M^2$ is well beyond the region where we can calculate $r(q^2)$ from theory, we will make the subtraction instead at zero kinetic energy, fitting the empirically known singlet scattering length and insuring correct threshold behavior.

We can also clear up another question by this discussion. In configuration space, the static approximation to single pion exchange gives a repulsive delta function at the origin (the Fourier transform of the constant term) in addition to the Yukawa potential. It has been conjectured that this repulsion might be spread out by relativistic effects to form a "hard core." However, since we now see clearly that this repulsive interaction can only be revealed (roughly speaking) by particles of sufficient energy to explore dimensions of the order of a nucleon Compton wavelength, it is much too small to account for the physical effects that the hard core has been used to explain. Consequently we expect that these effects have their origin in some dynamical feature of the system, such as the three-pion state or the appropriate neutral vector boson of Sakurai, rather than coming from relativistic kinematics. We wish to emphasize in this connection that the present treatment includes relativistic effects ex-

[15] R. Cziffia, M. J. Moravcsik, M. H. MacGregor, and H. P. Stapp, Phys. Rev. 114, 330 (1959).

actly, insofar as inelastic processes and electromagnetic interactions can be ignored.

III. PHENOMENOLOGICAL EXTENSION

We have noted that present techniques allow the function $r(q^2)=-\frac{1}{4}G^2yR(y)$ which specifies the two-nucleon interaction to be calculated only for $y>\frac{1}{9}$. Therefore we could introduce phenomenological parameters simply by specifying $R(y)$ below this value (e.g., by power series). This has the practical disadvantage that whenever a parameter is changed, a new solution for the integral equation must be calculated, placing the least squares adjustment of the parameters beyond the reach of any electronic computer likely to be developed in the near future. But note that in practice the solution of the integral equation is achieved by replacing the integral with a finite sum. This is equivalent to replacing $R(y)$ by $\sum_i \Delta_i R(y_i) \times\delta(y-y_i)$, or replacing the continuous cut by a finite number of poles. Therefore, if we wish to introduce a finite number of parameters for the unknown part of the cut, we can rewrite Eq. (7) as

$$D(-\mu^2/4y)=1+\frac{G^2\mu}{4M}\sum_i \frac{\Delta_i D(-\mu^2/4y_i)[t(y_i)-t(y)]}{y_i-y}R_i + \frac{G^2\mu}{4M}\int_0^1 dz\, R(z)f(z)\frac{t(z)-t(y)}{z-y}, \quad (12)$$

where the points y_i must be less than $\frac{1}{9}$, and $R(y)$ is now to be interpreted as that part of the interaction which we can compute from meson theory. Since the points y_i are arbitrary, we can equally well take our parameters to be $\alpha_i=R_i\Delta_i D(-\mu^2/4y_i)$. Then the function D becomes a sum of independent terms

$$D(-\mu^2/4y)=f(y)+\sum_i \alpha_i g_i(y), \quad (13)$$

where $f(y)$ is still the solution of Eq. (7) [with the new interpretation of $R(y)$], and the g_i are defined by the equations

$$g_i(y)=\frac{t(y_i)-t(y)}{y_i-y}+\frac{G^2\mu}{4M}\int_0^1 dz\, R(z)g_i(z)\frac{t(z)-t(y)}{z-y}. \quad (14)$$

The phase shift is then given by

$$\frac{(M^2+q^2)^{\frac{1}{2}}}{q}\tan\delta=\frac{E(q^2)+\sum_i \alpha_i F_i(q^2)}{G(q^2)+\sum_i \alpha_i H(q^2)}, \quad (15)$$

where

$$E(q^2)=\tfrac{1}{4}G^2\int_0^1 dy\frac{R(y)f(y)}{1+4yq^2/\mu^2},$$

$$F_i(q^2)=\frac{1}{1+4y_iq^2/\mu^2}+\tfrac{1}{4}G^2\int_0^1 dy\frac{R(y)g_i(y)}{1+4yq^2/\mu^2},$$

$$G(q^2)=1+\frac{G^2\mu q^2}{M\mu^2}\int_0^1 \frac{R(y)f(y)t(y)}{1+4yq^2/\mu^2}dy-L(q^2)E(q^2), \quad (16)$$

$$H_i(q^2)=\frac{4q^2t(y_i)/\mu^2}{1+4y_iq^2/\mu^2}+\frac{G^2\mu q^2}{M\mu^2}\int_0^1 \frac{R(y)g_i(y)t(y)}{1+4yq^2/\mu^2} dy-L(q^2)F_i(q^2).$$

We see that this equation has the desired properties: (a) the parameters α_i are independent and (b) the functions E, F_i, G, H_i can be computed prior to and independent of the least squares adjustment of the α_i. Of course the positions y_i are still arbitrary, except for the requirement that they lie in the range $0\leq y\leq\frac{1}{9}$ [or $\frac{1}{4}$ if we have not computed the two-pion $R(y)$]. If it is indeed true that over some finite range of energies the scattering amplitude is sensitive only to some average feature (or features) of the multiparticle cuts, our predictions should be insensitive to these values. How good this approximation is can only be tested by actual calculation. We note that since the parameterization corresponds to an even number of subtractions, we have assumed that the phase shift goes to zero at infinity. We can, however, obtain the case of a constant phase shift at infinity by simply taking one of the y_i to be zero.

One feature of this choice of parameterization should be stressed. If we consider the integrals in Eqs. (15) and (16) to be replaced by finite sums, we have N poles in the physical sheet of the Riemann surface and $2N$ parameters, (i.e., their positions and residues). At least in the nonrelativistic limit, there can only be N poles on the unphysical sheet. If we had chosen instead to determine the parameters directly by making subtractions using empirical values of the phase shift, we would have no such guarantee, and might end up with more poles on the unphysical sheet than on the left cut. Then, if we let the interaction go to zero, some of these would remain, giving a finite scattering amplitude and violating our physical assumptions. This situation would correspond to the ambiguity pointed out by Castillejo, Dalitz, and Dyson[16]; our parameterization insures that this cannot happen. We do, however, have to check that the values of α_i determined from experiment predict no poles other than those we have introduced. These would be either ghosts or bound states, depending on the sign of their residues, and would not be consistent with our assumptions. [Of course we *should* get the deuteron pole automatically if $R(y)$ is physically correct; in practice we could either introduce this pole explicitly, or obtain it by our phemenological parameters.] Note that since we have achieved independence of the α_i by not taking them to be simply the residues of poles in $h(q^2)$, we will have to calculate these residues if we wish to find out what approximation to the multimeson cut our parameters imply.

It is perhaps of interest to point out the connection of our formula with the at-first-sight entirely different

[16] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

TABLE I. Contribution of single pion exchange to the interaction function $R(y)=-4r(q^2)/yG^2=-4\,\mathrm{Im}h^B(q^2)/\pi yG^2$ for $0\leq y\leq 1$; $y=-\mu^2/4q^2$.

State	Coefficient 1	y	y^2	y^3	y^4	y^5
$^1S_0$	1					
$^3P_0$	-1					
$^1P_1$	-3	6				
$^3P_1$	0	1				
$^1D_2$	1	-6	6			
$^3D_2$	0	-9	12			
$^1F_3$	-3	36	-90	60		
$^3F_3$	0	6	-20	15		
$^1G_4$	1	-20	90	-140	70	
$^3G_4$	0	-30	120	-315	168	
$^1H_5$	-3	90	-630	1680	-1890	756
$^3H_5$	0	15	-140	420	-504	210
$^3S_1$	1					
ϵ^1	$2^{\frac{1}{2}}$	$-2^{\frac{1}{2}}(3)$				
$^3D_1$	2	-3				
$^3P_2$	$-2/5$	$3/5$				
ϵ^2	$-6^{\frac{1}{2}}/5$	$6^{\frac{1}{2}}(9/5)$	$-6^{\frac{1}{2}}(2)$			
$^3F_2$	$-3/5$	$12/5$	-2			
$^3D_3$	$9/7$	$-36/7$	$30/7$			
ϵ^3	$3^{\frac{1}{2}}(6/7)$	$-3^{\frac{1}{2}}(108/7)$	$3^{\frac{1}{2}}(300/7)$	$-3^{\frac{1}{2}}(30)$		
$^3G_3$	$12/7$	$-60/7$	$180/7$	$-30/7$		
$^3F_4$	$-4/9$	$10/3$	$-20/3$	$35/9$		
ϵ^4	$-5^{\frac{1}{2}}(2/9)$	$5^{\frac{1}{2}}(20/3)$	$-5^{\frac{1}{2}}(100/3)$	$5^{\frac{1}{2}}(490/9)$	$-5^{\frac{1}{2}}(28)$	
$^3H_4$	$-5/9$	$20/3$	$-160/9$	$280/9$	-14	
$^3G_5$	$15/11$	$-180/11$	$480/11$	$-840/11$	$378/11$	
ϵ^5	$30^{\frac{1}{2}}(3/11)$	$-30^{\frac{1}{2}}(135/11)$	$30^{\frac{1}{2}}(1050/11)$	$-30^{\frac{1}{2}}(2640/11)$	$30^{\frac{1}{2}}(3396/11)$	$-30^{\frac{1}{2}}(126)$
$^3I_5$	$18/11$	$-315/11$	$1680/11$	$-3780/11$	$3780/11$	$-1386/11$

approach through dispersion relations at fixed angle given by Cini, Fubini, and Stanghellini[17] in the first application of the Mandelstam representation to nucleon-nucleon scattering. Since $L(q^2)\rightarrow 0$ in the nonrelativistic limit, our formula states that $q\cot\delta$ is a rational function of q^2, which was the practical approximation they used. In fact, if we represent the one-meson cut by a single pole at $y=\frac{1}{2}$ with the appropriate (y-average) residue, and the multimeson cut by a single pole adjusted to fit the scattering length and effective range, we obtain exactly their formula for the $^1S_0$ phase. Fubini and Stanghellini[18] have also given similar expressions for the ^3P phases. While this method is quite useful for giving simple formulas valid over a limited energy range, we find that it is computationally simpler to solve integral equations than the equivalent algebraic equations if more than three or four poles are used.

So far we have done nothing to insure that $\tan\delta_l$ goes to zero as q^{2l+1}. If we knew $R(y)$ exactly, this property would be guaranteed. Since we do not, we are at liberty to determine l of our phenomenological parameters α_i by this requirement. As this is equivalent to making l subtractions at $q^2=0$ in our original dispersion relations, this will correspondingly suppress the contribution of the (unknown) portion of the cut for large q^2. This is equivalent to the well-known fact that higher angular momentum states are sensitive only to the longest range part (lightest quanta) of the interaction. We can also determine additional α_i by requiring the phase shift to take on empirically determined values at specific energies. This amounts to a redefinition of the functions E, F, G, H in Eq. (15), and reduces the number of parameters without destroying their independence. Explicit formulas are given in the Appendix. The values of $R(y)$ for single-pion exchange through $J=5$ are given in Table I. We note that if we use only the l conditions at $q^2=0$ and leave no free parameters, our formulas give a straightforward answer to the question of what unitary expression should be used for the scattering amplitude due to single-pion exchange.

We have so far ignored the coupling between triplet states of the same j but different l. In terms of the "nuclear bar" phase shifts defined by Stapp[19] with the simplified notation $\delta_{j\pm 1,j}\equiv\delta_{\pm}$, $\epsilon^j\equiv\epsilon$, we must discuss the amplitudes

$$h_{\pm}(q^2)=(e^{2i\delta_{\pm}}\cos 2\epsilon-1)(M^2+q^2)^{\frac{1}{2}}/2iq,$$
$$h_0(q^2)=(e^{i(\delta_++\delta_-)}\sin 2\epsilon)(M^2+q^2)^{\frac{1}{2}}/q. \quad (17)$$

We see that if we neglect the coupling parameter ϵ as a first approximation, the $h_{\pm}(q^2)$ are precisely of the form already considered, so we can compute δ_+ and δ_- by the method developed above. As has already been noted by Wong,[20] the Stapp form for the coupling

[17] M. Cini, S. Fubini, and A. Stanghellini, Phys. Rev. **114**, 1633 (1959).

[18] S. Fubini and A. Stanghellini (private communication).

[19] H. P. Stapp, T. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

[20] D. Y. Wong, Phys. Rev. Letters **2**, 406 (1959).

amplitude then tells us this function has a known cut on the left and a known phase on the right. But our D functions have the negative of this phase on the right and are real on the left, so following Omnès[21] we can immediately write down an exact expression for $h_0(q^2)$, and $\sin 2\epsilon$:

$$\frac{(q^2+M^2)}{q}\sin 2\epsilon=\frac{(q^2+M^2)\sin\delta_+\sin\delta_-}{q^2N_+N_-}\left[\sum_i\frac{\alpha_i{}^0}{1+4y_i{}^0q^2/\mu^2}+\tfrac{1}{4}G^2\int_0^1\frac{R(y)D^+(-\mu^2/4y)D^-(-\mu^2/4y)}{1+4yq^2/\mu^2}dy\right]$$
$$=\frac{(q^2+M^2)\sin\delta_+\sin\delta_-}{q^2N_+N_-}\left[\sum_i\frac{\alpha_i{}^0}{1+4y_i{}^0q^2/\mu^2}+E^0(q^2)+\sum_i\alpha_i{}^+F_i{}^0(q^2)+\sum_i\alpha_i{}^-E_i{}^0(q^2)+\sum_{ij}\alpha_i{}^+\alpha_j{}^-H_{ij}{}^0(q^2)\right].\quad(18)$$

In this expression $N^\pm$ or $D^\pm$ are defined by Eqs. (13) or (15), and E^0, F^0, G^0, H^0 by

$$E^0(q^2)=\tfrac{1}{4}G^2\int_0^1\frac{R(y)f^+(y)f^-(y)}{1+4yq^2/\mu^2}dy,$$
$$F_i{}^0(q^2)=\tfrac{1}{4}G^2\int_0^1\frac{R(y)g_i{}^+(y)f^-(y)}{1+4yq^2/\mu^2}dy,$$
$$G_i{}^0(q^2)=\tfrac{1}{4}G^2\int_0^1\frac{R(y)f^+(y)g_i{}^-(y)}{1+4yq^2/\mu^2}dy,\quad(19)$$
$$H_{ij}{}^0(q^2)=\tfrac{1}{4}G^2\int_0^1\frac{R(y)g_i{}^+(y)g_i{}^-(y)}{1+4yq^2/\mu^2}dy.$$

Having now a first approximation for the phase shifts and coupling parameter, we can make use of the exact unitarity condition for the coupled states. For δ_- we can use

$$\mathrm{Im}h_\pm{}^{-1}(q^2)=-\frac{q}{(q^2+M^2)^{\frac{1}{2}}}\left[1+\frac{\frac{1}{2}\sin^2 2\epsilon}{\cos^2\delta\cos 2\epsilon-1}\right]^{-1},\quad q^2>0\quad(20)$$

to compute a new kernel for the $D^\pm$ equations, namely,

$$K^1(p^2,q^2)=\frac{1}{\pi}\int_0^\infty ds^2\Big/\left[1+\frac{\frac{1}{2}\sin^2 2\epsilon}{\cos 2\delta\cos 2\epsilon-1}\right]s(s^2+M^2)^{\frac{1}{2}}(s^2-p^2).\quad(21)$$

The equations can then be solved to obtain better values for δ_-. Since $\mathrm{Im}h_+{}^{-1}$ diverges as $q/\sin^2\epsilon$ as $q^2\to 0$, we cannot use this method as it stands without a mathematical investigation of the meaning of the contour integral defining K. Alternatively, for either δ_+ or δ_- we can use the nonlinear integral equation

$$\mathrm{Re}h_\pm(q^2)=\tfrac{1}{2}(q^2+M^2)^{\frac{1}{2}}\sin^2\delta_\pm\cos 2\epsilon/q$$
$$=\int_{-\infty}^{-\frac{1}{4}}dp^2\frac{r(p^2)}{p^2-q^2}-\frac{P}{\pi}\int_0^\infty\frac{dp^2}{p}\times\frac{(p^2+M^2)^{\frac{1}{2}}(\cos 2\delta_\pm\cos 2\epsilon-1)}{(p^2-q^2)},\quad(22)$$

which can be solved by iteration, assuming we know ϵ and the residues of the phenomenological poles in $r(q^2)$. Knowing δ_+ and δ_-, we can then compute $\sin 2\epsilon$ from Eq. (18), and repeat the procedure until convergence is obtained. However, for the least squares adjustment of the α_i we must clearly remain content with the first approximation given above if these parameters are to be varied independently. After the adjustment to experiment has been carried out, we can then carry through the iteration procedure just described and discover whether further adjustment of the parameters to conform to the unitarity condition is required. As before, we can insure the correct behavior at $q^2=0$ and can fit the phase shifts and coupling parameter at specific energies by a redefinition of E, F, G, H (see Appendix).

For single-pion exchange it is possible to include static Coulomb effects, the charged-neutral pion mass difference, and differences between the renormalized charged and neutral coupling constants, as will be discussed in a separate paper by Wong and Noyes.[22]

CONCLUSION

We have shown that it is possible to give explicit formulas for the nucleon-nucleon phase shifts in terms of functions which are readily calculable from an integral equation. This equation depends on meson theory through a single function for each partial wave, which is exhibited for single-pion exchange and calculable for two-pion exchange. Phenomenological parameters may be incorporated to fit threshold behavior and empirical phase shifts; the formulas can still be expressed in terms of functions which can be calculated without knowledge of the parameters. The parameters are therefore suitable for a least squares adjustment to empirical data. The formulas are relativistic and unitary, onsofar as inelastic processes are negligible. Any field-theoretic description of the two-nucleon interaction compatible with the Mandelstam representation can be incorporated into this framework.

[21] R. Omnès, Nuovo cimento 8, 316 (1958).
[22] D. Y. Wong and H. P. Noyes, Bull. Am. Phys. Soc. 5, 50 (1960).

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APPENDIX

Assume that $(q^2+M^2)^{\frac{1}{2}}\tan\delta/q$ is to go as q^{2l+1} as $q^2\to 0$ and to take on $K-l$ values X_k at energies $q_k^2(l+1\leq k\leq K)$. Assume further that we have introduced I phenomenological poles, $I\leq K$. Then we can rewrite Eq. (15) as

$$\frac{(q^2+M^2)^{\frac{1}{2}}}{q}\tan\delta=\frac{A(q^2)+\sum_{r=1}^{R}\alpha_r B_r(q^2)}{C(q^2)+\sum_{r=1}^{R}\alpha_r D_r(q^2)}, \tag{A-1}$$

where $R=I-K$ and

$$\begin{aligned}A(q^2)&=E(q^2)+\sum_{i=1}^{K}\beta_i F_i(q^2),\\ B_r(q^2)&=F_r(q^2)+\sum_{i=1}^{K}\beta_{ir}F_i(q^2),\\ C(q^2)&=G(q^2)+\sum_{i=1}^{K}\beta_i H_i(q^2)-L(q^2)A(q^2),\\ D_r(q^2)&=H_r(q^2)+\sum_{i=1}^{K}\beta_{ir}H_i(q^2)-L(q^2)B_r(q^2).\end{aligned} \tag{A-2}$$

The coefficients β_i and β_{ir} are given by

$$\begin{aligned}\beta_i&=\sum_{k=l+1}^{K}(S^{-1})_{ik}[X_kG(q_k^2)-E(q_k^2)]-\sum_{k=1}^{l}(S^{-1})_{ik}E^{k-1},\\ \beta_{ir}&=\sum_{k=l+1}^{K}(S^{-1})_{ik}[X_kH_{K+r}(q_k^2)-F_r(q^2)]-\sum_{k=1}^{l}(S^{-1})_{ik}F_{K+r}{}^{k-1},\end{aligned} \tag{A-3}$$

where the matrix S_{ki} whose inverse appears in Eq. (A-3) is defined by

$$\begin{aligned}S_{ki}&=F_i{}^{k-1}, && 1\leq k\leq l\\ &=F_i(q_k^2), && l+1\leq k\leq K,\end{aligned} \tag{A-4}$$

and E^{k-1}, F_i^{k-1} denote the $(k-1)$st derivatives of $E(q^2)$ and $F_i(q^2)$ evaluated at $q^2=0$; that is

$$\begin{aligned}E^{k-1}&=\tfrac{1}{4}G^2\int_0^1(-4y/\mu^2)^{k-1}R(y)f(y)dy,\\ F_i{}^{k-1}&=(-4y_i/\mu^2)^{k-1}+\tfrac{1}{4}G^2\int_0^1(-4y/\mu^2)^{k-1}R(y)g_i(y)dy.\end{aligned} \tag{A-5}$$

The expression for the coupling parameter [Eq. (18)] can be similarly redefined to insure that $\sin 2\epsilon$ goes to zero as q^{2j+1} and to fit experimental values of $\sin 2\epsilon/\sin\delta_+\sin\delta_-$. In this case, we must assume that the values $\beta_i{}^{\pm}, \beta_{ir}{}^{\pm}$ have already been obtained by the method given above. The resulting formulas are lengthy and not particularly illuminating. Explicit equations including Coulomb and mass difference effects are now being prepared, and will be supplied by the author on request.

The above approximate formulas for the coupled amplitudes contain an additional defect beside the approximate treatment of unitarity. Although the uncoupled amplitudes have only the Mandelstam singularities, as asserted, Goldberger *et al.*[8] find that the coupled amplitudes have an additional kinematic branch cut starting at $q^2=-M^2$, if one goes beyond the one-meson exchange approximation. This does not alter the validity of the above formulas, but in practice requires that the iteration which is needed to make the phenomenological poles consistent with unitarity would also have to include a calculation of this additional contribution to $R(y)$. Under these circumstances it would be more elegant to use the helicity amplitudes, which do not suffer frm this defect, rather than the Stapp amplitudes. The above parameterization could equally well be used for the helicity amplitudes, but the unitarity condition would take a different form. An iteration scheme for this case is given by Goldberger, MacDowell, Grisaru, and Wong.[8]