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# Analytic Structure of Collision Amplitudes in Perturbation Theory*

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Some methods are developed for studying the singularities of collision amplitudes in perturbation theory as functions of two of the invariant energies, $s$, $t$, and $u$. It is shown that:

(i) There are no singularities other than normal thresholds in the physical regions of the physical sheet.
(ii) For the equal-mass case there are no singularities in the Euclidean region of the physical sheet.
(iii) The only straight lines of singularities on the real boundary of the physical sheet are normal singularities in the equal-mass case, and in the general-mass case are either normal singularities or they intersect the Euclidean region.
(iv) The curves of singularities on the real $s,t$ plane in the physical sheet do not connect to surfaces extending into the region $s$ real, $t$ complex except at turning points of the curves.
(v) Turning points of curves of singularities in the physical sheet may occur either when sufficient coincident singularities become also end-point singularities, or when there is an accidental relation between the Feynman variables at coincident singularities. The former correspond to anomalous thresholds; the latter are called spurious turning points.
(vi) For the equal-mass case there are no anomalous thresholds and no anomalous turning points in the curves of singularities.
(vii) Spurious turning points do occur in negative spectral regions, but here it appears that they may not lead to complex singularities on the physical sheet. There are no spurious turning points in positive spectral regions in low orders in perturbation theory and to all orders for some types of diagram. It is plausible that there are none for any diagram, but this is not proved.

The relation of this work to the Mandelstam representation is discussed. All the proven results in this paper are consistent with this representation. Some points are noted which require further investigation before the validity of the representation can be established to all orders in perturbation theory.

## 1. INTRODUCTION

THE following hypothesis underlies recent work on strong interactions of elementary particles:

(1) Collision amplitudes can be determined from the unitary condition, the location of singularities of the amplitudes in the physical sheet of the complex invariant energies, and some parameters related to the residues at poles of the amplitude or to its value at an arbitrary point. The parameters must be found from experiment.

For the special case of collision amplitudes involving only two particles incident and two outgoing and with certain restrictions on the masses of the particles that may be formed in the collision, Mandelstam[1] has proposed a further hypothesis:

(2) All singularities in the physical sheet lie on its real boundary.

For practical solution of the coupled equations resulting from these assumptions, a third hypothesis is necessary:

(3) The form of the collision amplitude is dominated by the nearest singularities in the physical sheet.

In this paper, we study the location of singularities of terms in the perturbation series for a scattering amplitude. We will be concerned in particular with the singularities on the physical sheet or on its boundary. The aim of such a study of perturbation terms is to see whether it is possible to deduce a form for the analytic structure of the amplitude itself by showing that the structure is a characteristic of all terms in the series. The Mandelstam representation is an example of such a structure, and in this paper a number of results are obtained which are necessary for the validity of the representation, and which go some way towards establishing sufficient conditions for its validity. In examples where the Mandelstam representation does not apply, a form of integral representation will still be required for use in conjunction with the unitary condition. It is hoped that the results of this paper will be useful in setting up methods to determine singularities from which more general integral representations can be obtained.

The development of integral representations for collision amplitudes, and in particular the proof of the Mandelstam representation, requires information about the singularities of the physical branch of the amplitude. The physical branch is determined by taking the three invariant energies[1] $s$, $t$, and ($u$) to be real, and associating a small negative imaginary part, $-i\epsilon$, with each mass in an internal line of a Feynman diagram. The physical sheet of two of the variables $s$, $t$, and $u$ is obtained by considering the physical branch with one of these variables real (sometimes it must be given a small positive or negative imaginary part), and letting the argument of the other vary from 0 to $2\pi$. The three variables are related by a mass condition,[1] so that only two can be varied independently.

The procedure for obtaining information about singularities in complex parts of the physical sheet is

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741 (1959); **115**, 1752 (1959).

based on a succession of steps, in which information about singularities is transferred from one part of the sheet to another by analytic continuation. It is shown in Sec. 3 that the only singularities in the physical scattering regions of the physical sheet are at normal thresholds for production of extra particles. This result follows from the unitary condition, though some use is also made of the perturbation series. In Sec. 4 a result of Mandelstam is quoted to show that for interactions of equal-mass particles there are no singularities in the Euclidean region.[2] This result is proved again by an independent method later in the paper in Sec. 8 example ($f$). Any straight line of singularities in the physical sheet must intersect either the physical scattering regions or the Euclidean region. In the former case it must coincide with a normal threshold, in the latter it must coincide with an anomalous threshold. In the equal-mass case there are no anomalous thresholds, and hence all the straight lines of singularities are known (Sec. 5). In Sec. 5 we discuss the manner in which curves of singularities are obtained for $s$, $t$, ($u$) real and in the physical sheet. A classification of curves of singularities is introduced which permits us to study generalized Feynman diagrams in which all lines are on the mass shell on the curves of singularities. In the Feynman integral this means that we need consider only coincident singularities. It is noted (Sec. 5) that the anomalous thresholds in fourth order occur when the curves of singularities have turning points.[3,4] At these points the coincident singularities are also end-point singularities, and the tangents to the curves are straight lines of singularities. The absence of this type of turning point in the equal-mass case is established in Sec. 7 to all orders in perturbation theory. It is essentially due to the fact that the only lines of singularities are given by the normal thresholds.

In Sec. 6 we study the general properties of the denominator of the Feynman integral for a general term in perturbation theory. This denominator is the discriminant[5] of the quadratic form in the internal momenta of the corresponding Feynman diagram. The importance of the discriminant $D(\alpha,s,t)$ is that it is stationary and zero at singularities of the amplitude. The singularities on the physical sheet are identified by the requirement that all the Feynman parameters $\alpha$ should be positive.[5a] The discriminant is a linear function of $s$ and $t$,

$$D(\alpha,s,t)=sf(\alpha)+tg(\alpha)-K(\alpha). \qquad (1.1)$$

In the equal-mass case, $K$ depends on the mass only through a factor $m^2$. It is shown in Sec. 7 that a curve of singularities can have a turning point only when the coefficient of $s$ or $t$ vanish on the curve. It will be noted that on the curve all variables are a function of a single parameter, which can be $s$ for example. Similar results hold when turning points are considered as functions of $t$, $u$ or of $u$, $s$.

The importance of determining the turning points (if any) is that only at a turning point of the curve of singularities ($s$, $t$ real), can there be an extension on to a surface of singularities which intersects the physical sheet for $s$ real and $t$ complex (or $t$ real and $s$ complex).

The vanishing of $f$ or of $g$ in Eq. (1.1) may occur either when a sufficient number of the $\alpha$ variables become zero (these correspond to anomalous thresholds), or when a general factor of $f$ or $g$ vanishes because of a special relation between the $\alpha$ variables at a point on the curve. The latter will be called a spurious turning point, since it does not correspond to a threshold. For the equal-mass case, there are no anomalous thresholds and no turning points of the first type. If a spurious turning point occurs in the region $s>0$, $t>0$ of the real part of the physical sheet, there will be a curve of singularities extending from that point into the complex part of the physical sheet. This would cause a breakdown of the Mandelstam representation. It is therefore important to prove that there are no spurious turning points in certain parts of the real $s$, $t$ plane.

In Sec. 8 a number of examples are worked out for the equal-mass case which illustrate explicitly most of the features of the general theory of Secs. 6 and 7. In particular, spurious turning points can be shown to be absent for all ladder diagrams. A spurious turning point will always occur when a diagram has singularities in more than one spectral region but it is necessary only to show they are absent from the spectral region where the relevant two variables are positive. This is proved for a ladder diagram with two crossed rungs and for the fully symmetric crossed eighth-order diagram (reduced). As an illustration of the general theory, a proof is given that there are no singularities in the Euclidean region for the equal-mass case.

In Sec. 9 the general form of the discriminant is studied further in an attempt to show that there are no spurious turning points in any order in perturbation theory. A reduction formula is obtained which permits the discriminant for any diagram to be expressed in terms of simpler diagrams in which one or more lines have been removed. The reduction formula is adequate to prove the absence of spurious turning points for certain classes of diagrams. The discriminant for a general diagram is analyzed and a plausibility argument is given for the absence of spurious turning points from positive spectral regions.

In Sec. 10 the relation between the results of this paper and the Mandelstam representation is discussed. If it is assumed that the absence of spurious turning points of curves of singularities has been made plausible, two further points must be considered. These are the

[2] S. Mandelstam, Nuovo cimento 4, 658 (196 ).
[3] R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. 114, 376 (1959).
[4] J. Tarski, J. Math. Phys. 1, 149 (1960).
[5] R. Chisholm, Proc. Cambridge Phil. Soc. 48, 300 (1952).
[5a] This identification is made as an assumption in this paper, it requires proof, and the conditions for its validity are discussed in a forthcoming paper: R. J. Eden, "The Problem of Proving the Mandelstam Representation," UCRL report 9254.

possibility of disconnected complex singularities, and the double application of Cauchy's theorem when all three spectral regions contain singularities. The former problem can also be discussed in terms of the vanishing of the coefficients of $s$ or of $t$ in $D(\alpha,s,t)$. The latter requires, for example, the consideration of singularities when $s$ and $t$ have small positive imaginary parts. It is expected that this will suffice to take the contour of integration along the "safe" side of the branch points at $u=$constant, and thereby avoid the complex surfaces of singularities that extend from the curve of singularities in the $u$, $s$ spectral region at the spurious turning points. However this point is not analyzed in detail. A number of general techniques for discussing singularities are used in the paper. Those developed elsewhere are described briefly in Sec. 2 so that notation and nomenclature will be accessible without constant reference to other papers.

## 2. GENERAL METHODS

### (A) Definition of the Physical Sheet

A term in the perturbation series for a collision amplitude for scalar particles will have the form,

$$F=\lim_{\epsilon\to 0+} c\int dk_1\cdots\int dk_l \prod_{i=1}^{n}\frac{1}{(q_i^2-m_i^2+i\epsilon)}. \quad (2.1)$$

The variable $q_i$ is the 4-momentum of the line $i$ in the corresponding Feynman diagram, and $m_i$ is the mass of the particle in this line. The $q_i$ are linear functions of the internal momenta $k_i$ and of the external momenta $p_k$. The term $F$ will be a function of the scalar products of the $p_k$, but these are not all independent. When the collision process involves Fermions or pseudoscalar particles there will be more complicated numerators in Eq. (2.1). The form of amplitude for this case has been described by Chisholm.[5] In this paper we are concerned with singularities of the amplitude, and, apart from possible complications or cancellations due to selection rules (which can be taken into account in special cases), it is sufficient for this purpose to consider only scalar particles.

When $F$ describes a reaction,

$$1+2\to 3+4, \quad (2.2)$$

it will be a function $F(s,t,u)$, where

$$s=(p_1+p_2)^2,\quad t=(p_1+p_4)^2,\quad u=(p_1+p_3)^2, \quad (2.3)$$

and

$$p_k^2=m_k^2,\quad \sum_1^4 p_k=0,\quad s+t+u=\sum_1^4 m_k^2. \quad (2.4)$$

The physical branch of $F(s,t,u)$ where $s$, $t$, $u$ are real is defined by the choice $\epsilon>0$ in Eq. (2.1). The physical sheet is obtained by analytic continuation of $F$ in the range

$$0<\arg s<2\pi, \quad (2.5)$$

keeping $t$ real. The variable $u$ is defined by Eq. (2.4) when $s$ and $t$ are given. Similarly the physical sheet includes the region in which $F$ is continued analytically from its physical branch in the range of Eq. (2.5) with $u$ real, and

$$0<\arg t<2\pi,\quad s \text{ or } u \text{ real}, \quad (2.5a)$$

$$0<\arg u<2\pi,\quad t \text{ or } s \text{ real}. \quad (2.5b)$$

It will be noted that the term physical sheet is defined here with a view to relating it to the Mandelstam representation. The chosen definition is more convenient for the methods in this paper than that used by Tarski.[4] The real $s$, $t$ plane is on the boundary of the physical sheet.

The transformation of Eq. (2.1) by means of Feynman parameters and subsequent integration over the internal momenta has been investigated by Chisholm.[5] The transformation introduces Feynman parameters $\alpha_1,\cdots,\alpha_n$ and gives a denominator which contains the function,

$$\psi(k,\alpha,s,t)=\sum_1^n \alpha_i(q_i^2-m_i^2+i\epsilon). \quad (2.6)$$

The function $\psi$ is a quadratic form in the internal momenta $k_j$. Let $D(\alpha,s,t)$ be the discriminant of $\psi$ as a function of the $k_j$, and let $C(\alpha)$ be the discriminant of the quadratic form $\psi_0$ obtained from $\psi$ by putting $m_i=0$, $s=0$, $t=0$, and $u=0$. Then Chisholm shows that the integral (2.1) becomes

$$F(s,t)=c(i\pi^2)^l\frac{(n-2l-1)!}{(n-1)!}\int_0^1 d\alpha_1\cdots\int_0^1 d\alpha_n \times\frac{\delta(1-\sum\alpha_i)[C(\alpha)]^{n-2l-1}}{[D(\alpha,s,t)]^{n-2l}}. \quad (2.7)$$

This integral representation of the terms in the collision amplitude has been used by a number of authors to study analytic properties. Particularly important developments have been made by Nambu,[6] Nakanashi,[7] Landau [see part (D) of this section],[8] Bjorken,[9] and, from a different representation, by Symansik,[10] and Taylor.[11] For the applications in this paper, we shall frequently use the method of coincident singularities and end-point singularities first used by the author[12] and later developed by Tarski,[4] and by Polkinghorne and Screaton.[13] These occur when the discriminant $D(\alpha,s,t)$ in Eq. (2.7) becomes zero for coincident roots of one of the $\alpha_i$, or becomes zero when one of the $\alpha_i$ is zero.

[6] Y. Nambu, Nuovo cimento **6**, 1064 (1957); **9**, 610 (1958).
[7] N. Nakanashi, Progr. Theoret. Phys. (Kyoto) **17**, 401 (1957).
[8] L. D. Landau, Nuclear Phys. **13**, 181 (1959).
[9] J. D. Bjorken (to be published).
[10] K. Symanzik, Progr. Theoret. Phys. (Kyoto) **20**, 690 (1958).
[11] J. C. Taylor, Phys. Rev. **117**, 261 (1960).
[12] R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952).
[13] J. C. Polkinghorne and G. R. Screaton, Nuovo cimento **15**, 289 (1960).

### (B) End-Point and Coincident Singularities

A function $f(z)$, defined by

$$f(z)=\int_0^1 d\alpha\, g(\alpha,z), \tag{2.8}$$

may become singular at $z=x$ if either (a) $g(0,z)$ is singular as $z$ approaches $x$, or (b) $g(\alpha,z)$ has two singularities, one on each side of the path of integration, which tend to coincidence as $z$ approaches $x$. The first condition is called an "end-point" or $E$ singularity, and the second is a "coincident" or $C$ singularity. In practice the singularities of the integrand are either poles or branch points, and they appear always to cause a singularity in the integrand when condition (a) or (b) holds.

It is evident that if

$$D(\alpha,s,t)=0, \tag{2.9}$$

and

$$\text{either } \partial D/\partial\alpha_i=0, \text{ or } \alpha_i=0, \quad i=1,\cdots,n, \tag{2.10}$$

then $F(s,t)$ given by Eq. (2.7) may be singular. If Eqs. (2.9) and (2.10) hold for values of $\alpha_i$ satisfying

$$\alpha_i>0, \quad \sum\alpha_i=1, \tag{2.11}$$

there will be either an $E$ singularity or a double singularity at each stage of the integration. If the double singularities are in fact $C$ singularities (and not both on the same side of the contour), the integral $F(s,t)$ will be singular.[13] We will consider later the problem of showing that Eqs. (2.9) to (2.11) lead to $C$ singularities.

The boundary of the physical sheet ($s$ and $t$ both real) is obtained by letting $\epsilon\to 0$ in the terms $(m_i^2-i\epsilon)$ of Eq. (2.1). This associates each frequency, for which the lines in Eq. (2.1) are on the mass shell, with a definite side of the contour of integration:

$$q_i^0=\pm[(m_i^2+\mathbf{q}_i^2-i\epsilon)]^{\frac{1}{2}}. \tag{2.12}$$

These relations lead to singularities in the momentum-space integration which may be either $C$ or $E$. In the physical scattering regions of the physical sheet, these singularities can be directly interpreted. The 3-momenta give $E$ singularities so that they correspond to particles at relative rest, and the sign of $i\epsilon$ in Eq. (2.12) ensures that singularities are coincident only when the particles concerned have positive energy. It will be shown later that these are the only singularities in physical regions.

If instead of Eq. (2.1) some of the internal masses are written $(m^2+i\epsilon)$, the singularities coming from momentum-space integration no longer have a simple interpretation in general, even in physical scattering regions. Clearly, some of them will correspond to some particles having positive and some negative energy while at relative rest, but others will not require the 3-momenta to give an $E$ singularity. The integral expressed in terms of Feynman variables will no longer have the form of Eq. (2.7) but will be another branch of the same function. We will consider now how to obtain these different branches without carrying out the integrations.

### (C) Analytic Continuation of Integral Representations

An integral representation of a function, Eq. (2.8) for example, can be analytically continued by varying $z$ in the integrand provided always that the path of integration is suitably distorted so that no singularity of $g(\alpha,z)$ crosses the path between $\alpha=0, \alpha=1$. It is permitted that a singularity goes round an end point of the path of integration to the other side of the contour, and in general this will give a different branch of the function. A number of possibilities are shown in Fig. 1. The path of integration is from $\alpha=0$ to $\alpha=1$ on the real axis for the physical branch of the function; this determines on which side of the integration contour the singularities lie if, for example, $z=x+i\epsilon$ gives the physical branch. Figure 1 (i) shows a typical $C$ singularity, where the location $a$ and $b$ of the singularities depends on $z$. If $z$ follows a path which causes $a$ and $b$ to move as in (ii) the final position is not a $C$ singularity. It will be noted that to remove the $C$ singularity, $a$ has gone around $\alpha=0$ which is an $E$ singularity. Figure 1 (iii) shows how a $C$ singularity may disappear when it passes through an $E$ singularity, and (iv) shows how it may be retained by taking $a$ around the $E$ singularity. Figures 1, (v) and (vi) illustrate how a singularity $c$, which may never enter the range of integration $0<\alpha<1$, may still lead to a singularity in the integral corresponding to another branch of the function. These analytic continuations under the integral sign have been extensively used by Tarski.[4]

### (D) The Landau-Bjorken Conditions

By considering the transformation from Eq. (2.1) to Eq. (2.7), Landau[8] and Bjorken[9] have shown that the conditions (2.9) and (2.10) are equivalent to the conditions:

$$\text{either} \quad q_i^2=m_i^2, \quad \text{or} \quad \alpha_i=0, \tag{2.13}$$

and

$$\sum\alpha_j q_j=0, \tag{2.14}$$

where $q_i$ is a 4-momentum in an internal line, and the sum in Eq. (2.12) is taken over any closed circuit in the diagram.

If $\alpha_i=0$ for any line, the condition for a singularity can be obtained from the reduced diagram in which the line $i$ is "short-circuited" (or reduced to a point). It should be noted though that although the reduced diagram will determine this singularity of the "parent" diagram correctly, it will not determine other singularities of the parent diagram so that it cannot be used to determine the functional dependence of the integral except near the particular singularity concerned.

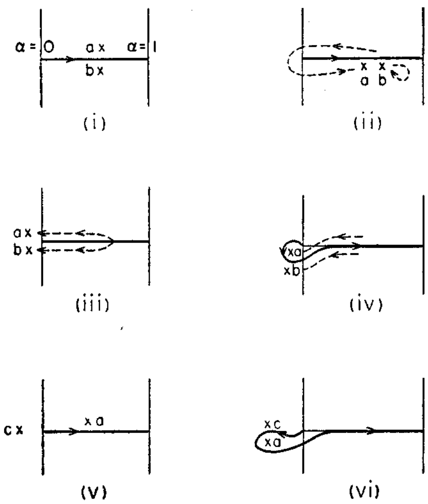


FIG. 1. Analytic continuation by moving singularities of the integrand and distortion of the path of integration in the complex $\alpha$ plane.

The Landau conditions determine the location of singularities of all branches of the function associated with a given diagram. Only when all $\alpha_i$ for a particular solution lie inside $[0,1]$, (with $\sum\alpha_i=1$), will the singularity lie in the physical sheet. Those solutions with some $\alpha_i$ outside $[0,1]$ are analytic continuations on to nonphysical sheets and are $C$ singularities with a suitably distorted path of integration (see footnote 5a).

Landau[8] and Taylor[11] have used conditions (2.13), and (2.14) to construct dual diagrams consisting of directed vectors of length $m_i$ which satisfy the "equilibrium" condition. The dual diagrams were proposed also by Karplus, Sommerfield, and Wichmann[3] in connection with third- and fourth-order terms. In these examples, the dual diagrams were used to investigate the nearest singularities only. They can also be used to determine the algebraic relations that given higher-order singularities, not necessarily the nearest ones.[14] Mathews[15] and Bjorken[9] have also investigated equations similar to (2.13) and (2.14) using the analogy with electric-circuit theory.

### 3. SINGULARITIES IN PHYSICAL SCATTERING REGIONS

It will be shown in this section that the only singularities in the physical scattering regions are normal thresholds for real competing processes. We will consider for definiteness the region

$$s>(m_1+m_2)^2, \quad u<0, \quad t<0. \tag{3.1}$$

With $S=1+R$, the unitary condition has the form,

$$(p_1,p_2|(R^\dagger+R)|p_3,p_4)=-\sum_{n_1,n_2,\cdots}(p_1,p_2|R^\dagger|n_1n_2\cdots)\times(n_1n_2\cdots|R|p_3,p_4). \tag{3.2}$$

Total energy and momentum must be conserved in all matrix elements. The intermediate states $|n_1n_2\cdots)$ include any number of particles, whose total rest mass satisfies

$$(\sum m_j)^2\leq s. \tag{3.3}$$

There may also be selection rules that further restrict these states but they will not be considered here. It is because we are considering the physical region (3.1) that we can make the restriction (3.3) which implies that each intermediate particle has positive energy greater than its rest mass.

In a perturbation solution for $R$, the unitary condition is satisfied to each order in the coupling constant. Thus if $R_l$ denotes $R$ to order $l$, we have

$$(R_l^\dagger+R_l)=-P_l\{\sum_{l_1}\sum_{n_1,n_2,\cdots}R_{l_1}^\dagger|n_1n_2\cdots)(n_1n_2\cdots|R_{l_1}\}, \tag{3.4}$$

where $l_1\leq(l-2)$, and $P_l$ selects those terms on the right-hand side which are of order $l$ or lower.

#### Theorem 3A

A necessary and sufficient condition for the amplitude to have a branch point at $s=s_c$ in the physical region (3.1) is that $s_c$ is a normal threshold for a competing production process.

If $R$ has a branch point at $s=s_c$, let $R_l$ be the term of lowest order in the perturbation expansion to have this branch point. Then the left-hand side of Eq. (3.4) is nonanalytic at $s=s_c$. Each term on the right-hand side is analytic there since $l_1<l$. Hence the sum must change on the right-hand side so that an extra term is included, thus giving nonanalytic behavior to match that on the left. This extra term can arise only when it corresponds to a new competing process. Hence, $s_c$ is a normal threshold and we have

$$s_c=(\sum m_i)^2, \tag{3.5}$$

the sum being over all particles in the state which is newly allowed as $s$ exceeds $s_c$.

It will be noted that this argument involves not only analyticity of the matrix elements,

$$(d,c|R_{l_1}|a,b), \tag{3.6}$$

but also analyticity of the production amplitude,

$$(d,c)|R_{l_1}^\dagger|n_1n_2\cdots), \tag{3.7}$$

for $l_1<l$. If the production amplitude were not an analytic function of $s$, then we could consider the equation

$$(d,c|R_{l_1}^\dagger+R_{l_1}|n_1n_2\cdots)=-P_{l_1}\{\sum_{l_2}\sum_{n_1'n_2'}(d,c|R_{l_2}^\dagger|n_1'n_2'\cdots)\times(n_1'n_2'\cdots|R_{l_2}|n_1n_2\cdots)\}, \tag{3.8}$$

[14] R. J. Eden, C. Enz, and J. Lascoux, Bull. Am. Phys. Soc. 5, 284 (1960).

[15] J. Mathews, Phys. Rev. 113, 381 (1959).

and deduce that the production amplitude,

$$(d,c|R_{l_2}{}^\dagger|n_1'n_2'\cdots), \tag{3.9}$$

for $l_2<l_1<l$ was not analytic. By repetition of this process, the production amplitude of lowest order can be obtained. It must be nonanalytic at a point $s=s_c$ if $R_{l_1}$ is not analytic there. But the lowest-order production amplitude does not have any branch points. Hence, in the physical scattering region all the matrix elements of $R_{l_1}$ must be analytic except at production thresholds.

Conversely if a new intermediate state is allowed on the right-hand side of Eq. (3.4) at $s=s_c$, let $l$ be the lowest-order term for which this competing process can enter the sum. The right-hand side is nonanalytic; hence, in order to make the left also nonanalytic $R_l$ must have a branch point at $s=s_c$. A branch point in $l$th order cannot be cancelled by a branch point in higher order. Hence $R$ must have a branch point at $s=s_c$.

These singularities in the physical regions can be interpreted in terms of the conditions for singularities described in Sec. 2. This then permits an extension of theorem 3A to give information about nonphysical regions of the physical sheet. It is useful to introduce some definitions:

An "$s$-partition" of a diagram is defined as a partition along a single dividing line intersecting only internal lines of the diagram across which flows the total 4-momentum $(p_1+p_2)$, where $s=(p_1+p_2)^2$. Similar definitions are made for $t$-partitions and $u$-partitions, and $s$, $t$ partitions are illustrated in Figs. 2(i) and 2(ii).

### Theorem 3B

If a diagram corresponds to a singularity in the physical region where $s$ is the square of the energy, it is always possible to make at least one $s$-partition in which every line cut by the partition corresponds to a particle on its mass shell.

From the Landau conditions, if a diagram corresponds to a singularity, either every line is on the mass shell, or the Feynman variables are zero for some lines and the diagram can be reduced. The fully reduced diagram has every line on the mass shell. Either an $s$-partition can be made of the fully reduced diagram or the 4-momentum $(p_1+p_4)$ passes through a single vertex. If the latter is true the position of the singularity will be a function of $t$ only, (or alternatively of $u$ only), and a $t$-partition can be drawn. However, from theorem 3A the only singularities are at $s=s_c$, so there cannot be any that depend only on $t$. Neither can a $t$-partition be made of the fully reduced diagram, therefore an $s$-partition is possible. The insertion of lines at the vertices of the fully reduced diagram does not affect the $s$-partition and the theorem follows.

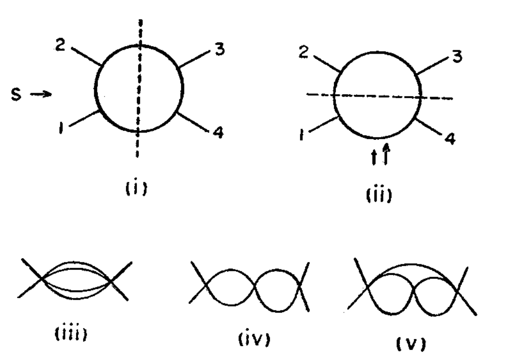


FIG. 2. Diagrams (i) and (ii) show $s$- and $t$-partitions; (iii), (iv) and (v) show a generalized self-energy part and chains of these parts.

### Theorem 3C

The fully reduced diagram corresponding to a singularity in the physical region is always a "generalized self-energy part" or a chain of these parts. [These are illustrated in Fig. 2, diagrams (iii), (iv), and (v).]

This theorem follows from the fact that the fully reduced diagrams (in which by definition all lines are on the mass shell) which give singularities at normal thresholds in the physical regions must correspond to all particles at rest relative to each other and with positive energies [the latter is a "causality" requirement following from the use in Eq. (2.1) of $(m^2-i\epsilon)$]. This excludes the insertion of lines on the mass shell which change the self-energy diagram into a vertex diagram. The possibility of accidental coincidences of self-energy and vertex singularities is not considered as it would occur only for very special values for the masses of particles, which do not appear to occur in nature. A chain of self-energy parts must for similar reasons have identical links. This type of singularity (of chain diagrams) is related to the unitary Eq. (3.4), and corresponds to a value of $s$ for which an additional state becomes allowed in the sum on the right-hand side, the left-hand side is nonanalytic, and also the terms on the right-hand side have branch points. It was to exclude this complication that the proof of theorem 3A made use of the lowest-order term which has the branch point.

To summarize: In the physical scattering regions (a) unitarity and the positive energies of physical particles requires that all singularities (branch points) occur at normal thresholds; (b) the use of $(m^2-i\epsilon)$, which is related to causality, requires that all singularities correspond to fully reduced diagrams which are generalized self-energy parts.

## 4. THE EUCLIDEAN REGION

This region is defined by

$$s\leq(m_1+m_2)^2,\quad t\leq(m_1+m_4)^2,\quad u\leq(m_1+m_3)^2. \tag{4.1}$$

The definitions of $s$, $t$, and $u$ in Eqs. (2.3) can be satisfied in this region by taking the external 4-momenta to be Euclidean vectors satisfying Eqs. (2.4).

For the equal-mass case, it has been shown by Mandelstam[2] that analyticity in the Lehman ellipses can be used to show that inside the complex region (in the physical sheet),

$$|stu| < 288m^6, \tag{4.2}$$

and

$$s+t+u=4m^2, \tag{4.3}$$

the amplitude $A(s,t,u)$ is analytic except for the expected poles at $m^2$ and branch points at $4m^2$. This proves analyticity in the Euclidean region, and the result is independent of perturbation theory.

An independent perturbation-theory proof of analyticity in the Euclidean region will be given in Sec. 8, example ($f$). This proof is also restricted to the equal-mass case, but it makes very plausible the possibility that the amplitude will be analytic in the Euclidean region in the general-mass case provided the fourth-order term is analytic there. This result is also made plausible by the work of Bjorken[9] and Taylor.[11] The general mass case will be discussed in more detail in another paper.

An independent proof in perturbation theory is needed because Mandelstam's result does not imply that each term of the perturbation series has the same property of analyticity. In fact our later result shows this to be true for all diagrams in which the masses in the internal lines are not smaller than those of the external lines. This is sufficient for all diagrams in the equal mass case since reduction by short-circuiting internal lines cannot reduce any masses.

## 5. SINGULARITIES ON THE REAL BOUNDARY OF THE PHYSICAL SHEET

We next consider singularities of terms in the perturbation series for an amplitude in the nonphysical, non-Euclidean regions of the real $s$, $t$, $u$ plane. This is on the boundary of the physical sheet when all masses are written $(m^2-i\epsilon)$ in the Feynman integral, $s$, $t$, $u$ being real.

### (A) Lines of Singularities

These are defined as arising from singularities whose position depends on one of $s$, $t$, $u$ only. From the discussion of Sec. 3, normal thresholds correspond to lines of singularities in the physical regions. These are given by reduced diagrams which are generalized self-energy parts or chains of these parts. The original diagram will have all those lines on the mass shell which are cut by one or more independent $s$-partitions. The Feynman variables of these lines will all lead to $C$ singularities. Those for lines not cut by the $s$-partition will give $E$ singularities since the lines are not on the mass shell. The location in the Feynman integrand of these $C$ and $E$ singularities depends (for example) on $s$ only; hence they will remain $C$ and $E$ singularities as $t$ varies into nonphysical regions of the physical sheet. Therefore the normal thresholds in physical regions extend to lines of singularities for all real values of $t$; this is the boundary region of the physical sheet. They also extend to "planes of singularities" for $t$ complex, and $s=s_c$, real.

There may also be lines of singularities which arise from reduced diagrams that are generalized vertex parts (or chains of these parts). The location of singularities in the Feynman integrand depends on one variable only, e.g., $s$, and therefore if these lines are in the physical sheet in one part of the real $s$, $t$ plane, they will remain in the physical sheet in all parts. But from Sec. 3 we know that the only singularities in the physical regions are normal thresholds. Excluding accidental coincidences for special mass values, we conclude that vertex parts cannot lead to lines of singularities which enter the physical region. These results are summarized in the following theorem.

### Theorem 5A

The only lines of singularities on the real boundary of the physical sheet are (a) normal singularities meeting the physical region at normal thresholds, (b) anomalous singularities, which, if they are present at all, do not intersect the physical regions but do intersect the Euclidean region.

*Corollary*

In the equal-mass case the only lines of singularities in the physical sheet correspond to normal thresholds.

### (B) Curves of Singularities: Preliminary Discussion

Curves of singularities are obtained when the lines cut by both an $s$- and a $t$-partition (or more than two such partitions) give rise to $C$ singularities in the Feynman integration. The resulting curves of singularities are on the physical sheet when the Feynman parameters at the $C$ singularities are in the range [0,1]. The number of $s$-partitions and $t$-partitions associated with a given singularity give a measure of the complexity of its structure. The simplest reduced diagrams are obtained when just one $s$- and one $t$-partition is made. These are indicated in Fig. 3, diagrams (a), ($b$), and (c). The internal lines of these diagrams in the equal-mass case may be any integral multiple of the elementary-particle mass. It is instructive to consider

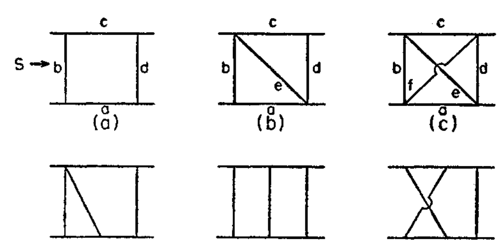


FIG. 3. Examples of low-order, fully reduced diagrams.

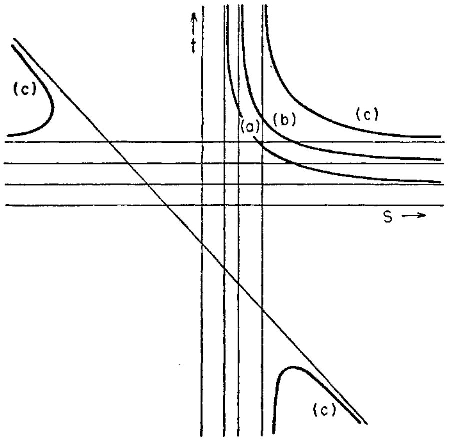


FIG. 4. Curves of singularities for diagrams (a), (b), and (c) of Fig. 3. No scale is shown, as only the general form of the curves is required.

these diagrams further, since they illustrate characteristics which we shall later establish more generally. Diagrams (d), (e), and (f) in Fig. 3 illustrate some of the reduced diagrams associated with two $s$-partitions and one $t$-partition.

In the equal-mass case, Fig. 3(a) has only one branch in the physical sheet. The fact that the internal lines have masses $\geq m$ while the external lines each have mass $m$ is sufficient to exclude the possibility of anomalous thresholds. The form of the curve of singularities is shown in Fig. 4 [curve (a)], though the location of its asymptotes will vary with the values of the internal masses.

In the general-mass case, Fig. 3(a) may give a curve of singularities with anomalous form.[1,3,4] This can be tested by reducing one of lines $a$, $b$, $c$, or $d$ to a point and considering whether the resulting vertex part has singularities on the physical sheet. If diagram 3(a) does not have anomalous form when its internal masses are least, it will not have anomalous form at all. If only the reduction of line $a$ gives a vertex singularity when $b$, $c$, and $d$ have least masses, then Fig. 3(a) will have anomalous form for all values of the mass in the lines $a$, those in $b$, $c$, and $d$ being fixed. This shows that some types of anomalous threshold remain to arbitrary order in perturbation theory when they are present in lowest order.

The diagram 3(b) can lead to a singularity for an $s$-partition, and a $t$-partition, but not for a $u$-partition. Its singular curve for the equal-mass case therefore lies in the region $s>s_c$, $t>t_c$, where $s_c$ and $t_c$ are the singular asymptotes obtained by reducing two of the lines $a$, $b$, $c$, and $d$ to a point. This curve can either be evaluated explicitly,[14] or by the more general arguments given later in this paper. It is shown as curve (b) in Fig. 4.

For the general-mass case in diagram 3(b), the possibility of anomalous thresholds must be considered. These can be checked by considering the vertex part in which one of $a$, $b$, $c$, or $d$ is reduced. This vertex part has no singularities in the physical sheet in the equal-mass case, so a further reduction must be made. It will be noted that if Fig. 3(a) does not give an anomalous threshold, neither will Fig. 3(b) since the corresponding vertex part of the latter has at least one internal mass larger than the former. The same argument also applies (more strongly) to diagram 3(c) where the vertex parts will have two lines with additional mass and cannot have any singularities in the physical sheet if Fig. 3(a) is not anomalous (selection rules are not considered here).

A new feature arises with Fig. 3(c), since it can give singularities for an $s$-, $t$-, or $u$-partition. The curve of singularities therefore has three branches in the physical sheet. For a normal case, these are indicated by curves (c) in Fig. 4. The same principle as before determines the asymptotes for which pairs of lines $(b,d)$, $(a,c)$, or $(e,f)$ must be reduced to points. The indicated form of the singular curve 4(c) will be justified in more detail later [Sec. 8(d)].

In the equal-mass case diagram 3(d) cannot give a singular curve on the physical sheet when all internal lines are on the mass shell and correspond to single masses. This follows from the fact that it contains an internal vertex part which has no singularities in the physical sheet. Thus its only singularities involve the wrong frequency condition $(m^2+i\epsilon)$ on one of its lines, and this would impose the same requirement on 4(d) itself. This result can also be obtained by an algebraic method given in Sec. 9.

Another type of singular curve to which special attention must be given in considering general terms is illustrated by the anomalous-thresholds curves from the fourth-order term, diagram 3(a). In the general-mass case, its regular form is shown by Fig. 4(a). Other possible forms of curve along which $\lim_{\epsilon\to 0}A(s,t,m_i^2-i\epsilon)$ is singular (analytic continuation with $s$ and $t$ real is made before the limit is taken) are shown in Fig. 5, curves (a) and (b).[4] Curve 5(a) has a single branch which goes on to a different Riemann sheet at each point of tangency to a line singularity. It should however be noted that the complex-conjugate amplitude $A^+(s,t)$ will be singular on the broken part of the curve. Therefore the continuous plus the broken part of the curve form the boundary of the spectral function. For this type of singular curve, the Mandelstam representation still applies to the fourth-order term.[1,4] Curve 5(b) illustrates anomalous thresholds of type II. It has two branches in the physical sheet and the Mandelstam representation does not apply to the fourth-order term when this type of singularity occurs.[1,4]

It has been noted by Tarski in connection with the anomalous curves shown in Fig. 5 that the slope of the curve $\Gamma$ determines the relative sign of the imaginary parts of $s$ and $t$ on the surface of singularities $\Sigma(s,t)$ near its intersection with $\Gamma$.[4] Thus, when $(ds/dt)$ is negative along $\Gamma(s,t)$, the imaginary parts of $s$ and $t$

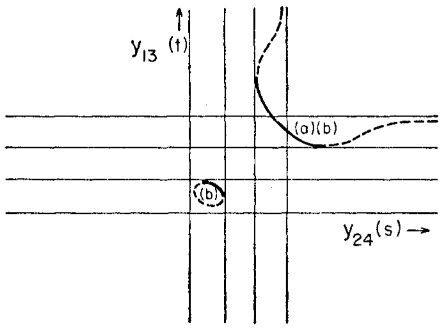


FIG. 5. Anomalous thresholds of type I give a curve of singularities of form (a); those of type II (super-anomalous) give curves of type (b). The broken lines denote curves on nonphysical sheets.

have opposite sign on $\Sigma(s,t)$. We shall later require a particular investigation of points where $(ds/dt)$ is zero (or infinite) since at these "turning points" it is possible for $t$ (or $s$) to become complex while $s$ (or $t$) remains real.

At the turning points illustrated in Fig. 5, one $C$ singularity has moved to $\alpha_i=0$ so that it is also an $E$ singularity. At this point, therefore, the curve of singularities coincides with the related line of singularities. On the broken part of the curve, the singularity in the integrand of $A(s,t)$ has slipped off the contour of integration [0,1], as illustrated in Fig. 1 (iii). For the complex-conjugate function $A^+(s,t)$ however, analytic continuation will lead to a distorted contour as in Fig. 1(iv).

## 6. PROPERTIES OF THE DISCRIMINANT, $D(\alpha,s,t)$

In this section the general form of the discriminant $D$ of the quadratic $\psi$, Eq. (2.6), will be studied. The information obtained will be shown in Sec. 7 to be useful in determining the absence (or existence) of turning points in curves of singularities. The importance of turning points is that sometimes they lead to complex singularities in the physical sheet. Our main objective is therefore to prove their absence under certain conditions which will be described later.

The discriminant is defined by the transformation of Eq. (2.6) to diagonal form,

$$\psi=\sum \alpha_i(q_i^2-m_i^2+i\epsilon)=\sum c_j k_j'^2+D(\alpha,s,t). \quad (6.1)$$

The 4-momentum $q_i$ in a typical internal line is a linear function of the external momenta $p_k$ and the internal momenta $k_j$ with coefficients 0, $\pm 1$. The coefficients $c_j$ are functions of the $\alpha$, and are positive when all the $\alpha$'s are positive. This follows from the fact that the left-hand side of Eq. (6.1) excluding the mass terms is a positive definite form in the internal-momentum variables. The transformed variables $k_j'$ on the right-hand side of Eq. (6.1) are linear functions of the $k_j$ and the external momenta $p_j$. The discriminant $D$ can be expressed uniquely in terms of any pair of the invariant energies $s$, $t$, $u$ given by Eqs. (2.3) and (2.4). Its form depends on which pair is chosen. The symbol $\alpha$ will be used to denote $\alpha_1, \alpha_2, \cdots, \alpha_n$, collectively. The masses $m_i$ in Eq. (6.1) will not be assumed equal unless this is explicitly stated.

The main technique to be used for discussing the properties of $D(\alpha,s,t)$ is based on its invariance under different choices for the paths of the external momenta through a given diagram and under different choices of circuits for the internal momenta. A second technique to be used later is based on the relation between $D(\alpha,s,t)$ and $D(\alpha,t,u)$ which can be obtained by substitution from Eq. (2.4).

The left-hand side of Eq. (6.1) has the form,

$$\sum a_{ij}k_ik_j+2\sum b_ik_i+c. \quad (6.2)$$

The coefficients $a_{ij}$, $b_i$, $c$ all depend linearly on $\alpha$. The $a_{ij}$ do not depend on other variables. The $b_i$ depend linearly on the external momenta. The term $c$ depends linearly on the squares and products of external momenta and on the squares of the internal masses. The discriminant of Eqs. (6.1) or (6.2) is

$$D=\begin{vmatrix} a_{11} & a_{12}\cdots & b_1 \\ a_{21} & a_{22} & b_2 \\ \vdots & & \\ b_1 & b_2 & c \end{vmatrix}. \quad (6.3)$$

Expanding by the last row and last column, we have

$$D=-\sum A_{ij}(\alpha)b_ib_j+C(\alpha)c, \quad (6.4)$$

where

$$C(\alpha)=\begin{vmatrix} a_{11} & a_{12}\cdots & a_{1l} \\ a_{21} & a_{22} & a_{2l} \\ \vdots & & \\ a_{l1} & a_{l2} & a_{ll} \end{vmatrix}. \quad (6.5)$$

The coefficient $A_{ij}(\alpha)$ in Eq. (6.4) is the co-factor of $a_{ij}$ in $C(\alpha)$.

The products $b_ib_j$ in Eq. (6.4) involve either scalar products of the external momenta or the squares of the external masses. They can therefore be expressed in terms of $s$, $t$, and the squares of external masses, giving

$$D(\alpha,s,t)=sf(\alpha)+tg(\alpha)-K(\alpha,m). \quad (6.6)$$

Each term in Eq. (6.6) is homogeneous in the $\alpha$ and of degree $(l+1)$, where $l$ is the number of internal-momentum variables $k_j$.

*Lemma 6A*

The discriminant $D(\alpha,s,t)$ is quadratic in each $\alpha_i$.

Choose the internal circuits for $k_j$, and paths for the external momenta so that a particular line, with momentum $q_1$ say, has $q_1=k_1$. This is always possible for diagrams with singularities depending on both $s$ and $t$, and we are not considering others here. Then we can write

$$a_{11}=\alpha_1+x_{11} \quad c=c'-\alpha_1 m_1^2, \quad (6.7)$$

where $x_{11}$ and $c'$ are independent of $\alpha_1$, and no other

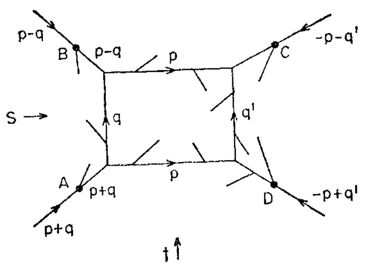


FIG. 6. Choice of 4-momenta in studying the general form of the discriminant.

term in Eq. (6.3) contains $\alpha_1$. This proves $D(\alpha,s,t)$ is quadratic in $\alpha_1$, and since this line was arbitrarily chosen, Lemma 6A follows.

From Eq. (6.7) we also obtain Lemma 6B.

*Lemma 6B*

The term in $D(\alpha,s,t)$ which is independent of $s$ and $t$ has the form

$$K(\alpha,m)=\sum_{i=1}^{n}\alpha_i^2 m_i^2 K_i(\alpha)-\sum_{i=1}^{n}\alpha_i m_i^2 K_i'(\alpha)-\sum_{j=1}^{4}K_j''(\alpha)M_j^2, \quad (6.8)$$

in which $m_i$ denotes an internal mass, and $M_j$ an external mass. The coefficients $K_i(\alpha)$, $K_i'(\alpha)$, and $K_j''(\alpha)$ are sums (or differences) of products of the $\alpha$ in which no $\alpha_k$ occurs more than once. They are homogeneous of order $(l-1)$, $l$, and $(l+1)$, respectively. Similarly we obtain Lemma 6C.

*Lemma 6C*

The coefficients $f(\alpha)$ and $g(\alpha)$ of $s$ and $t$ in $D(\alpha,s,t)$ are each the sums (or difference) of products linear in each $\alpha$ and of order $(l+1)$.

Further information about the association of different $\alpha_i$ in $f$ and $g$ can be obtained by a particular choice of variables. This is illustrated in Fig. 6. The internal variables are chosen so that the external momenta appear only in the lines shown in the diagram. The remaining internal lines are not shown, though the fact that there may be junctions with the lines carrying external momenta across the diagram is indicated. In each of the lines shown the external variables are combined linearly with the internal variables. The points of entry (or leading intersections) of the external lines with the diagram are marked $A$, $B$, $C$, and $D$.

Using mass shell conditions and definitions [Eq. (2.3)], we obtain

$$(p+q)^2=M_a^2,\quad (p-q)^2=M_b^2, \quad (6.9)$$

$$p^2+q^2=\tfrac{1}{2}(M_a^2+M_b^2),\quad p^2+q'^2=\tfrac{1}{2}(M_c^2+M_d^2), \quad (6.10)$$

$$2pq=\tfrac{1}{2}(M_a^2-M_b^2),\quad 2pq'=\tfrac{1}{2}(M_c^2-M_d^2), \quad (6.11)$$

and

$$s=4p^2,\quad t=(q+q')^2,\quad u=(q-q')^2. \quad (6.12)$$

Hence we have

$$2qq'=t-\tfrac{1}{2}(\sum_a^d M_j^2)-\tfrac{1}{2}s. \quad (6.13)$$

This equation shows that we can identify the coefficient of $t$ from the coefficient of $2qq'$ in formula (6.6) for $D$. The 4-momenta $q$ and $q'$ occur only in the quantities $b_i$ in $D$. [$q^2$ and $q'^2$ also occur in $c$ but can be eliminated by Eq. (6.10) in terms of $s$ given by Eq. (6.12).] From Fig. 6 we see that if $\alpha_i$ multiplies $q$ it does not multiply $q'$. This is a consequence of our construction in Fig. 6, where the lines carrying $q$ and $q'$ do not have any internal line in common. If there was an internal line in common, $qq'$ would apparently involve a coefficient $\alpha_1^2$ from that line. However from Lemma 6C, $t$ and therefore $qq'$ must only have coefficients linear in $\alpha_1$; therefore this $\alpha_1^2$ term must cancel with another. For this reason we shall restrict the $q$ and $q'$ lines to have no internal line in common.

*Lemma 6D*

Each $\alpha_i$ in the $q$ line of Fig. 6 must be associated with an $\alpha_j$ in the $q'$ line, in the product $(q\cdot q')\alpha_i\alpha_j\cdots$. This result holds for every independent pair of $q$, $q'$ lines, and gives Lemma 6E.

*Lemma 6E*

The coefficient $g(\alpha)$ of $t$ in $D(\alpha,m,s,t)$ is a sum of products of $\alpha$. Each product contains one $\alpha_i$ from each independent $q$ line in the diagram. A $q$ line is one of any pair carrying the 4-momentum whose square is equal to $t$.

It should be noted that there may be cancellation of some terms so that Lemma 6E does not mean that every $\alpha_i$ from each independent $q$ line must occur in $g(\alpha)$. This cancellation will be considered further after considering the relation of the form of $D$ to the form of the curves of singularities.

The form of $f(\alpha)$ can be studied in a similar manner by interchanging the external vectors in Fig. 6. This is necessary since, with the labeling of Fig. 6, the quantity $s$ occurs not only in products $p^2$, but also in $q^2$, $q'^2$, and $qq'$, so that there is in general a lot of cancellation. Relabeling Fig. 6, we take now

$$p_a=p+q,\qquad p_b=p'-q, \quad (6.14a)$$

$$p_c=-p'-q,\quad p_d=-p+q. \quad (6.14b)$$

This gives

$$s=(p_a+p_b)^2=(p+p')^2=2pp'+\tfrac{1}{2}t+\sum_a^d M_j^2. \quad (6.15)$$

The coefficient of $s$ in $D$, namely $f(\alpha)$, can now be identified from the $p$ and $p'$ lines which connect $A$ to $D$ and $B$ to $C$, respectively, in Fig. 6.

We shall also require the form of $D$ when expressed in terms of $u$, $s$, or in terms of $t$ and $u$. This may be obtained in two ways and the fact that both must give

the same answer gives further information about $D$. The first way is obtained from Eq. (2.4),

$$t=\sum_a^d M_j^2-s-u. \tag{6.16}$$

Then Eq. (6.6) gives

$$D(\alpha,u,s)=u\{-g(\alpha)\}+s\{f(\alpha)-g(\alpha)\}\\ -K(\alpha,m)-g(\alpha)\sum M_j^2. \tag{6.17}$$

Similarly, we obtain

$$D(\alpha,t,u)=t\{g(\alpha)-f(\alpha)\}+u\{-f(\alpha)\}\\ -K(\alpha,m)-f(\alpha)\sum M_j^2. \tag{6.18}$$

It should be noted that the convention of Mandelstam has been adopted in the notation of Eqs. (6.6), (6.17), and (6.18).[1] The form of the function $D$ depends on the variables in the bracket; thus $D(\alpha,u,s)$ has not the same form as $D(\alpha,s,t)$ but is related to it by Eq. (6.16). When $s$, $t$, and $u$ satisfy Eq. (6.16) the numerical values of the expressions in Eqs. (6.6), (6.17), and (6.18) are the same.

The second method of obtaining Eq. (6.17) is to study the form of $D(\alpha,u,s)$ in terms of $\alpha$ products by using diagrams similar to Fig. 6. From Fig. 6 itself we obtain

$$u=(p_a+p_c)^2=(q-q')^2=-2qq'+\tfrac{1}{2}s+\tfrac{1}{2}\sum_a^d M_j^2. \tag{6.19}$$

Thus the coefficient of $u$ in $D(\alpha,u,s)$ is given by the coefficient of $(-2qq')$. This is simply $-g(\alpha)$, as we had already obtained by means of the transformation Eq. (6.16) for Eq. (6.6) to Eq. (6.17). However new information will be obtained if we consider the coefficient of $s$ in

$$D(\alpha,u,s), \tag{6.20}$$
$$h(\alpha)=f(\alpha)-g(\alpha).$$

We must now choose variables in Fig. 6 so that

$$p_a=p+q,\quad p_b=p'-q, \tag{6.21a}$$

and

$$p_c=-p+q,\quad p_d=-p'-q. \tag{6.21b}$$

Then the coefficient of $s$ in $D(\alpha,u,s)$ is the coefficient of $2pp'$ in the expression (6.3) for $D$. Hence $h(\alpha)$ is a sum of products of the $\alpha$ variables. Each product contains one $\alpha_i$ from each line joining $A$ to $C$, and one from each line joining $B$ to $D$. As before, not every $\alpha_i$ from these lines will occur as some may cancel. From Eq. (6.20) we see that every product term in $f(\alpha)$ which does not contain an $\alpha_i$ from each $A$ to $C$ line and each $B$ to $D$ line, must also occur in $g(\alpha)$ with the same coefficient and the same sign.

Examples of these general properties will be given in Sec. 8, and they will be further utilized in Sec. 9. We will consider next in more detail the terms in $D(\alpha,s,t)$ which is independent of $s$ and $t$. It is convenient now to restrict all external masses to have the same value $M$. Then from Eq. (6.6), we have

$$K(\alpha,m)=-D(\alpha,\,s=0,\,t=0). \tag{6.22}$$

This can be evaluated, in principle, with the labeling used in Fig. 6 and $p=0$, $q=-q'$, giving

$$p_a=-p_b=p_c=-p_d=q, \tag{6.23}$$

and

$$q^2=M^2. \tag{6.24}$$

Since the internal masses occur only in the combination $\sum \alpha_i m_i^2$, and the external momenta satisfy Eq. (6.23) and Eq. (6.24), we can write $K(\alpha,m)$ in the form

$$K(\alpha,m)=\sum \alpha_i m_i^2 K_2(\alpha)-M^2K_1(\alpha). \tag{6.25}$$

This form is consistent with Eq. (6.8), and from Lemma 6B, $K_1(\alpha)$ and $K_2(\alpha)$ are each homogeneous in the $\alpha$ and linear in each $\alpha_i$.

We can obtain $K_1(\alpha)$ by putting all internal masses $m_i$ equal to zero; and $K_2(\alpha)$ by taking all external masses to be zero. Take first

$$m_i=0,\quad i=1,2,\cdots,n. \tag{6.26}$$

Then $M^2K_1(\alpha)$ is the discriminant of

$$\psi_1=\sum \alpha_i q_i^2, \tag{6.27}$$

where, from Eq. (6.23),

$$q_i=e_iq+\sum_j e_{ij}k_j, \tag{6.28}$$

and

$$e_i=0 \text{ or } \pm 1,\quad e_{ij}=0 \text{ or } \pm 1. \tag{6.29}$$

Each $q_i$ contains at least one internal momentum, and some contain also the external momentum $q$. In the discriminant $M^2K_1(\alpha)$, the 4-momentum $q$ will occur only in the form $q^2=M^2$. Hence in considering properties of the discriminant rather than the quadratic form we can replace the 4-momenta $k_j$ by scalars $x_j$ and the external 4-momentum $q$ by $M$. This gives, instead of Eq. (6.27),

$$\psi_2=\sum \alpha_i\{\sum_j (e_{ij}x_j+e_iM)\}^2. \tag{6.30}$$

When the $\alpha$'s are all positive, $\psi_2$ is a positive definite quadratic in the $x_j$ and must therefore have a positive discriminant. Hence we have

$$K_1(\alpha)>0 \quad\text{when}\quad \alpha_i>0,\quad i=1,2,\cdots,n. \tag{6.31}$$

A similar condition can be obtained for $K_2(\alpha)$ by taking

$$m_i\neq 0,\quad M=0. \tag{6.32}$$

Then $\sum \alpha_i m_i^2 K_2(\alpha)$ is the discriminant of

$$\psi_3=\sum \alpha_i(\sum_j e_{ij}k_j)^2+\sum \alpha_i m_i^2. \tag{6.33}$$

and $K_2(\alpha)$ is the discriminant of

$$\psi_4=\sum_j \alpha_i(\sum e_{ij}x_j)^2. \tag{6.34}$$

This is positive when the $\alpha$'s are positive, and hence we have

$$K_2(\alpha)>0 \text{ when } \alpha_i>0, \quad i=1,2,\cdots,n. \qquad (6.35)$$

We consider next the relative magnitude of the two terms on the right-hand side of Eq. (6.25). The masses will now be restricted so as to include the general reduced diagram arising for equal-mass interactions. The unreduced diagrams have equal masses $m$ in every internal line, and the external lines each have mass $m$. The reduced diagrams may have larger masses than $m$ by a factor of an integer $\geq 1$; their external masses are unchanged. We therefore assume that

$$M\leq m_i, \quad i=1,2,\cdots,n. \qquad (6.36)$$

The discriminant $M^2K_1(\alpha)$ of $\psi_2$ in Eq. (6.30) is a determinant of the form,

$$\begin{vmatrix} a_{ij}\cdots & b_i' \\ \vdots & \\ b_j' & c' \end{vmatrix}, \qquad (6.37)$$

in which $a_{ij}=a_{ji}$, and for all $\alpha_i>0$ we have

$$\|a_{ij}\|=C(\alpha)>0, \qquad (6.38)$$

where $C(\alpha)$ is defined in Eq. (6.6). The $b_i'$ in expression (6.37) is a special value of the $b_i$ in Eq. (6.4).

Expression (6.37) gives

$$M^2K_1(\alpha)=c'C(\alpha)-\sum_{i,j} b_i'b_j'A_{ij}, \qquad (6.39)$$

where $A_{ij}$ is the co-factor of $a_{ij}$ in the determinant $(a_{ij})$. The second term in Eq. (6.39) has a discriminant

$$\|A_{ij}\|=\|a_{ij}\|^{n-1}=\{C(\alpha)\}^{n-1}>0. \qquad (6.40)$$

Hence, from Eq. (6.39), we have

$$M^2K_1(\alpha)<c'C(\alpha) \text{ when } \alpha_i>0, \quad i=1,\cdots,n. \qquad (6.41)$$

From Eq. (6.30) and expression (6.37), we obtain

$$c'=\sum_{i=1}^{n}\alpha_i(e_iM)^2. \qquad (6.42)$$

From expression (6.5) and Eq. (6.33), we obtain

$$K_2(\alpha)=C(\alpha). \qquad (6.43)$$

From the definition of $e_i$, and restriction (6.36) on the masses, when all $\alpha_i$ are positive we have

$$\sum \alpha_i(e_iM)^2<\sum \alpha_iM^2\leq\sum \alpha_im_i^2. \qquad (6.44)$$

Combining expression (6.41), (6.43), and (6.44) and using the definition in Eq. (6.25), we obtain

$$K(\alpha,m)>0, \text{ for } \alpha_i>0, \quad i=1,\cdots,n. \qquad (6.45)$$

We state this result as a theorem,

**Theorem 6A**

The discriminant $D(\alpha,s,t)$ for any reduced or unreduced diagram in an equal-mass system, evaluated at the point $s=t=0$, is always negative when all $\alpha$'s are positive.

We consider next the values of the derivatives of $K(\alpha,m)$ with repsect to $\alpha_i$, $i=1,\cdots,n$. First we note that $K_2(\alpha)$, [or $C(\alpha)$], is linear in each $\alpha_i$, and it is the discriminant of Eq. (6.34). By a suitable choice of the internal momenta, Eq. (6.34) becomes

$$\psi_4=\alpha_1x_1^2+\sum_{i\geq 2}^{n}\alpha_i(\sum_j e_{ij}x_j)^2. \qquad (6.46)$$

The coefficient of $\alpha_1$ in $K_2(\alpha)$ is the discriminant of $\psi_2$ when $x_1=0$ and is positive when the $\alpha$ are positive. Hence we have

$$\frac{\partial}{\partial\alpha_i}K_2(\alpha)>0, \text{ when } \alpha_j>0, \quad j=1,\cdots,n. \qquad (6.47)$$

From Eqs. (6.25), (6.39), (6.42), and (6.43), we can write

$$K(\alpha,m)=\sum \alpha_im_i^2K_2(\alpha)-\sum \alpha_i(e_iM)^2K_2(\alpha)+\sum A_{ij}b_ib_j \qquad (6.48)$$

$$=\sum \alpha_i(m_i^2-e_i^2M^2)K_2(\alpha)+\sum A_{ij}b_ib_j. \qquad (6.49)$$

We have shown already that the first term on the right of Eq. (6.49) is positive [see Eqs. (6.35) and (6.44)], and increases with each $\alpha_i$ [Eq. (6.47)]. The second term is positive [see Eq. (6.40)]. In order to study the derivative of this term we note that $M^2K_1(\alpha)$ given by Eq. (6.39) is the discriminant of $\psi_2$ given by Eq. (6.30). We can choose the internal momenta and the path of the external momenta of Eq. (6.23) through the diagram so that we have

$$\psi_2=\alpha_1x_1^2+\sum_{i\geq 2}\alpha_i(\sum_j e_{ij}x_j+e_iM)^2. \qquad (6.50)$$

The discriminant of $\psi_2$ is of the form Eq. (6.39) in which is clear from Eq. (6.50) that the $b_i$ will not depend on $\alpha_1$. Hence the derivative,

$$\frac{\partial}{\partial\alpha_1}\sum_{i,j} A_{ij}b_ib_j, \qquad (6.51)$$

will be the discriminant of the quadratic form

$$\sum_{i\geq 2}\alpha_i(\sum_j e_{ij}x_j)^2, \qquad (6.52)$$

in which $x_1$ is put equal to zero. This is positive definite when the $\alpha$'s are positive, and we obtain for $\alpha$ positive (since $\alpha_1$ was chosen arbitrarily),

$$\frac{\partial}{\partial\alpha_k}\sum A_{ij}b_ib_j>0. \qquad (6.53)$$

Since both terms on the right-hand side of Eq. (6.49) increase with $\alpha_i$, we obtain Theorem 6B.

### Theorem 6B

The discriminant $D(\alpha,s,t)$ for any reduced or unreduced diagram in an equal-mass system, evaluated at the point $s=t=0$, is a negative and decreasing function of each $\alpha_i$, when all the $\alpha$'s are positive.

In terms of $K(\alpha,m)$, this gives for $\alpha_j>0$, $j=1,\cdots,n$,

$$\frac{\partial}{\partial\alpha_i}K(\alpha,m)>0, \quad i=1,\cdots,n. \tag{6.54}$$

The next section will describe the determination of some general characteristics of the curves of singularities from the discriminant $D$. Some of the properties of $D$ obtained in this section will be illustrated in applications in Sec. 8, and used further in discussing the general term in the perturbation series in Sec. 9.

## 7. SINGULARITIES AND THE DISCRIMINANT

### (A) Turning Points in Curves of Singularities

A turning point in a curve of singularities is defined as a point where the tangent to the curve is parallel to one of the coordinate axes, $s$, $t$, or $u$. Their importance is due to their connection with complex singularities in the physical sheet. This will be described later in this section [Part (E)].

The curves of singularities are obtained in principle by solving the equations

$$\partial D(\alpha,s,t)/\partial\alpha_i=0, \quad i=1,\cdots,n, \tag{7.1}$$

where $D(\alpha,s,t)$ is the discriminant for a fully reduced diagram. Since these equations are homogeneous in $\alpha$, they lead to a condition on $s$ and $t$ which is the equation of the curve of singularities, say

$$t=t(s). \tag{7.2}$$

When Eq. (7.2) is satisfied, the actual values of the $\alpha$ at a singular point are obtained by solving Eq. (7.1) together with

$$\sum_1^n \alpha_i=1. \tag{7.3}$$

These give each $\alpha_i$ as a function of $s$:

$$\alpha_i=\alpha_i(s), \quad i=1,\cdots,n. \tag{7.4}$$

Since $D$ is homogeneous and of order $l$ in the $\alpha$, we have

$$\sum_1^n \alpha_i\frac{\partial D(\alpha,s,t)}{\partial\alpha_i}=lD(\alpha,s,t). \tag{7.5}$$

Thus Eq. (7.4) gives a function $D[\alpha(s),s,t]$ which is zero along the curve of Eq. (7.2). Hence we have

$$\begin{aligned}0&=\frac{dD(\alpha,s,t)}{ds}\\ &=\frac{\partial D(\alpha,s,t)}{\partial s}+\frac{\partial D(\alpha,s,t)}{\partial t}\frac{dt}{ds}+\sum_{i=1}^n\frac{\partial D(\alpha,s,t)}{\partial\alpha_i}\frac{d\alpha_i}{ds}.\end{aligned} \tag{7.6}$$

The last term in Eq. (7.6) is zero on the curve, from Eq. (7.1), and $D(\alpha,s,t)$ has the form from Eq. (6.6),

$$D(\alpha,s,t)=sf(\alpha)+tg(\alpha)-K(\alpha,m). \tag{7.7}$$

Hence along the curve of singularities, we have

$$dt/ds=-f(\alpha)/g(\alpha). \tag{7.8}$$

This leads to Theorem 7A.

### Theorem 7A

If a curve of singularities has a tangent, $t=$constant, in the $s$, $t$, plane, then the coefficient of $s$ in the discriminant $D(\alpha,s,t)$ must vanish at the point of tangency,

$$\partial D(\alpha,s,t)/\partial s=f(\alpha)=0. \tag{7.9}$$

This theorem applies to the general-mass case.

The point of tangency will be called a "turning point." Theorem 7A has a similar form for turning points $s=$constant in the $s$, $t$ plane, and analogs for turning points in $u$.

There are two distinct ways in which the turning-point theorem 7A may be satisfied. These will be called "anomalous turning points," and "spurious turning points." The former are so called because they are always associated with anomalous thresholds. The latter are not associated with any thresholds at all. It is important to distinguish between these two types of turning point.

### (B) Anomalous Turning Points

These will occur if, along a curve of singularities $\Gamma(s,t)$ in the real $s$, $t$ plane in the physical sheet, $f(\alpha)$ becomes zero because sufficient $C$ singularities become also $E$ singularities. From Lemma 6E this requires at least that all the $\alpha_i$ in $f(\alpha)$ which are associated with a particular $p$ or $p'$ line (in the sense of Lemma 6D) become zero at the turning point. Let us assume there is such a turning point. Then when this set of $\alpha_i$ is zero (denoted by $\alpha'$), we will have

$$D(\alpha'',\alpha'=0,s,t)=tg(\alpha'')-K(m,\alpha''), \tag{7.10}$$

where $\alpha''$ denotes those $\alpha$'s which are not included in $\alpha'$. Since the turning point is on $\Gamma(s,t)$, the right-hand side of Eq. (7.10) will satisfy the Landau conditions with all the $\alpha''$ giving $C$ singularities. But this expression is the discriminant for a reduced diagram that depends only on $t$. Hence it gives a line of singularities which is in

the physical sheet if the turning point is in the physical sheet. Clearly at the turning point the line of singularities is tangent to $\Gamma(s,t)$.

We know that in the general-mass case, the diagrams whose singularities depend only on $t$ must, when fully reduced, have the momentum $(q+q')$ passing through a single vertex somewhere in the diagram. It must therefore reduce to a vertex part or chain of these parts at least (possibly of course to self-energy parts). This permits us to extend Lemma 6E, giving Lemma 7A.

*Lemma 7A*

The coefficient $f(\alpha)$ in $D(\alpha,s,t)$ is a sum of products of $\alpha$. Each product contains one $\alpha_i$ from each independent "$s$-path" in the diagram. An $s$-path is any connected set of lines which when reduced to a point gives a reduced diagram depending only on $t$.

No $\alpha_i$ occurs more than once in any product. The sum of momenta in all independent $s$ paths is $(p_a+p_b)$. Similar lemmas hold for $g(\alpha)$ and $t$ paths, and $h(\alpha)$ and $u$ paths.

From theorem 7A and Lemma 7A we obtain Lemma 7B.

*Lemma 7B*

If a curve of singularities in the physical sheet has a turning point with tangent parallel to the $s$ axis where the $\alpha$ in just one $s$-path becomes zero, the curve changes from the physical sheet to a nonphysical sheet at the point of tangency.

This lemma follows from the fact that $f(\alpha)$ is an odd function of those values of $\alpha$ which are zero at the turning point, and $f(\alpha)$ changes sign since $(dt/ds)$ changes sign. Hence, some $\alpha$ values become negative, which means we have a nonphysical branch on one side of the turning point.

### (C) Tangency at Normal Thresholds

In the equal-mass case, there are no anomalous thresholds in the physical sheet. Hence the only lines of singularities at which a curve may have a turning point are the normal thresholds. At a normal threshold, $t=$constant, all the $\alpha_i$ in $f(\alpha)$ become zero except those (if any) which are in the self-energy part of $g(\alpha)$. It can be verified in any example that this causes $f(\alpha)$ to go to zero faster than the right-hand side of Eq. (7.10) (a self-energy part now), in the neighborhood of the normal threshold. Hence $s$ must tend to infinity at the point of tangency. This will also be true for normal thresholds in the general mass case.

*Lemma 7C*

Tangency at normal thresholds occurs only asymptotically.

### (D) Spurious Turning Points

When there is a sufficient degree of symmetry between $s$ and $u$ paths, it will be possible to have

$$f(\alpha)=0, \tag{7.11}$$

without any of the $\alpha$'s becoming zero. This leads to a turning point which, since it involves no $E$ singularities, is not associated with a line of singularities $t=$constant. An example of such a turning point is given by curve (c) in Fig. 4 in the region $u>0$, $s>0$. The factor which become zero at this point is given in Sec. 8, example (d). When Eq. (7.11) holds, the expression (6.17) has the form

$$D(\alpha,u,s)=u\{-g(\alpha)\}+s\{-g(\alpha)\} -K(\alpha,m)-g(\alpha)\sum M_j^2. \tag{7.12}$$

Then we can write

$$du/ds=-1. \tag{7.13}$$

Any curve of singularities of the normal form in the $u>0$, $s>0$, spectral region will at some point satisfy Eq. (7.13), and will have a tangent line $t=$constant. This line does not give a threshold value. These spurious turning points in $t$ for $t<0$ occur in fourth-order perturbation theory and do not prevent the proof of the Mandelstam representation in that order. We will consider later their implications for higher-order terms (Sec. 10).

A more dangerous possibility is the occurrence of a spurious turning point at $t=$constant$>0$. These will be considered in some specific examples in Sec. 8 and will be proved not to occur. They will be considered in Sec. 9 for the general term in the equal-mass case and an argument for their absence will be given, but not a proof. The particular danger from them will be indicated in Part (E) of this section.

### (E) Turning Points and Complex Singularities

A general diagram has the discriminant, Eq. (6.6),

$$D(\alpha,s,t)=sf(\alpha)+tg(\alpha)-K(a,m). \tag{7.14}$$

The path of integration over the variables $\alpha$ in Eq. (2.7) is from 0 to 1 along the real axis, unless analytic continuation forces a distortion of the path of integration. We consider the circumstances in which $s$ can become complex with a small imaginary part while $t$ remains real. We take

$$s=s_1+is_2, \tag{7.15}$$

and let $s_2$ be small enough to consider first-order terms only, but large compared with any imaginary parts of $K$ coming from $(m^2-i\epsilon)$. Then if $D$ remains zero, and the amplitude singular, when $s$ becomes complex while $t$ and $\alpha$ remain real, we must have

$$f(\alpha)=0. \tag{7.16}$$

This result can also be obtained by considering the intersection of the two-dimensional surface $\Sigma(s,t)$, in the four-dimensional space ($s$ and $t$ complex) with the curve $\Gamma(s,t)$ for which $s$ and $t$ are real.[4] Both $\Sigma$ and $\Gamma$ satisfy the equation

$$t=t(s). \tag{7.17}$$

The derivative $(dt/ds)$ is independent of the direction of differentiation, and hence if $t$ remains real on $\Sigma(s,t)$ near $\Gamma(s,t)$, we must have

$$dt/ds=0. \tag{7.18}$$

### Theorem 7B

Singularities on curves in the real part of the physical sheet do not extend into the complex part of this sheet (one variable real) except at turning points.

It is not always the case that turning points lead to complex singularities in the physical sheet. For example in fourth order for one type of anomalous threshold, the turning points do not lead to complex singularities in the physical sheet, but for another type they do. Evidently in the former case the $C$ singularities, which become also $E$ singularities, fall off the contour of integration as $s$ goes complex. In the latter they drag the contour with them.

The absence of anomalous turning points in the physical sheet excludes the possibility of one type of complex singularity. We have therefore to consider those from spurious turning points, and also to consider the possibility of complex singularities that are not connected to any singular curve in the real part of the physical sheet. These will be discussed further in Sec. 10.

## 8. APPLICATIONS AND EXAMPLES

In this section the general theory is illustrated by examples that are selected so as to bring out a number of features characteristic of more general diagrams.

### (A) Normal Thresholds

A generalized self-energy part, with $n$ lines joining two vertices, has an integrand giving, under transformation to Feynman variables, a denominator

$$\psi=\alpha_1(k_1-p)^2+\sum_2^{n-1}\alpha_i(k_i-k_{i+1})^2+\alpha_n k_n^2-\sum_1^n\alpha_i m_i^2. \tag{8.1}$$

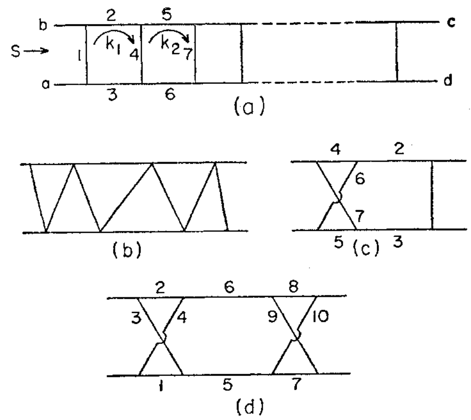


FIG. 7. Diagrams studied in worked examples. The numbers indicate labeling by Feynman parameters $\alpha_1, \alpha_2, \cdots, \alpha_{10}$.

If we write $p^2=s$, the discriminant is

$$D(\alpha,s)=\alpha_1\alpha_2\cdots\alpha_n s-\sum_{i=1}^n\alpha_i m_i^2\sum_{j=1}^n\frac{(\alpha_1\cdots\alpha_n)}{\alpha_j}. \tag{8.2}$$

The coefficient of $s$ gives a simple illustration of Lemma 6E.

From Eq. (8.2) we obtain

$$D(\alpha,s)\leq\alpha_1\alpha_2\cdots\alpha_n\{s-(\sum_1^n m_i)^2\} \tag{8.3}$$

for $\alpha_i>0$, $i=1,\cdots n$. It is clear that for

$$s<(\sum m_i)^2, \quad \text{we have} \quad D<0. \tag{8.4}$$

When $s$ is greater than $(\sum m_i)^2$ there is a region in $\alpha$ space in which $D$ is positive. Hence there is a singularity of the amplitude at

$$s=s_c=(\sum_1^n m_i)^2. \tag{8.5}$$

This is the normal threshold above which production of particles with masses $m_1, m_2, \cdots m_n$ is allowed.

### (B) Simple Ladder Diagrams

These are ladder diagrams which do not contain any crossed lines. Label the lines as in Fig. 7(a), giving

$$\begin{aligned}\psi=&\alpha_1(k_1+p)^2+\alpha_4(k_2-k_1)^2+\alpha_7(k_3-k_2)^2+\cdots\\&+\alpha_2(k_1+q)^2+\alpha_5(k_2+q)^2+\cdots\\&+\alpha_3(k_1-q')^2+\alpha_6(k_2-q')^2+\cdots-\sum_1^n\alpha_i m_i^2.\end{aligned} \tag{8.6}$$

The discriminant is

$$D(\alpha,s,t)=\begin{vmatrix}\alpha_1+\alpha_2+\alpha_3+\alpha_4 & -\alpha_4 & 0 & \alpha_1 p+\alpha_2 q-\alpha_3 q' \\ -\alpha_4 & \alpha_4+\alpha_5+\alpha_6+\alpha_7 & -\alpha_7 & \alpha_5 q-\alpha_6 q' \\ 0 & -\alpha_7 & \alpha_7+\cdots & \cdots \\ \vdots & & & \\ \alpha_1 p+\alpha_2 q-\alpha_3 q' & \alpha_5 q-\alpha_6 q' & \cdots & \cdots\end{vmatrix}. \tag{8.7}$$

From Eq. (6.14), we have

$$s=2qq'+\tfrac{1}{2}t+\tfrac{1}{2}\sum M_a^2, \tag{8.8}$$

and

$$t=4p^2. \tag{8.9}$$

The coefficient of $t$ in Eq. (8.7) is simply evaluated by Lemma 7A, which gives, in the notation of Eq. (6.5),

$$g(\alpha)=\alpha_1\alpha_4\alpha_7\cdots\alpha_{3n+1}. \tag{8.10}$$

The coefficient of $s$ is the coefficient of $2qq'$ in Eq. (8.7). This gives

$$f(\alpha)=\alpha_2\alpha_3d_{23}+\alpha_5\alpha_6d_{56}+\cdots, \tag{8.11}$$

where $d_{23}$, $d_{56}$, $\cdots$ are positive when the $\alpha$'s are positive, and

$$d_{23}=\sum (4,5,6,7)(7,8,9,10)(10,11,12,13)\cdots. \tag{8.12}$$

Each term in the sum is a product $\alpha_i\alpha_j\alpha_k\cdots i\neq j$, $i\neq k$, $j\neq k$, $\cdots$, with $i$ taken from the first bracket, $j$ from the second bracket, etc.

Since both $f(\alpha)$ and $g(\alpha)$ are positive when the $\alpha$'s are positive, we obtain the result: if a fully reduced simple ladder diagram has a curve of singularities in the physical sheet, then along the curve we have

$$dt/ds<0 \text{ when } \alpha_i>0, \quad i=1, 2, 3, \cdots. \tag{8.13}$$

From this result and Theorem 7A, we obtain Lemma 8A.

*Lemma 8A*

Curves of singularities in the physical sheet and corresponding to simple ladder diagrams do not have turning points except possibly at end-point singularities.

End-point singularities may occur at

$$s=s_c, \quad \text{when} \quad \alpha_{3j+1}=0, \tag{8.14}$$

for some values of $j$, or at

$$t=t_c, \text{ when } \alpha_{3j}=0 \text{ and (or) } \alpha_{3j+2}=0 \tag{8.15}$$

for all values of $j$. These singularities will occur only for particular values of the masses. It is clear from Sec. 5 that the only singularities in the physical sheet have $s>0$ and $t>0$. This could also be proved by explicit consideration of $K(\alpha,m)$ given by Eq. (8.7), or by the method of Sec. 9(b).

For equal internal and external masses, in the original unreduced diagrams, it has been shown (see the corollary to Theorem 5A) that the only lines of singularities in the physical sheet are given by normal thresholds. For these, we have either all $\alpha_{3j+1}=0$ in Eq. (8.14), or all $\alpha_{3j}$ and $\alpha_{3j+2}=0$ in Eq. (8.15). These give the asymptotes

$$s=4m^2, \quad t=(nm)^2, \tag{8.16}$$

and the curve has negative slope everywhere in the physical sheet. More generally, if the diagram has been reduced, these asymptotes will involve higher integer multiples of $m$.

### (C) Partly Reduced Ladder Diagrams

For many of these diagrams there will be no curve of singularities in the physical sheet; an example is given by diagram (d) in Fig. 3 which was discussed in Sec. 5. For our present purpose we need only know that the curves of singularities in the physical sheet have certain characteristics if they exist. It is therefore not necessary to consider the form of $K(m,\alpha)$ in general. For the equal-mass case a fully reduced diagram which may be expected to have a curve of singularities is shown in Fig. 7(b). Like all diagrams in this class its discriminant is derived from Eq. (8.7) by taking some of $\alpha_{3j}$ and $\alpha_{3j+2}$ to be zero. Since $g(\alpha)$ and $f(\alpha)$ remain positive when the other $\alpha$'s are positive, condition (8.13) still applies, with $\alpha_i$ now referring only to the nonreduced lines of the diagram. The fact that there are no anomalous lines of singularities means that $dt/ds$ becomes zero or infinite only asymptotically. The asymptotes for the curve from Fig. 7(b) will be

$$s=9m^2, \quad t=(7m)^2. \tag{8.17}$$

### (D) The Symmetric Crossed Diagram

This diagram is shown in Fig. 3(c). It is obtained from an eighth-order diagram with some reduced lines. It leads to a discriminant

$$D(\alpha,s,t)=4\alpha_1\alpha_3(\alpha_5\alpha_6-\alpha_2\alpha_4)s +4\alpha_2\alpha_4(\alpha_5\alpha_6-\alpha_1\alpha_3)t-K(\alpha,m). \tag{8.18}$$

The parameters $\alpha_1, \alpha_2,\cdots,\alpha_6$ correspond to lines $a, b,\cdots,f$ in Fig. 3 diagram (c) taken in the same order.

With external masses $m$ and internal masses $m_l$, we have

$$K(m,\alpha)=-16m^2\alpha_1\alpha_2\alpha_3\alpha_4 +4\sum_{i<j<k}\alpha_i\alpha_j\alpha_k\sum\alpha_lm_l^2. \tag{8.19}$$

For $m_l\geq m$, we have, from Eq. (8.19) or from Theorems 6A and 6B,

$$K(m,\alpha)>0, \quad\text{and}\quad \partial K(m,\alpha)/\partial\alpha_i>0, \quad\text{for}\quad \alpha_i>0. \tag{8.20}$$

This condition is satisfied for all diagrams of type (c) in Fig. 3, which are formed by reducing higher-order diagrams in the equal-mass case.

Now Eq. (7.1) holds on a curve of singularities and we have

$$\partial D/\partial\alpha_1=4\alpha_3(\alpha_5\alpha_6-\alpha_2\alpha_4)s-4\alpha_2\alpha_4\alpha_3t-\partial K/\partial\alpha_1. \tag{8.21}$$

From Eqs. (8.20), (8.21), and (7.1), for $\alpha>0$ and

$$s>0 \quad\text{and}\quad t>0, \tag{8.22}$$

we have

$$\alpha_5\alpha_6>\alpha_2\alpha_4. \tag{8.23}$$

Similarly, when Eq. (8.22) holds, we have

$$\alpha_5\alpha_6>\alpha_1\alpha_3. \tag{8.24}$$

Hence, along the curve of singularities in the region of the physical sheet given by Eq. (8.22), we have

$$dt/ds=-f(\alpha)/g(\alpha)<0, \tag{8.25}$$

where $f(\alpha)$ and $g(\alpha)$ are the coefficients of $s$ and $t$ in Eq. (8.18).

The result Eq. (8.25) proves that in the region given by Eq. (8.22) the only turning points are at $E$ singularities. But the only $E$ singularities are given by normal thresholds for the equal-mass case, and these are asymptotes. Applying similar arguments to the regions $u>0$, $s>0$ and $t>0$, $u>0$, we obtain the curve ofsingularities having the three branches marked (c) in Fig. 4.

At the turning point $(dt/ds)=0$ in the region $t<0$, the curve of singularities connects to a surface of singularities on which for $t$ real $s$ becomes complex. However, this is a feature also of the fourth-order diagram in which the internal lines are crossed when $s$ and $t$ are positive and $t$ negative. In this region the double-dispersion relation is obtained by working in terms of variables $s$ and $u$ and not $s$ and $t$. Then the diagram (c) of Fig. 3 has

$$du/ds<0 \tag{8.26}$$

along the singular curve, since in this region we have

$$\alpha_1\alpha_3>\alpha_5\alpha_6, \tag{8.27a}$$

and

$$\alpha_1\alpha_3>\alpha_2\alpha_4. \tag{8.27b}$$

For a general diagram it is still necessary to show that singularities in all three spectral regions from one diagram do not prevent application of Cauchy's theorem. This problem will not be considered further here.

### (E) Crossed Rungs in a Ladder Diagram

A simple example in which only a single pair of rungs is crossed is shown in Fig. 7 diagram (c). The discriminant is

$$\begin{aligned} D(\alpha,s,t) &= s\{\alpha_2\alpha_3(\alpha_4+\alpha_5+\alpha_6+\alpha_7)+\alpha_3\alpha_4\alpha_6+\alpha_2\alpha_5\alpha_7-\alpha_1\alpha_4\alpha_5\} \\ &\quad +t\{\alpha_1(\alpha_6\alpha_7-\alpha_4\alpha_5)\}-K_{st}(\alpha,m), \end{aligned} \tag{8.28}$$

where $K_{st}(\alpha,m)$ is written with the suffixes $s$, $t$ to denote the form of $D$ with which it is associated. From Eq. (8.28) and Theorem 6B, we see that

$$\partial D(\alpha,s,t)/\partial\alpha_1=0, \tag{8.29}$$

which gives

$$-s(\alpha_4\alpha_5)+t(\alpha_6\alpha_7-\alpha_4\alpha_5)-\partial K_{st}/\partial\alpha_1=0, \tag{8.30}$$

leads to the inequality

$$\alpha_6\alpha_7-\alpha_4\alpha_5>0,\text{ for } s>0,\ t>0,\text{ and } \alpha>0. \tag{8.31}$$

Similarly, by differentiating with respect to $\alpha_4$, we deduce

$$\alpha_2\alpha_3+\alpha_3\alpha_6-\alpha_1\alpha_5>0,\text{ for } s>0,\ t>0,\text{ and } \alpha>0. \tag{8.32}$$

Inequalities (8.31) and (8.32) show that for $s>0$, $t>0$, and $\alpha>0$, we have

$$f(\alpha)>0 \quad\text{and}\quad g(\alpha)>0. \tag{8.33}$$

Hence the only turning points in this region must occur at $E$ singularities if any. But from Secs. 3 and 4 there are no such $E$ singularities. Hence we have

$$dt/ds<0,\text{ for } s>0 \text{ and } t>0 \tag{8.34}$$

in the physical sheet.

A similar result can be proved for the region $s>0$, $u>0$ in which we have

$$\begin{aligned} D_{su}&=s\{\alpha_2\alpha_3(\alpha_4+\alpha_5+\alpha_6+\alpha_7)+\alpha_3\alpha_4\alpha_6+\alpha_2\alpha_5\alpha_1-\alpha_1\alpha_6\alpha_7\} \\ &\quad +u\{\alpha_1(\alpha_4\alpha_5-\alpha_6\alpha_7)\}-K_{su}(\alpha,m). \end{aligned} \tag{8.35}$$

However in the region $t>0$, $u>0$, we have

$$\begin{aligned} D_{tu}&=t\{\alpha_1\alpha_6\alpha_7-\alpha_3\alpha_4\alpha_6-\alpha_2\alpha_5\alpha_7-\alpha_2\alpha_3(\alpha_4+\alpha_5+\alpha_6+\alpha_7)\} \\ &\quad +u\{\alpha_1\alpha_4\alpha_5-\alpha_3\alpha_4\alpha_6-\alpha_2\alpha_5\alpha_7-\alpha_2\alpha_3(\alpha_4+\alpha_5+\alpha_6+\alpha_7)\} \\ &\quad -K_{tu}(\alpha,m). \end{aligned} \tag{8.36}$$

We obtain

$$\begin{aligned} \partial D_{tu}/\partial\alpha_3&=-(u+t)\{\alpha_4\alpha_6+\alpha_2(\alpha_4+\alpha_5+\alpha_6+\alpha_7)\} \\ &\quad -\partial K_{tu}/\partial\alpha_3. \end{aligned} \tag{8.37}$$

This is clearly negative for $u>0$, $t>0$, and $\alpha>0$. Hence there are no singularities in the physical sheet in this region.

### (F) The Euclidean Region

From Theorem 6A, for a general diagram, with $\alpha>0$, we can write

$$D(\alpha,\ s=0,\ t=0)=-K_{st}(\alpha,m)<0. \tag{8.38}$$

A similar result holds when we put $u=0$, $t=0$, in

$$D(\alpha,u,t)=(4m^2-t-u)f(\alpha)+tg(\alpha)-K_{st}(\alpha,m). \tag{8.39}$$

Hence, we have

$$4m^2f(\alpha)-K_{st}(\alpha,m)<0. \tag{8.40}$$

From Eqs. (8.38) and (8.40), for $0<u<4m^2$, we can write

$$(4m^2-u)f(\alpha)-K_{st}(\alpha,m)<0. \tag{8.41}$$

Similarly for $0<t<4m^2$, we have

$$tg(\alpha)-K_{st}(\alpha,m)<0. \tag{8.42}$$

Writing $s$ for $(4m^2-u)$ in Eq. (8.41) and combining Eqs. (8.41) and (8.42), we obtain

$$D(\alpha,s,t)=sf(\alpha)+tg(\alpha)-K_{st}(\alpha,m)<0, \tag{8.43}$$

provided that

$$0<s+t<4m^2. \tag{8.44}$$

This gives an independent derivation of the result of Sec. 4, namely, Lemma 8B.

*Lemma 8B*

For the equal-mass case, there are no singularities in the Euclidean region of the physical sheet.

### (G) The Double-Crossed Ladder Diagram

This diagram is illustrated in Fig. 7(d). It can either be eighth-order with all masses equal or a higher-order reduced diagram with internal masses larger than the external masses. The discriminant is quite a lengthy expression, but in order to study it not every term is required. Either by direct evaluation, or using the methods of Sec. 6, we obtain in $D(\alpha,s,t)$

$$f(\alpha)=(\text{positive terms})-\alpha_1\alpha_2\alpha_9\alpha_{10}-\alpha_3\alpha_4\alpha_7\alpha_8, \quad (8.45)$$

$$g(\alpha)=(\alpha_3\alpha_4-\alpha_1\alpha_2)(\alpha_9\alpha_{10}-\alpha_7\alpha_8), \quad (8.46)$$

and

$$-K(m,\alpha)<0, \quad \text{for} \quad \alpha>0. \quad (8.47)$$

The asymptotes of the curve of singularities (assuming the curve to have a branch on the physical sheet—if it does not the diagram can be ignored) will be given, in the equal-mass case, by

$$s=s_c=4m^2, \quad (8.48)$$

and

$$t=t_c=16m^2. \quad (8.49)$$

Near $s=s_c$, only $\alpha_5$ and $\alpha_6$ are appreciably different from zero; near $t=t_c$ only $\alpha_3$, $\alpha_4$, $\alpha_5$, and $\alpha_6$ appreciably differ from zero. Near $s=s_c$, $f(\alpha)$ will be of second order in the small $\alpha_i$, $g(\alpha)$ will be fourth-order, and $K(m,\alpha)$ will be second-order. Near $t=t_c$, $f(\alpha)$ will be of second order in the small $f_i$, and $g(\alpha)$ and $K(m,\alpha)$ will be independent of them.

We see that near $s=s_c$, $f(\alpha)$ will be dominated by the positive term containing $\alpha_5\alpha_6$ as a factor. It is clearly positive, as indeed it must be if the amplitude is to be singular on the asymptote. Near $t=t_c$, it is also clear that $g(\alpha)$ is positive, since it is dominated by the terms $\alpha_3\alpha_4\alpha_9\alpha_{10}$. From the symmetry of the diagram, the two factors in $g(\alpha)$ will be equal on the curve of singularities, and hence $g(\alpha)$ is positive.

Although $f(\alpha)$ is positive on the curve near $s=s_c$, it is not immediately evident that it cannot become negative elsewhere. However by the symmetry of the diagram on the curve of singularities, we must have $\alpha_1=\alpha_2=\alpha_7=\alpha_8$; $\alpha_3=\alpha_4=\alpha_9=\alpha_{10}$; and $\alpha_5=\alpha_6$. The negative terms in Eq. (8.45) then cancel with two of the positive terms, for example with $\alpha_3\alpha_9\alpha_1\alpha_8$ and $\alpha_4\alpha_{10}\alpha_2\alpha_7$.

We conclude that both $f(\alpha)$ and $g(\alpha)$ are positive on the curve of singularities in the region $s>0$, $t>0$ of the physical sheet. Hence there are no spurious turning points in this spectral region. The other spectral resions can be examined similarly.

## 9. THE GENERAL TERM FOR EQUAL-MASS INTERACTIONS

### (A) A Simplification Formula

Any diagram whose singularities depend on both $s$ and $t$ can be labeled so that the quadratic in the 4-momenta has the form

$$\psi=\alpha_1k_1^2+\sum_{j=2}^{n}\alpha_jq_j^2-\sum_{j\geq 1}\alpha_jm_j^2. \quad (9.1)$$

Here (as in Sec. 6), we have

$$q_j=\sum_{i=2}^{n}e_{ji}k_i+\sum e_{jl}'p_l, \quad (9.2)$$

with $e_{ji}$ appropriately chosen as 0, $\pm1$ for the internal circuits, and $e_{jl}'$ similarly chosen for the external momenta. The discriminant from Eq. (9.1) is

$$D(\alpha,s,t)=\begin{vmatrix} \alpha_1\times\sum_{j\geq2}(e_{j1})^2\alpha_j & \sum_{j\geq2}e_{j1}e_{j2}\alpha_j & \cdots & \sum_{j\geq2}e_{j1}e_{jl}'\alpha_jp_l \\ \sum_{j\geq2}e_{j1}e_{j2}\alpha_j & \sum_{j\geq2}(e_{j2})^2\alpha_j & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{j\geq2}e_{j1}e_{jl}'\alpha_jp_j & \cdots & \cdots & -\sum_{j\geq1}\alpha_jm_j^2 \end{vmatrix} \quad (9.3)$$

$$=sf(\alpha)+tg(\alpha)-K(\alpha,m). \quad (9.4)$$

Since $\alpha_1$ occurs only in the first row, first column, and in the last term of the last row, last column, $D$ can readily be differentiated with respect to $\alpha_1$. This gives

$$\partial D(\alpha,s,t)/\partial\alpha_1=D(\alpha,s,t;\alpha_1^{-1})-\alpha_1m_1^2K(\alpha;\alpha_1^{-1})-m_1^2K_2(\alpha). \quad (9.5)$$

The notation $\alpha_1^{-1}$ indicates that the line labeled $\alpha_1$ is to be removed before evaluating the expression concerned. The removal may leave another line previously internal as part of the external line; in such a case the line plays no part in the diagram and its label is redundant. The removal may alternatively leave two lines as part of the same internal line when they were previously distinct. In this case the parameters must be added and both retained. The line $\alpha_1$ in Eq. (9.1) can be any line in the diagram.

From Eqs. (9.4) and (9.5) we see that

$$\partial f(\alpha)/\partial\alpha_i=f(\alpha,\alpha_i^{-1}), \quad (9.6)$$

and

$$\partial g(\alpha)/\partial\alpha_i=f(\alpha,\alpha_i^{-1}). \quad (9.7)$$

Expressing the discriminant as a function of $t$ and $u$ (see Sec. 6), we obtain similarly

$$\partial h(\alpha)/\partial\alpha_i=h(\alpha,\alpha_i^{-1}). \quad (9.8)$$

Since $f(\alpha)$ is homogeneous in the $\alpha$, of degree $l$, Eq. (9.6) gives

$$f(\alpha)=\frac{1}{l}\sum_1^n \alpha_i\frac{\partial f(\alpha)}{\partial\alpha_i}=\frac{1}{l}\sum_1^n \alpha_i f(\alpha,\alpha_i^{-1}). \qquad (9.9)$$

Similar relations hold for $g(\alpha)$ and $h(\alpha)$.

The procedure of removing a given line from a diagram will be called "simplification" of the diagram. This is to distinguish it from "reduction" of the diagram in which one or more lines are reduced to points. Since we have shown that end-point singularities do not lead to anomalous curves in the equal-mass case, we are concerned now only with curves of singularities for diagrams that are fully reduced and for which all the parameters $\alpha$ are positive.

### (B) Singularities Not in the Physical Sheet

From Theorem 6B and Eqs. (9.6) and (9.7) we obtain

$$\frac{\partial D(\alpha,s,t)}{\partial\alpha_i}=sf(\alpha,\alpha_i^{-1})+tg(\alpha,\alpha_i^{-1})-\frac{\partial K(\alpha,m)}{\partial\alpha_i}, \qquad (9.10)$$

in which the last term is negative.

Since Eq. (9.10) must be zero on any curve of singularities it is necessary that

$$sf(\alpha,\alpha_i^{-1})+tg(\alpha,\alpha_i^{-1})>0, \qquad (9.11)$$

on the curve. Hence in order to show that there is no curve of singularities in the physical sheet in the region $s>0, t>0$, it is sufficient to show that, for $\alpha_j>0$, $j=1,\cdots,n$,

$$f(\alpha,\alpha_i^{-1})<0 \quad\text{and}\quad g(\alpha,\alpha_i^{-1})<0, \qquad (9.11\text{a})$$

for any one of the $n$ possible simplified diagrams. This is sufficient to show, for example, that a ladder diagram contributes to only one spectral region. It will show similarly that some diagrams have no singularities anywhere in the physical sheet [for example diagram (d) of Fig. 3].

### (C) Spurious Turning Points

For some diagrams, the result (9.9) is sufficient to prove that on a curve of singularities, with $\alpha_i>0$, we have

$$f(\alpha)>0, \quad\text{and}\quad g(\alpha)>0, \qquad (9.12)$$

for $s>0$ and $t>0$. We require that every term in $f(\alpha,\alpha_i^{-1})$ shall be positive for positive $\alpha$. This holds for all simple ladder diagrams, and provides an alternative method for obtaining the results of Sec. 8(B).

A more general situation has been illustrated by the symmetric crossed diagram in Sec. 8(D). If the simplification procedure of removing one particular line leaves the "dangerous factor" of $f(\alpha)$ in the term $f(\alpha,\alpha_i^{-1})$ and only the associated negative term in $g(\alpha,\alpha_i^{-1})$; we can deduce from

$$g(\alpha,\alpha_i^{-1})<0, \qquad (9.13)$$

that for $s>0$, $t>0$ we have

$$f(\alpha,\alpha_i^{-1})>0. \qquad (9.14)$$

This shows that the dangerous factor in $f(\alpha)$ is positive. With a little ingenuity this method appears to apply to all sixth- and eighth-order diagrams and could be used to prove the absence of spurious turning points in the "positive" spectral regions to this order. Spurious turning points will occur in the "negative" spectral regions. For example a turning point in $t(s)$ will occur for several of these diagrams in the negative spectral region $s>0$, $t<0$ but not in the positive spectral region $s>0$, $t>0$ (nor in $u>0$, $t>0$).

In the remainder of this section a "plausibility argument" will be given for the absence of spurious turning points in positive spectral regions for a general term in perturbation theory. The unbelieving reader may be able to devise a rigorous proof along the lines indicated; he is also advised that a counter example will invalidate the Mandelstam representation. The basis of the argument is that the dominant terms in $f(\alpha)$ and $g(\alpha)$ in the region $s>0$, $t>0$, are those associated with the asymptotes. Since these asymptotes are normal thresholds, the Feynman parameters in the dominant terms must correspond to the lines in the generalized self-energy parts that determine the asymptotes. The other Feynman parameters tend to zero near the asymptote. The Feynman parameters can be divided into four classes for any given diagram:

(a) Those that do not tend to zero at either asymptote. These will be denoted by $\alpha_i$, and they correspond to lines that are in the generalized self-energy parts for both asymptotes. An example is given by the pair $\alpha_5$, $\alpha_6$ of Sec. 8(D).

(b) Those that do not tend to zero near the asymptote $s=s_c$, but do tend to zero near $t=t_c$. We denote these as $\beta_i$. An example is the pair $\alpha_1$, $\alpha_3$ in Sec. 8(D).

(c) Those that tend to zero near $s=s_c$, but not near $t=t_c$. We denote these by $\gamma_i$. An example is $\alpha_2$, $\alpha_4$ in Sec. 8(D).

(d) Those that tend to zero near both asymptotes. We denote these as $\delta_i$. An example is $\alpha_4$, $\alpha_5$ in Fig. 7(b), example (E) of Sec. 8.

We seek to show that any pair of parameters giving a negative term in $f$ or in $g$ can be associated with a pair having the same coefficient but having a positive sign and dominating the negative term. The coefficient of any set of four lines 1, 2, 3, 4, can be determined by repeated application of the simplification formula (9.10). If the removal from the diagram of a set of lines $a_1$, $a_2,\cdots,a_{l-1}$ leaves only the lines 1, 2, 3, 4 arranged as in Fig. 8(a), then Eq. (9.10) shows that $sf+tg$ will contain a term

$$\alpha_1{}^a\alpha_2{}^a\cdots\alpha_{l-1}{}^a\{\alpha_1\alpha_3 s+\alpha_2\alpha_4 t\}. \qquad (9.15)$$

(Note that the $\alpha_i$ do not yet have the meaning indicated

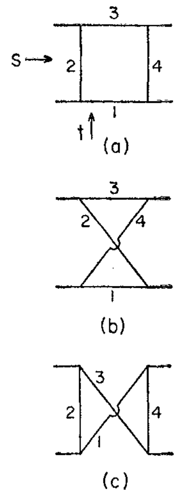


FIG. 8. Residual diagrams obtained by repeated differentiation of the general discriminant.

in (a) above.) By summing all such terms we obtain

$$sf+tg\supset a_{24}{}^{13}\{\alpha_1\alpha_3 s+\alpha_2\alpha_4 t\}. \tag{9.16}$$

The symbol $\supset$ means "contains the term," and $a_{34}{}^{12}$ is the sum of products of parameters that have the property of the set in Eq. (9.15), namely that removal of their lines leaves Fig. 8(a). Similar results are obtained for those sets of lines whose removal leads to Fig. 8(b) or to Fig. 8(c). Let $b_{34}{}^{12}$ and $c_{34}{}^{12}$ denote the appropriate sums of products of parameters. Then we have

$$sf+tg\supset b_{24}{}^{13}\{-\alpha_1\alpha_3 s+(\alpha_2\alpha_4-\alpha_1\alpha_3)t\}, \tag{9.17}$$

and

$$sf+tg\supset c_{24}{}^{13}\{(\alpha_1\alpha_3-\alpha_2\alpha_4)s-\alpha_2\alpha_4 t\}. \tag{9.18}$$

These expressions are best obtained by first evaluating the contributions to $D(\alpha,u,t)$ and $D(\alpha,s,u)$ of diagrams (b) and (c), respectively, in Fig. 8 and then transforming to $D(\alpha,s,t)$ as described in Sec. 6.

It should be noted that there is a lot of freedom of choice in pairing negative terms with positive ones in $f$ and $g$. This is because most terms are positive [as can be seen from part (B) of this section] if there is a curve of singularities in the physical sheet for $s>0$, $t>0$. We now use the notation $\alpha$, $\beta$, $\gamma$, $\delta$ as described above in the paragraph preceding Eq. (9.15). An $\alpha$ line must be dominantly associated with positive coefficients near each asymptote, and will occur in combinations similar to lines 2, and 4, in Eq. (9.17), or 1 and 3 in Eq. (9.18) since only these lines in Fig. 8 can contribute to both asymptotes. A $\beta$ line may occur in combination of lines 1 and 3 of Eq. (9.16) or 1 and 3 in Eq. (9.17). The former gives a positive term, so it need not be considered in that arrangement of terms in $D$. The latter gives a negative contribution to $f$ and also to $g$, and this contribution does not tend to zero near $s=s_c$. In $g$ we have the term from Eq. (9.17),

$$b_{24}{}^{13}(\alpha_2\alpha_4-\beta_1\beta_3), \tag{9.19}$$

in which $b_{24}{}^{13}$ is positive on the physical sheet. At the asymptote $t=t_c$, the parameters $\beta_1$ and $\beta_3$ are zero; at the other asymptote they tend to equality and to equality with $\alpha_2$ and $\alpha_4$. Thus near one asymptote the expression (9.19) is certainly positive, while it tends to zero near the other asymptote. It is plausible that this change takes place monotonically since it requires a change from perpendicular to parallel 4-momenta in the lines 2, 4 and 1, 3 in diagram (b) for Fig. 8. The term (9.19) would then be positive. For many (but not all) diagrams this can be proved directly by differentiating $D(\alpha,s,t)$ as in examples 8(d) and 8(e).

In addition to Eq. (9.19) which gives part of the coefficient of $t$, we have a term $-b_{24}{}^{13}\beta_1\beta_3 s$. However the diagram must contain a positive term to balance this, otherwise $\partial D/\partial\beta_1$ would be negative. Therefore the discriminant $D$ can be rearranged so that $\beta_1\beta_3$ is paired with a different two lines from $\alpha_2\alpha_4$. Since $\beta_1$ and $\beta_3$ contribute to the asymptote $s=s_c$, their product must have a positive coefficient near this line. Therefore we can find a term such that

$$(a_{56}{}^{13}-b_{24}{}^{13})\beta_1\beta_3 \tag{9.20}$$

is positive near the asymptote. Clearly the difference between the lines in $a_{56}{}^{13}$ and $b_{56}{}^{13}$ causes a "twisting" of the lines 1 and 3 so that their directions are reversed when the "$b$-lines" are removed, but not when the "$a$-lines" are removed. This can be achieved only if the $a$-lines and $b$-lines contain a crossed pair. By fixing our attention on this crossed pair instead of on $\beta_1\beta_3$, we can obtain a term similar to Eq. (9.19) but now in $f$ and having the lines $\beta_1$ and $\beta_3$ as coefficients. Again this term changes from a positive value to zero along the curve, and it seems likely to be positive throughout the range. Similar arrangements of products appear always to be possible for any negative terms in $f$ and $g$ coming from $\gamma$ or $\delta$ lines.

It should be noted that the difficulty in making the above argument rigorous is associated with the occurrence of many positive terms in $f$ and $g$ rather than with the negative terms. If there is a minimal number of positive terms as in example 8(d), the method of considering the first derivatives of $D$ suffices to prove that the negative terms can be adequately paired against positive terms. When there are many positive terms, it is very plausible that they will more easily dominate the dangerous negative terms, but it is not possible to prove dominance in a single pairing. Instead one has to use the multiple pairing described in the above paragraphs.

### 10. FURTHER REQUIREMENTS OF THE MANDELSTAM REPRESENTATION

The Mandelstam representation[1] (denoted M.R.) contains Born terms, single dispersion integrals, and double dispersion integrals. We will be concerned only with the latter and will indicate some of the assump-

tions which are implied by this form of integral representation. We will then consider to what extent these assumptions have been justified in perturbation theory by the preceding sections, and will indicate some further points that require study before the representation can be proved.

The double dispersion integrals are

$$A(s,t,u)=\frac{1}{\pi^2}\int\int\frac{A_{12}(s',t')}{(s-s')(t-t')}ds'dt' +\frac{1}{\pi^2}\int\int\frac{A_{23}(t',u')}{(t-t')(u-u')}dt'du' +\frac{1}{\pi^2}\int\int\frac{A_{31}(u',s')}{(u-u')(s-s')}du'ds', \quad (10.1)$$

where

$$s+t+u=4m^2, \quad (10.2)$$

in the equal-mass case. The integrals in Eq. (10.1) are over a part of the real $s$, $t$ plane. Small positive imaginary parts of $s$, $t$, $u$ specify how the surface of integration passes the singularities of the integrands.

From Eq. (10.1), single dispersion relations can be derived. For example, we have[1]

$$A(s,t,u)=\frac{1}{\pi^2}\int_{4m^2}^{\infty}\frac{A_1(s,t')dt'}{t-t'}+\frac{1}{\pi^2}\int_{4m^2}^{\infty}\frac{A_2(s,u')du'}{u-u'}. \quad (10.3)$$

The assumption of a real domain of integration has the following consequences:

(1) $A(s,t,u)$ is analytic when $s$, $t$, $u$ are real except (a) at points of discontinuity of $A_1(s,t')$ as a function of $s$, or $A_2(s,u')$, and similar points with respect to $t$ and to $u$; or (b) on curves of discontinuity of $A_{12}(s,t)$ as functions of $s$, $t$ or $A_{23}(t,u)$ or $A_{31}(t,s)$.

(2) $A(s,t,u)$ is analytic when $s$ is real and $t$ complex with $0<\arg t<2\pi$, and similarly for other pairs of variables.

From these it follows that $A(s,t,u)$ is analytic when one of $s$, $t$, or $u$ is real and the others are limited by cuts in their complex planes from $4m^2$ along the real axis to infinity. This region is called the physical sheet.

We have shown for the equal-mass case that the only singularities of $A$ whose location depends on one variable are given by normal thresholds. For $s$, $t$, $u$ real, the singularities lie on curves having normal thresholds as asymptotes. For $s$ real, none of the singular curves for which $t$ is real are connected to singularities for which $t$ is complex except possibly at spurious turning points.

Let us consider a spurious turning point as a function of $t(s)$ but with $t<0$. In order to approach this spurious turning point by analytic continuation we need to pass the asymptotes $s=s_c$ and $u=u_c$ of the curve, which themselves determine singular points and which lie on the cuts in the $s$, $u$ planes. If we keep $t$ real, $s$ can become complex with a positive imaginary part at the spurious turning point. But, from Eq. (10.2), this will cause $u$ to pass through the cut in its plane on to a nonphysical sheet since it must acquire a negative imaginary part. Similarly if $u$ gets a positive imaginary part, $s$ will go off the physical sheet. This aspect of spurious turning points certainly requires more detailed consideration, but this suggests that they will probably not give any serious trouble to the validity of the M.R.

If, however, a spurious turning point was to occur in the region $s>0$, $t>0$ for a curve $t=t(s)$; when $s$ becomes complex above its branch cut, $t$ could remain real while $A(s,t,u)$ was singular. Now $u$ would acquire a negative imaginary part, but would not go off the physical sheet since its real part would be below the onset of the branch cut. This would cause complex singularities in the physical sheet and would invalidate the M.R. For this reason it is necessary to extend the discussion of spurious turning points in positive spectral regions so as to exclude them rigorously to all orders in perturbation theory.

The only other major point that has been consciously omitted from this paper concerns complex singularities that are disconnected from the real part of the $s$, $t$, $u$ plane. Singularities are determined by the vanishing of $D(\alpha,s,t)$, where

$$D(\alpha,s,t)=sf(\alpha)+tg(\alpha)-K(\alpha,m). \quad (10.4)$$

If $t$ and $\alpha$ are real and $s$ complex, then $D$ is not zero unless $f(\alpha)$ is zero. But $f(\alpha)$ can be zero only on normal thresholds $t=t_c$, where $s$ can be complex without invalidating the M.R., or at a spurious turning point. This suggests that spurious turning points must be eliminated also for complex $s$. A second way for $D(\alpha,s,t)$ to become zero is that when $s$ is complex some of the $\alpha$'s are also complex. This requires that a $C$ singularity has dragged the contour into the complex plane of one (or more) of the $\alpha$ variables. It seems unlikely that this will happen without there being a connection with a corresponding singularity with $s$ real. However, although neither of these possibilities seem very likely, they both require further investigation before the M.R. can be proved. It is hoped that some of these points will be discussed in later papers.


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