short compared to the distance between the wormhole mouths in the approximating Euclidean space. However, as seen in Fig. 2, it is impossible to send a signal through the throat in such a way as to contradict the principle of causality; in effect the throat "pinches off" the light ray before it can get through. This pinch-off effect presents fundamental issues of principle which require further investigation.

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Bound States and Dispersion Relations*

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A vertex closely related to the Bethe-Salpeter amplitude is discussed in the one meson exchange approximation by dispersion theory methods. Both scalar and spinor cases are treated. The relation between anomalous thresholds and the Schrödinger equation is discussed in some detail. It is shown that dispersion methods can be used to determine bound state parameters. An estimate is made of the asymptotic $(D-S)$ ratio for the deuteron.

1. INTRODUCTION

It has been customary in the field-theoretic discussions of bound states to introduce the multiparticle amplitudes of Gell-Mann and Low[1] and Schwinger.[2] In the case of the two-particle bound state, which we shall call a deuteron for definiteness, the relevant amplitude is

$$\langle 0|T(\psi_N(N)\psi_p(p))|D\rangle. \tag{1.1}$$

This type of amplitude and the integral equation that it satisfies[3] have been used in discussions of the relativistic corrections to the binding energies[4] and to the electromagnetic structure[5] of bound states. The amplitude in which the deuteron has been replaced by a scattering state has been used in deriving, from field theory, the interparticle potential to be used in a Schrödinger equation discussion of nucleon-nucleon scattering.[6] This latter problem has not been satisfactorily solved, due in part to the ambiguities concerning the treatment of the relative time dependence of the amplitudes.

The Fourier transform of the two-particle amplitude depends on two variables, the center-of-mass and relative momentum. Wick[7] and Cutkosky[8] have discussed the analytic structure of this amplitude in terms of both variables and were able to solve the problem of two scalar particles interacting via massless mesons in the ladder approximation. The problem of scalar nucleon and antinucleon system and massless mesons has been solved by Okubo and Feldman.[9] It has proven very difficult to extend this approach to more realistic problems, although Wanders[10] has given a discussion in the ladder approximation of the case with mesons of finite mass.

We would like to point out that many of the properties of the bound state can be examined in terms of an amplitude much simpler than (1.1). This new type of amplitude allows the use of the powerful methods of dispersion theory. In order to apply this approach to a large class of problems it is necessary to know how to handle dispersion theory in the presence of anomalous thresholds. This problem has been recently clarified by the work of Mandelstam and Nambu and Blankenbecler.[11,12] The spinor amplitude which we wish to consider is

$$\langle N|f_p(0)|D\rangle, \tag{1.2}$$

where $f_p(0)$ is the proton current operator. This vertex is a function of only one variable and satisfies a dispersion relation in that variable. It will be shown that if one considers an unsubtracted dispersion relation for this amplitude, then the deuteron parameters are determined in terms of the masses and interactions of the neutron and proton. Renormalization effects are easily dealt with. Thus, this work complements the recent work of Haag,[13] Nishijima,[14] and Zimmerman[15] on the bound-state problem.

This vertex will be shown to be very simply related to the Schrödinger wave function. The work of refer-

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ences 11 and 12 on dispersion relations and anomalous thresholds will be clarified by showing that the wave function for a bound state has nothing but anomalous thresholds and that these two different approaches yield very similar results in the anomalous region. This vertex allows an unambiguous definition of the nucleon-nucleon potential for the calculation of bound state properties. This potential is chosen so that when used in conjunction with the Schrödinger equation, it will yield the exact vertex (1.2). While there seems to be no overwhelming argument to support this definition of the potential, there seem to be no overwhelming arguments against it either. This potential is very close in spirit to the potential of Charap and Fubini,[16] but small differences occur.

Finally, a rough calculation of the asymptotic $(D-S)$ ratio for the deuteron will be carried out using dispersion methods in the lowest approximation.

2. THE DEUTERON VERTEX FUNCTION

In this section dispersion relations for a vertex function closely related to the covariant two-body amplitude will be discussed. We shall see that these dispersion relations provide an integral equation analogous to the Salpeter-Bethe[3] equation and to the Wick[7]-Cutkosky[8] parametrization; in fact, these relations lead to the determination of the bound-state parameters. In order to clarify the physical ideas underlying our calculations and to illustrate our treatment of the anomalous threshold, the cases of scalar and spinor particles will be treated in detail. The algebraically simpler scalar case will be treated first.

A. The Scalar Vertex

In the following we will treat the neutron, proton, and deuteron as scalar particles. Consider the vertex

$$\Gamma(x)=(2D^0 2N^0)^{\frac{1}{2}}\langle N|f_p(0)|D\rangle, \tag{2.1}$$

where $f_p(0)$ is the current operator for the proton, and we consider this vertex as a function of the scalar variable x,

$$x=-(D-N)^2. \tag{2.2}$$

If the neutron is contracted, we are led to the expression

$$\Gamma(x)=i(2D^0)^{\frac{1}{2}}\int d^4y\, e^{-iN\cdot y}\langle 0|[f_N(y),f_p(0)]\theta(y_0)|D\rangle.$$

This form suggests that $\Gamma(x)$ is an analytic function in the lower-half x-plane, and the absorptive part of $\Gamma(x)$ is given as

$$\mathrm{Im}\Gamma(x)=-\pi(2D^0)^{\frac{1}{2}}\sum_s\langle 0|f_p(0)|s\rangle \times\langle s|f_N(0)|D\rangle\delta(s+N-D), \tag{2.3}$$

where s is a complete set of states. The lowest mass state of interest is that of a nucleon and pion, and if we restrict our attention to this particular state, the absorptive part is

$$\mathrm{Im}\Gamma(x)=-\frac{\pi(2D^0)^{\frac{1}{2}}}{(2\pi)^3}\int d^3l d^3k\langle 0|f_p(0)|lk\rangle \times\langle lk|f_N(0)|D\rangle\delta(l+k-D+N). \tag{2.4}$$

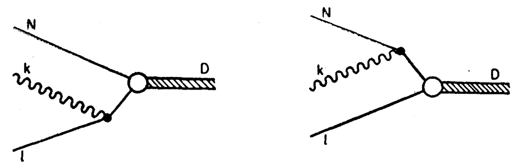

FIG. 1. Graphs for the amplitude M with one-nucleon intermediate states.

In discussing the second matrix element in (2.4), it is convenient to introduce the invariant function,

$$M=(2D^0 2l^0 2k^0)^{\frac{1}{2}}\langle kl|f_N(0)|D\rangle. \tag{2.5}$$

If the nucleon l is contracted and the equal-time commutator neglected, M becomes

$$M=i(2D^0 2k^0)^{\frac{1}{2}}\int d^4y\, e^{-il\cdot y}\langle k|[f_l(y),f_N(0)]\theta(y_0)|D\rangle.$$

The absorptive part of M is

$$\mathrm{Im}M=\pi(2D^0 2k^0)^{\frac{1}{2}}\sum_S\{\langle k|f_l(0)|S\rangle\langle S|f_N(0)|D\rangle \times\delta(l+k-S)-\langle k|f_N(0)|S\rangle\langle S|f_l(0)|D\rangle\delta(l+S-D)\}.$$

By neglecting rescattering corrections in the intermediate state, we may then introduce the pion mass, μ, and the coupling constant, g, defined by the following matrix elements on the energy shell

$$g\mu=(2l^0 2k^0)^{\frac{1}{2}}\langle 0|f_p(0)|lk\rangle,\quad g\mu=(2p^0 2l^0)^{\frac{1}{2}}\langle p|J_k(0)|l\rangle.$$

If only the one-nucleon states are retained, corresponding to the graphs in Fig. 1, and the vertex $\Gamma(x)$ on the energy shell, $\Gamma(M^2)=\Gamma_0$, is introduced, then one finds

$$M=+\mu g\Gamma_0\left[\frac{1}{(l+k)^2+M^2}+\frac{1}{(N+k)^2+M^2}\right]. \tag{2.6}$$

The absorptive part of $\Gamma(x)$ then becomes,

$$\frac{\mathrm{Im}\Gamma(x)}{\mu^2 g^2\Gamma_0}=-\frac{\pi}{(2\pi)^3}\int\frac{d^3l d^3k}{2l^0 2k^0}\delta(l+k-D+N) \times\left[\frac{1}{(l+k)^2+M^2}+\frac{1}{(N+k)^2+M^2}\right].$$

Introducing relative and center-of-mass coordinates according to

$$P=l+k,\quad Q=\tfrac{1}{2}(l-k),$$

[16] J. Charap and S. Fubini, Nuovo cimento 14, 540 (1959).

leads one to

$$\frac{\mathrm{Im}\Gamma(x)}{\mu^2 g^2 \Gamma_0} = -\frac{\pi}{(2\pi)^3}\int d^4P\, d^4Q$$

$$\times\theta(l_0)\theta(k_0)\delta(\tfrac{1}{2}P^2+2Q^2+M^2+\mu^2)$$

$$\times\delta\left(-P\cdot Q-\frac{M^2-\mu^2}{2}\right)\delta(P-D+N)\left[\frac{1}{M^2-x}\right.$$

$$\left.+\frac{1}{\frac{1}{2}(x-M_D{}^2+M^2-2\mu^2)+2N^0Q^0-2\mathbf{N}\cdot\mathbf{Q}}\right],$$

or

$$=-\frac{\pi|\mathbf{Q}|}{4(2\pi)^3\sqrt{x}}\int d\Omega\left[\frac{1}{M^2-x}+\frac{1}{A-2\mathbf{N}\cdot\mathbf{Q}}\right]$$

$$=-\frac{\pi^2|\mathbf{Q}|}{(2\pi)^3\sqrt{x}}\left[\frac{-1}{x-M^2}\right.$$

$$\left.+\frac{1}{4|\mathbf{N}||\mathbf{Q}|}\ln\left(\frac{A+2|\mathbf{N}||\mathbf{Q}|}{A-2|\mathbf{N}||\mathbf{Q}|}\right)\right], \quad (2.7)$$

where

$$Q_0=(M^2-\mu^2)/2\sqrt{x};\quad N_0=(M^2+\mathbf{N}^2)^{\frac{1}{2}},$$
$$4x|\mathbf{Q}|^2=(x-(M-\mu)^2)(x-(M+\mu)^2),$$
$$4x|\mathbf{N}|^2=(x-(M_D-M)^2)(x-(M_D+M)^2),$$
$$A=\tfrac{1}{2}(x-M_D{}^2+M^2-2\mu^2)+2N^0Q^0.$$

The deuteron binding energy is ϵ. In the following we will neglect ϵ whenever possible.

It is seen that $|\mathbf{N}|$ is pure imaginary in the unphysical region and the discussion given in reference 11 instructs one to analytically continue the absorptive part from the physical region. Thus, defining $|\mathbf{N}|=-in$, the absorptive part becomes

$$\mathrm{Im}\Gamma(x)=-\frac{g^2\mu^2\Gamma_0}{8\pi\sqrt{x}}\left[\frac{-|\mathbf{Q}|}{x-M^2}+\frac{1}{2n}\tan^{-1}\left(\frac{2n|\mathbf{Q}|}{A}\right)\right]. \quad (2.8)$$

At approximately $x=3M^2$, A vanishes, and the arctangent passes through $\pi/2$. This is the signal which tells us that there is an anomalous threshold present,[12] since for $x<3M^2$, the arctangent must be continued up onto its second branch.

If there were no anomalous threshold present, the dispersion relation would be of the form,

$$\Gamma(x)=-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}dx'\frac{\mathrm{Im}\Gamma(x')}{x'-x+i\epsilon}.$$

The procedure in the case of an anomalous threshold is to deform the path of integration until the integral is taken along the contour shown in Fig. 2. The anomalous threshold is determined by requiring that the argument of the logarithm in (2.7) vanish, i.e., $A^2=4|\mathbf{N}|^2|\mathbf{Q}|^2$.

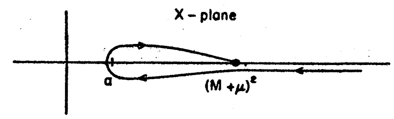

FIG. 2. Contour for the dispersion integral with an anomalous threshold.

The final result is

$$\Gamma(x)=\frac{-1}{\pi}\int_a^{\infty}dx'\frac{\mathrm{Im}\Gamma_c(x')}{x'-x+i\epsilon}, \quad (2.9)$$

where, $a\simeq M^2+2\mu(\mu+2\alpha)$, $\alpha^2=M\epsilon$, and

$$\mathrm{Im}\Gamma_c(x)=-\frac{g^2\mu^2\Gamma_0}{8\pi\sqrt{x}}\left[\frac{-|\mathbf{Q}|\theta[x-(M+\mu)^2]}{x-M^2}\right.$$

$$+\frac{\pi}{2n}\theta(x-a)\theta[(M+\mu)^2-x]$$

$$\left.+\frac{1}{2n}\tan^{-1}\left(\frac{2n|\mathbf{Q}|}{A}\right)\theta[x-(M+\mu)^2]\right]. \quad (2.10)$$

This value of the anomalous threshold is the same both numerically and physically, as one finds in perturbation theory.[12] The branch of the arctangent is chosen so that it is π at $x=(M+\mu)^2$.

For later comparison, let us consider $\mathrm{Im}\Gamma(x)$ in the anomalous region,

$$\mathrm{Im}\Gamma_c(x)=\frac{-g^2\mu^2\Gamma_0}{16\sqrt{x}|\mathbf{n}|}\simeq\frac{-g^2\mu^2\Gamma_0}{32M[(x-M^2+2\alpha^2)/2]^{\frac{1}{2}}}. \quad (2.11)$$

It will be shown that this form is extremely close to the value predicted by the Schrödinger equation with a simple Yukawa potential.

It is also noted that (2.9) leads to an eigenvalue condition if $\Gamma(x)$ is evaluated at $x=M^2$, since Γ_0 cancels from both sides of the equation. In the present case, this is a condition on g, M, and μ for a bound state of energy ϵ. Since only the one pion exchange diagram has been kept, this eigenvalue condition is, of course, very approximate. However, there is every expectation that if enough intermediate states were retained, this condition would allow one to calculate accurate binding energies in the dispersion theory formalism.

B. The Spinor Vertex

We turn now to the more involved spinor case before returning to a further discussion of the connection between this approach, anomalous thresholds and the Schrödinger equation. Let us consider the spinor amplitude,

$$\Gamma(x)C\bar{u}(N)=(2D^0N^0/M)^{\frac{1}{2}}\langle N|f_p(0)|D\rangle, \quad (2.12)$$

where C is the charge conjugation matrix; the other quantities are defined as before.[17] The matrix structure

[17] R. Blankenbecler, M. Goldberger, and F. Halpern, Nuclear Phys. 12, 647 (1959).

of Γ will be discussed later. In order to analyze $\Gamma(x)$, we contract on the neutron, and

$$\Gamma(x)Cu(N)=i(2D^0)^{\frac{1}{2}}\int d^4y\, e^{-iN\cdot y}u(N) \times\langle 0|[f_N(y),f_p(0)]\theta(y_0)|D\rangle;$$

the equal-time commutator makes no contribution. This form for $\Gamma(x)$ suggests that it is analytic in x, and the dispersion relation is

$$\Gamma(x)=-\frac{1}{\pi}\int_0^\infty dx'\frac{\mathrm{Im}\Gamma(x')}{x'-x+i\epsilon}. \tag{2.13}$$

We will make the bold assumption that no subtractions are necessary in this dispersion relation for $\Gamma(x)$, and show that this leads to an eigenvalue condition for the deuteron mass.

The absorptive part of Γ calculated in the canonical way is,

$$\mathrm{Im}\Gamma(x)C\bar{u}(N)=\pi(2D^0)^{\frac{1}{2}}\textstyle\sum_s\langle 0|f_p(0)|s\rangle \times\bar{u}(N)\langle s|f_N(0)|D\rangle\delta(N+S-D).$$

Again keeping the lowest mass state of a nucleon and a pion, we must evaluate

$$\mathrm{Im}\Gamma(x)C\bar{u}(N)=\pi(2D^0)^{\frac{1}{2}}\sum_{l\text{-spin}}\int\int\frac{d^3kd^3l}{(2\pi)^3} \times\langle 0|f_p(0)|^{(-)}lk\rangle u(N)\langle^{(-)}lk|f_N(0)|D\rangle \times\delta(D-N-l-k). \tag{2.14}$$

The pion-nucleon vertex is readily evaluated. Since the state $|lk(-)\rangle$ must be a $P_{\frac{3}{2},\frac{3}{2}}$ state, and since the rescattering effects should be small, this matrix element will be approximated by its value on the mass shell,

$$\langle 0|f_p(0)|lk^{(-)}\rangle=+ig(M/l^02k^0)^{\frac{1}{2}}\gamma_5(\tau\cdot T^*)u(l), \tag{2.15}$$

where τ and T are the isotopic spin operators of the nucleon and meson, respectively. The structure of the vertex, expressed in terms of the $P_{\frac{3}{2},\frac{3}{2}}$ phase shift, will be taken into account later.

In discussing the second matrix element which contributes to the absorptive part of $\Gamma(x)$, it is again convenient to introduce the invariant amplitude

$$M(k,l,D)=(2D^02k^0l^0/M)^{\frac{1}{2}}\bar{u}(N)\langle lk^{(-)}|f_N(0)|D\rangle. \tag{2.16}$$

Contracting the nucleon, we are led to

$$M=i(2D^02k^0)^{\frac{1}{2}}\int d^4y\, e^{-il\cdot y}\bar{u}(N)\bar{u}(l) \times\langle k|\{f_l(x),f_N(0)\}\theta(x_0)|D\rangle, \tag{2.17}$$

and then,

$$\mathrm{Im}M=\pi(2D^02k^0)^{\frac{1}{2}}\textstyle\sum_s\{\bar{u}(N)\langle k|f_N(0)|s\rangle\bar{u}(l) \times\langle s|f_l(0)|D\rangle\delta(l+S-D)+\bar{u}(l)\langle k|f_l(0)|s\rangle \times\bar{u}(N)\langle s|f_N(0)|D\rangle\delta(l+k-S)\}. \tag{2.18}$$

As before, this matrix element will be approximated by retaining only the one-nucleon state q. Thus

$$\mathrm{Im}M=\pi(2D^02k^0)^{\frac{1}{2}}\textstyle\sum_q\{\bar{u}(N)\langle k|f_N(0)|q\rangle\bar{u}(l) \times\langle q|f_l(0)|D\rangle\delta(l+q-D)+\bar{u}(l)\langle k|f_l(0)|q\rangle \times\bar{u}(N)\langle q|f_N(0)|D\rangle\delta(N+q-D)\}. \tag{2.19}$$

Introducing the coupling constants as the following matrix elements on the energy shell,

$$(2k^0q^0/M)^{\frac{1}{2}}\bar{u}(N)\langle k|f_N(0)|q\rangle =ig\bar{u}(N)\gamma_5(\tau\cdot T)u(q), \tag{2.20}$$

$$(2D^0q^0/M)^{\frac{1}{2}}\bar{u}(N)\langle q|f_N(0)|D\rangle=\bar{u}(q)\Gamma_0(N)C\bar{u}(N),$$

we find

$$\mathrm{Im}M=g\pi\textstyle\sum_q(M/q^0)\{\bar{u}(N)i\gamma_5(\tau\cdot T)u(q)\bar{u}(q) \times\Gamma_0(l)C\bar{u}(l)\delta(l+q-D)+\bar{u}(l)i\gamma_5(\tau\cdot T) \times u(q)\bar{u}(q)\Gamma_0(N)C\bar{u}(N)\delta(N+q-D)\}.$$

An evaluation of $\Gamma_0(n)$ has been performed in reference 17, to which the reader is referred for a discussion of the deuteron spin in terms of the pseudovector ξ_μ. The matrix form of $\Gamma_0(N)$ was found to be

$$\Gamma_0(N)=F_0i\gamma\cdot\xi+G_0N\cdot\xi; \tag{2.21}$$

and F_0 and G_0 were evaluated in terms of the nonrelativistic deuteron parameters.

Since (2.6) shows that M is analytic in the upper-half l-plane, and by making use of the Dirac equation, we may infer that

$$M=-g\left\{\frac{\bar{u}(N)i\gamma_5(\tau\cdot T)i\gamma\cdot k\Gamma_0(l)C\bar{u}(l)}{(N+k)^2+M^2} -\frac{\bar{u}(l)i\gamma_5(\tau\cdot T)i\gamma\cdot k\Gamma_0(N)C\bar{u}(N)}{(l+k)^2+M^2}\right\}, \tag{2.22}$$

and finally

$$\mathrm{Im}\Gamma(x)C\bar{u}(N)=-\frac{3\pi g^2}{(2\pi)^3}\int\frac{d^3kd^3l}{2k^02l^0}\delta(l+k-D+N) \times\left\{(M+i\gamma\cdot l)\left[\frac{i\gamma\cdot k\Gamma_0(N)}{(l+k)^2+M^2}+\frac{\Gamma_0(l)i\gamma\cdot k}{(N+k)^2+M^2}\right]C\bar{u}(N)\right\}.$$

The first term is recognized as a renormalization term and the second contains the structure of the bound state.

It is now an easy manner to discuss the general matrix form of $\Gamma(x)$. It is clear that it must have the structure

$$\Gamma(x)=F(x)i\gamma\cdot\xi+G(x)N\cdot\xi +[M+i\gamma\cdot(D-N)][H(x)i\gamma\cdot\xi+I(x)N\cdot\xi], \tag{2.23}$$

which reduces to (2.21) on the mass shell.

We shall be primarily interested in $\bar{u}(P)\Gamma(x)C\bar{u}(N)$ on the mass shell where $P=D-N$ and $x=M^2$. In this case the terms $H(x)$ and $I(x)$ will not contribute, and in the following we will restrict our attention to $F(x)$

and $G(x)$ exclusively. Formally, we can write

$$\bar{u}(D-N)\,\mathrm{Im}\Gamma(x)C\bar{u}(N)$$
$$=\bar{u}(D-N)[\mathrm{Im}F(x)i\gamma\cdot\xi+\mathrm{Im}G(x)N\cdot\xi]C\bar{u}(N)$$
$$=-\frac{3\pi g^2}{(2\pi)^3}\int\frac{d^3kd^3l}{2k^02l^0}\delta(l+k-D+N)$$
$$\times\left[\frac{\bar{u}(D-N)(-\mu^2)\Gamma_0(N)C\bar{u}(N)}{(l+k)^2+M^2}-\frac{\bar{u}(D-N)i\gamma\cdot k\Gamma_0(l)i\gamma\cdot kC\bar{u}(N)}{(N+k)^2+M^2}\right]. \quad (2.24)$$

We will denote the contribution of the first term in (2.24) to $\mathrm{Im}\Gamma$ by the subscript 1, and the second by 2. Thus

$$\bar{u}(D-N)\,\mathrm{Im}\Gamma_1(x)C\bar{u}(N)=\frac{3\pi g^2\mu^2}{(2\pi)^3}\int\frac{d^3kd^3l}{2k^02l^0}$$
$$\times\delta(l+k-D+N)\frac{\bar{u}(D-N)\Gamma_0(N)C\bar{u}(N)}{(l+k)^2+M^2}.$$

Introducing center-of-mass and relative momentum variables as before, we must evaluate

$$\bar{u}(D-N)\,\mathrm{Im}\Gamma_1(x)C\bar{u}(N)=\frac{+3\pi q^2\mu^2}{(2\pi)^3}\int d^4Pd^4Q$$
$$\times\delta(P-D+N)\delta(\tfrac{1}{2}P^2+2Q^2+M^2+\mu^2)\theta(k^0)\theta(l^0)$$
$$\times\delta(-P\cdot Q-\tfrac{1}{2}M^2+\tfrac{1}{2}\mu^2)\frac{\bar{u}(D-N)\Gamma_0(N)C\bar{u}(N)}{P^2+M^2}. \quad (2.25)$$

Comparing with (2.24), we find

$$\frac{\mathrm{Im}F_1(x)}{F_0}=\frac{\mathrm{Im}G_1(x)}{G_0}=-\frac{3g^2\mu^2|\mathbf{Q}|}{8\pi\sqrt{x}(x-M^2)}\theta(x-(M+\mu)^2), \quad (2.26)$$

where $|\mathbf{Q}|^2=(x-(M-\mu)^2)(x-(M+\mu)^2)/4x$. A calculation of $F_1(M^2)$ and $G_1(M^2)$ will require the integral

$$J_1=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}dx'\frac{[(x'-(M-\mu)^2)(x'-(M+\mu)^2)]^{\frac{1}{2}}}{x'(x'-M^2)^2}\simeq\frac{1}{2\pi M^2}\ln\left(\frac{M}{2\mu}\right). \quad (2.27)$$

Using J_1 we have, therefore,

$$\frac{F_1(M^2)}{F_0}=\frac{G_1(M^2)}{G_0}=+\frac{3g^2\mu^2}{16\pi}J_1=+\frac{3g^2}{32\pi^2}\left(\frac{\mu}{M}\right)^2\ln\left(\frac{M}{2\mu}\right). \quad (2.28)$$

Returning to (2.24), and using (2.21) and the definitions

$$\mathbf{N}\cdot\mathbf{Q}=|\mathbf{N}||\mathbf{Q}|z,\quad B(x)=A(x)/2|\mathbf{N}||\mathbf{Q}|,$$

we obtain

$$\bar{u}(D-N)\,\mathrm{Im}\Gamma_2(x)C\bar{u}(N)=+\frac{3\pi g^2}{8(2\pi)^3\sqrt{x}|\mathbf{N}|}\int\frac{d\Omega}{B-z}$$
$$\times\bar{u}(D-N)[S+\mathbf{V}\cdot\mathbf{Q}+\mathbf{Q}\cdot T\cdot\mathbf{Q}]C\bar{u}(N), \quad (2.29)$$

where

$$\begin{aligned}S&=-[\mu^2i\gamma\cdot\xi+\tfrac{1}{2}MN\cdot\xi-M\xi_0Q_0-\gamma_0Q_0N\cdot\xi\\&\quad+2\gamma_0\xi_0Q_0{}^2]F_0-[\tfrac{1}{2}\mu^2N\cdot\xi+\mu^2Q_0\xi_0]G_0,\\ \mathbf{V}&=-[M\boldsymbol{\xi}+iN\cdot\xi\boldsymbol{\gamma}-2i\xi_0Q_0\boldsymbol{\gamma}-2\gamma_0Q_0\boldsymbol{\xi}]F_0+\mu^2G_0\boldsymbol{\xi},\\ T&=-2i\boldsymbol{\gamma}\boldsymbol{\xi}F_0.\end{aligned} \quad (2.30)$$

The result of the angular integration in (2.29) is conveniently expressed in terms of the quantities

$$\begin{aligned}\alpha&=S+\tfrac{1}{2}|\mathbf{Q}|^2(\mathrm{Tr}T)-\tfrac{1}{2}|\mathbf{Q}|^2(\mathbf{N}\cdot T\cdot\mathbf{N})/|\mathbf{N}|^2,\\ \beta&=|\mathbf{Q}|\,\mathbf{V}\cdot\mathbf{N}/|\mathbf{N}|,\\ \gamma&=\tfrac{3}{2}|\mathbf{Q}|^2(\mathbf{N}\cdot T\cdot\mathbf{N})/|\mathbf{N}|^2-\tfrac{1}{2}|\mathbf{Q}|^2(\mathrm{Tr}T).\end{aligned} \quad (2.31)$$

Then

$$\bar{u}(D-N)\,\mathrm{Im}\Gamma_2(x)C\bar{u}(N)=+\frac{3\pi^2g^2\bar{u}(D-N)}{4(2\pi)^3\sqrt{x}|\mathbf{N}|}\times\left[(\alpha+\beta B+\gamma B^2)\ln\left(\frac{B+1}{B-1}\right)-2(\beta+\gamma B)\right]C\bar{u}(N). \quad (2.32)$$

In order to obtain expressions for $\mathrm{Im}F_2$ and $\mathrm{Im}G_2$ we must express (2.31) in terms of $(i\gamma\cdot\xi)$ and $(N\cdot\xi)$. The necessary results are

$$\begin{aligned}S&=-\mu^2F_0i\gamma\cdot\xi-N\cdot\xi\left[\frac{M}{2}\left(1-\frac{2Q_0}{\sqrt{x}}\right)^2F_0+\tfrac{1}{2}\mu^2G_0\left(1+\frac{2Q_0}{\sqrt{x}}\right)\right],\\ \mathbf{V}\cdot\mathbf{N}&=N\cdot\xi[-2MF_0(1+N_0/\sqrt{x})\\&\quad\times(1-2Q_0/\sqrt{x})+\mu^2G_0(1+N_0/\sqrt{x})],\\ (\mathrm{Tr}T)&=-2F_0i\gamma\cdot\xi-(2M/x)F_0N\cdot\xi,\\ \mathbf{N}\cdot\mathrm{T}\cdot\mathbf{N}&=-2MF_0(1+N_0/\sqrt{x})^2N\cdot\xi.\end{aligned} \quad (2.33)$$

Finally, the combination of (2.23), (2.32), and (2.33) yields expressions for $\mathrm{Im}F_2$ and $\mathrm{Im}G_2$.

The total imaginary part of $F(x)[=F_1(x)+F_2(x)]$, denoted by $\mathrm{Im}F_c(x)$, is obtained by performing the

analytic continuation to the physical Riemann sheet,

$$\mathrm{Im}F_c(x)=\frac{-3g^2F_0}{8\pi\sqrt{x}}\Bigg\{-\left[\frac{\mu^2|\mathbf{Q}|}{M^2-x}+\frac{A|\mathbf{Q}|}{4n^2}\right]\theta(x-(M+\mu)^2)$$
$$+[\mu^2+|\mathbf{Q}|^2(1-B^2)]\Bigg[\frac{\pi}{2n}\theta(x-a)\theta((M+\mu)^2-x)$$
$$+\frac{1}{2n}\tan^{-1}\left(\frac{2n|\mathbf{Q}|}{A}\right)\theta(x-(M+\mu)^2)\Bigg]\Bigg\}, \quad (2.34)$$

where, as before, $N=-in$ and $a=M^2+2\mu(\mu+2\alpha)$. Of course, an analogous expression can be obtained for $\mathrm{Im}G_c(x)$.

In the foregoing discussion, we have neglected pion-nucleon rescattering effects in the intermediate states. It is a simple matter to include these rescattering corrections, at least in an approximate manner, as follows. It is clear that only one angular momentum and isotopic spin state is relevant, the $P_{\frac{1}{2},\frac{1}{2}}$ state. An approximate expression which satisfies unitarity and retains the structure of the Born approximation is obtained by replacing g^2 in the absorptive part by $g^2\exp[2\,\mathrm{Re}\Delta(x)]$, where,

$$\pi\Delta(x)=(x-M^2)\int_{(m+\mu)^2}^{\infty}\frac{dx'\delta(x')}{(x'-x-i\epsilon)(x'-M^2)}, \quad (2.35)$$

and δ is the relevant pion-nucleon phase shift.

In contrast to the scalar case, Eq. (2.34) is not convergent, due to the term $A|\mathbf{Q}|/4n^2$, and thus does not provide an eigenvalue condition for determining the binding energy. This is not surprising since it is well known that a zero range wave function will lead to divergences for the expectation value of the Hamiltonian. It will be shown that our approximate expression for the absorptive part of Γ is closely related to a zero range approximation in Schrödinger theory. We would therefore expect that such difficulties would disappear if $\Gamma(l)$ rather than $\Gamma_0(l)$ were inserted into Eq. (2.22), since this is equivalent to keeping some of the higher meson exchange effects. This should then permit a consistent calculation of the binding energy. However, the one pion exchange approximation allows a determination of the asymptotic $(D-S)$ ratio ρ for the deuteron. In particular, the dispersion relation for $G(x)$ will permit us to express $G(x)$ as a linear function of F_0 and G_0, and thus obtain the ratio G_0/F_0 which is directly related to ρ. This calculation will be presented after a discussion of the nonrelativistic limit of the vertex.

3. THE NONRELATIVISTIC WAVE FUNCTION

The connection between the vertex $\Gamma(x)$ and the Schrödinger wave function ϕ has been discussed in reference 17 and more generally by Blankenbecler.[18]

[18] R. Blankenbecler, Nuclear Phys. 14, 97 (1959).

In particular, it has been shown that, aside from trivial normalization differences,

$$\Gamma(x)=(y+\alpha^2)\phi(y)=\frac{-1}{(2\pi)^3}\int d^3q\;V(\mathbf{p}-\mathbf{q})\phi(q^2), \quad (3.1)$$

where, $x=M^2-2(y+\alpha^2)$, y is the square of the relative momentum, and $\alpha^2=M\epsilon$. From the dispersion relation for $\Gamma(x)$, $\phi(y)$ is expected to have a cut for $y<-(\mu+\alpha)^2$. It will be shown that this is the case for a restricted class of potentials.

It is clear that the type of potential yielding results in closest agreement with field theory is the Yukawa or a linear superposition thereof. Therefore, let us try to solve the Schrödinger equation (3.1) with a wave function and potential of the form

$$(y+\alpha^2)\phi(y)=\Gamma(y)=\frac{1}{\pi}\int_{\beta^2}^{\infty}dy'\frac{\mathrm{Im}\Gamma(-y')}{y'+y}, \quad (3.2)$$

and

$$V(y)=-(2\pi)^3\lambda[y+\mu^2]^{-1}. \quad (3.3)$$

Defining $\mathrm{Im}\Gamma(-y')=\pi(y'-\alpha^2)\sigma(y')$ leads to

$$\phi(y)=\frac{N}{y+\alpha^2}-\int_{\beta^2}^{\infty}dy'\frac{\sigma(y')}{y'+y}, \quad (3.4)$$

where the asymptotic normalization N has been defined as

$$N=\int_{\beta^2}^{\infty}dy'\,\sigma(y'). \quad (3.5)$$

This relation will yield the eigenvalue condition for the Schrödinger equation.

Substitution of (3.4) into (3.1) and performance of the angular integration yields

$$\Gamma(y)=2\pi\lambda\int_{\beta^2}^{\infty}dy'\,\sigma(y')\int_0^1 dz\int_0^{\infty}dq$$
$$\times\{[q^2+yz(1-z)+\mu^2z+\alpha^2(1-z)]^{-1}$$
$$-[q^2+yz(1-z)+\mu^2z+y'(1-z)]^{-1}\}, \quad (3.6)$$

which shows that indeed, $\Gamma(y)$ has the representation (3.2). Thus it is seen that the bound-state wave function has a pole at $y=-\alpha^2$ and a cut from $(-\beta^2)$ to $(-\infty)$.

The equation for $\sigma(y)$ is easily determined to be

$$y^{\frac{1}{2}}(y-\alpha^2)\sigma(y)\theta(y-\beta^2)=\pi\lambda\int_{\beta^2}^{\infty}dy'$$
$$\times\sigma(y')\{\theta(y-(\mu+\alpha)^2)-\theta(y-[(y')^{\frac{1}{2}}+\mu]^2)\}. \quad (3.7)$$

It is clear that $\beta=\mu+\alpha$, a result in agreement with the anomalous threshold found in the field theory case. It is interesting to note that one can derive a very

similar field theoretic equation if in the evaluation of the scattering matrix M, Eq. (2.6), Γ_0 is replaced by the appropriate $\Gamma(x)$ in each of the two terms. The resulting dispersion relation is analogous to the Bethe-Salpeter equation in the ladder approximation. This replacement also improves the convergence properties of the dispersion relations as discussed at the end of Sec. 2.

It is possible to determine the solution to (3.7) in a simple iterative form. It turns out that in order to calculate $\sigma(y)$ on the left-hand side of (3.7) for, say $(N+1)\mu+\alpha>y^{\frac{1}{2}}>N\mu+\alpha$, $N=1, 2, \cdots$, it is sufficient to know $\sigma(y)$ under the integral for $y^{\frac{1}{2}}<N\mu+\alpha$. Thus knowing $\sigma(y)$ in the lowest region in y enables one to calculate it in successively higher regions by simple quadratures. The boundary conditions to be imposed on $\sigma(y)$ are obviously that it must be chosen to be continuous and to suffer discontinuities in slope at $y^{\frac{1}{2}}=N\mu+\alpha$. Physically, of course, this would correspond to the anomalous threshold for the exchange of N mesons.

For $(2\mu+\alpha)>y^{\frac{1}{2}}>(\mu+\alpha)$, we have

$$(y-\alpha^2)\sigma(y)=\pi\lambda N/y^{\frac{1}{2}}. \tag{3.8}$$

Comparing this relation with Eq. (2.11), which expresses $\mathrm{Im}\Gamma(x)$ for scalar particles, shows that they are of identical form for small y. An analogous result holds for the spinor case. Thus the one-pion exchange contribution yields just the Yukawa potential in the nonrelativistic limit. Corrections to this result will be discussed shortly.

It is to be noted that the bound-state wave function ϕ agrees with the field theoretic result in the anomalous region and further, ϕ has nothing but anomalous thresholds. This should be the case, since the Schrödinger equation is expected to be a valid description only if real particle production (or physical thresholds) are unimportant. Thus, the fact that the bound-state Schrödinger wave function is purely anomalous is to be expected.

Nambu has pointed out a very interesting example of the importance of anomalous thresholds in assuring a Schrödinger or Dirac description of a system. If one tries to calculate the electromagnetic structure of the hydrogen atom by the usual dispersion approach, there exists an anomalous threshold until the hydrogen atom mass M_{H}, the proton mass M_p, and the electron mass M_e are related by $M_{\mathrm{H}}^2=M_p^2+M_e^2$. This requires that the charge on the proton be $137e$, in the static limit. Thus the breakdown of the Dirac equation and the disappearance of the anomalous threshold are two ways of describing the same phenomena. Therefore it would seem that a hydrogen atom with $Z>137$ must be described field theoretically. It is tempting and perhaps not unreasonable to infer from this discussion that the vertex description would provide a calculational scheme for all Z.

The boundary condition stating that Γ of infinity must vanish does not permit the complete neglect of the negative cut.[19] It is interesting to see what happens if one approximates the effect of the negative cut in the wave function by an appropriately placed pole. According to (3.4), ϕ then becomes

$$\phi(y)=N(\gamma^2-\alpha^2)[(y+\alpha^2)(y+\gamma^2)]^{-1}, \tag{3.9}$$

which is recognized as the well-known Hulthén deuteron model. It is found for this model that one must choose $\gamma\approx 7\alpha$, which places the pole at the onset of the two pion cut, a most reasonable result. One could attempt to generalize this to the relativistic case by replacing the cut in x by a pole chosen to agree with (3.9) in the nonrelativistic limit, thus yielding a covariant Hulthén deuteron model. An obvious improvement would be to take the one pion cut into account exactly and to simulate the effects of the higher meson exchanges by a pole.

Thus it is possible by this scheme to calculate the deuteron wave function, and hence, the triplet potential, by dispersion methods. This formulation provides not only the potential but also the equation in which it is to be used, at least for the calculation of bound-state properties. There seems to be no ambiguities concerning a velocity-dependent force if one accepts (3.1) as a definition of the potential. Hence we are proposing a potential which is chosen to yield the bound-state properties, not the low-energy scattering properties, of field theory.

In detail, one would calculate the potential as follows. From dispersion theory the absorptive part of $\Gamma(x)$ is calculated, which involves the analytic continuation process described in reference 12. Then from the definition of $\sigma(y)$, we have

$$\mathrm{Im}\Gamma(-y)=\pi(y-\alpha^2)\sigma(y). \tag{3.10}$$

If it is assumed that the potential can be written in the form suggested by Charap and Fubini,[16]

$$V(y)=-(2\pi)^3\lambda\left[\frac{1}{y+\mu^2}+\int_{4\mu^2}^{\infty}dM^2\frac{v(M^2)}{y+M^2}\right], \tag{3.11}$$

then the Schrödinger equation for $y^{\frac{1}{2}}>2\mu+\alpha$ leads to [compare with (3.7)]

$$\begin{aligned}y^{\frac{1}{2}}(y-\alpha^2)\sigma(y)/\pi\lambda=&N\left[1+\int_{4\mu^2}^{(y^{\frac{1}{2}}-\alpha)^2}dM^2\,v(M^2)\right]\\&-\int_{(\mu+\alpha)^2}^{(y^{\frac{1}{2}}-\mu)^2}dy'\,\sigma(y')\Big[1+\theta\{[y^{\frac{1}{2}}-(y')^{\frac{1}{2}}]^2-4\mu^2\}\\&\times\int_{4\mu^2}^{[y^{\frac{1}{2}}-(y')^{\frac{1}{2}}]^2}dM^2\,v(M^2)\Big].\end{aligned} \tag{3.12}$$

[19] P. Noyes and D. Wong, Phys. Rev. Letters 3, 191 (1959), have also noticed a similar condition for the partial wave scattering amplitudes.

If we restrict our attention to the two-pion cut, $(3\mu+\alpha) > y^{\frac{1}{2}} > (2\mu+\alpha)$, then this reduces to

$$y^{\frac{1}{2}}(y-\alpha^2)\sigma(y)/\pi\lambda N = 1+\int_{4\mu^2}^{(y^{\frac{1}{2}}-\alpha)^2} dM^2 v(M^2) - \int_{(\mu+\alpha)^2}^{(y^{\frac{1}{2}}-\mu)^2} \frac{dy'}{(y')^{\frac{1}{2}}(y'-\alpha^2)} = y\,\mathrm{Im}\Gamma(-y)/\pi^2\lambda N. \tag{3.13}$$

A calculation of $\mathrm{Im}\Gamma$ in this variable range then determines the triplet potential weight function $v(M^2)$ in the two pion exchange region. Because of the effect of the normal threshold at $x=(M+\mu)^2$, which occurs in the two-pion exchange anomalous region, this definition of the potential would be expected to yield physically different results from the CF potential. This difference should be small except for small distances, where, of course, an evaluation considering only two-pion effects is poor. Apart from this small effect, which tends to weaken the one-pion force at small distances, the potential considered here is very close in spirit to the CF potential. The essential point is that one subtracts from the N-pion exchange contribution, the iterated effects of all smaller number of exchanged pions. Finally, if one were to approximate the higher meson exchange contributions by a pole, the effect on the potential is easily calculated.

4. CALCULATION OF ρ PARAMETER

In order to illustrate that the preceding discussions leads to an evaluation of the various bound state parameters, we shall obtain the asymptotic $(D-S)$ ratio, ρ, for the deuteron. As mentioned at the end of Sec. 2, if we had evaluated the higher meson exchange contributions, we could also calculate the binding energy, but in the present approximation this would not be a meaningful result.

As pointed out earlier, we begin by determining G_0 as a linear function of F_0 and G_0, and thus the ratio G_0/F_0, which is related to ρ by[16]

$$G_0/F_0 = -3\rho/2\epsilon + (\rho^2). \tag{4.1}$$

Rather than working with the entire expression for $\mathrm{Im}G(x)$, we propose to evaluate ρ by keeping only the Schrödinger limit. This means we take the nonrelativistic limit of $\mathrm{Im}G(x)$ in the anomalous region and extend it to infinity. The renormalization effects will be retained, however, because they tend to compensate for the errors incurred in taking the nonrelativistic limit. Returning to (2.32) and (2.33), we find for the terms contributing to G:

$$(\alpha+\beta B+\gamma B^2)_G = -MF_0\left[\frac{\mu^2}{y} - \frac{3}{4}\frac{(y+\mu^2)^2}{y^2}\right] + \mu^2 G_0\left[\frac{y-\mu^2}{2y}\right], \tag{4.2}$$

where $x=M^2+2(y-\alpha^2)$. We will neglect certain α^2 terms, and thereby commit a small error. Thus, setting $\mu=3\alpha$,

$$G_0 = \frac{-3g^2}{32M\pi}\int_{5\mu^2/3}^{\infty}\frac{dy}{y^{\frac{3}{2}}}\left\{MF_0\left[\frac{3}{4}\frac{(y+\mu^2)^2}{y^2} - \frac{\mu^2}{y}\right] + \mu^2 G_0\left[\frac{y-\mu^2}{2y}\right]\right\} - \frac{3g^2}{32\pi^2}\left(\frac{\mu}{M}\right)^2 G_0 \ln\left(\frac{M}{2\mu}\right), \tag{4.3}$$

or

$$\frac{G_0}{F_0} = \frac{-1.40}{8\mu}\frac{g^2}{4\pi}\left[1+(0.24)\left(\frac{\mu}{M}\right)\frac{g^2}{4\pi} + (0.146)\left(\frac{\mu}{M}\right)^2\frac{g^2}{4\pi}\right]^{-1}. \tag{4.4}$$

Finally,

$$\rho = +0.030 \tag{4.5}$$

for $g^2/4\pi = 14$. This result is in rough agreement with other estimates of the ρ value based on investigations of the two nucleon scattering problem.[20]

A more accurate evaluation of the dispersion integral using the relativistic absorptive part leads to a ρ value approximately ten percent smaller than that obtained above. The rescattering corrections of Eq. (2.36), on the other hand, tend to increase this value by about five percent for a reasonable phase shift. Therefore the net effect of these corrections is a slight reduction of the asymptotic $(D-S)$ ratio given above.

It is interesting to note that the result (4.4) depends only on fundamental parameters describing the pion-nucleon system, and the deuteron binding energy. This is to be compared with the results of reference (20) which explicitly contain experimentally determined nucleon-nucleon scattering parameters.

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[20] D. Wong, Phys. Rev. Letters **2**, 406 (1959); M. Goldberger and S. MacDowell (to be published).