# Application of Dispersion Relations to Meson-Nucleon Scattering*†

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Relativistic nonforward scattering dispersion relations are used to obtain information about low-energy meson-nucleon scattering. It is determined which of the $s$-, $p$-, and $d$-wave phase shifts are predicted by dispersion theory. Charge independence is assumed. The form of the dispersion relations used is justified by relating the asymptotic energy dependence of the dispersion relation amplitudes to the assumption of a finite range of interaction and to the choice of a particular meson current and the usual equal time commutation relations. The relevence of the analytic properties of the scattering amplitude as a function of momentum transfer is discussed in connection with the partial wave expansion of the dispersion amplitudes. The contribution to the dispersion integrals from energies above the 33 resonance is estimated.

## I. INTRODUCTION

DERIVATIONS of the low-energy $\pi$ meson-nucleon phase shifts from dispersion relations have been carried out by several authors. Oehme[1] has used dispersion relations for the derivatives of the forward scattering amplitude to obtain the low-energy $p$-wave phase shifts. Chew *et al.*[2] have used nonforward scattering dispersion relations to obtain the low-energy $s$-, $p$-, and $d$-wave phase shifts. These papers, however, do not justify the particular form of the dispersion relations used, particularly with respect to the high-energy behavior; do not justify the assumption that the 33 resonance exhausts the dispersion integrals; and do not justify the various partial wave expansions which are made.

In this paper we will extract the maximum information contained in the nonforward scattering dispersion relations about low-energy meson-nucleon scattering. That is, we will determine which of the $s$-, $p$-, and $d$-wave phase shifts are predicted by dispersion theory. We will attempt to justify the particular form of the dispersion relations used and the various Legendre expansions which we shall perform. The contribution to the dispersion integrals from amplitudes other than the 33-resonant amplitude will be estimated.

We assume charge independence to be valid. It is important to recognize that this may be a serious limitation when our results are compared with experiment. For example, an estimate of the charge dependent effects on the $p$-wave phase shifts by Greenberger[3] shows that these effects may easily be large enough to change the sign of some of the small nonresonant $p$-wave phase shifts.

Our program in more detail is as follows. In Sec. II we investigate the effect of the assumption of a finite range of interaction on the asymptotic energy dependence of the dispersion relation amplitudes and hence on the particular form of the dispersion relations. We also study the further restrictions implied by assuming equal time commutation rules for the meson field and a particular form for the meson current.

Section III contains a discussion of the validity of the Legendre expansions necessary to determine the partial wave amplitudes. This discussion is based on the analytic properties of the scattering amplitude as a function of momentum transfer.

The actual derivation of equations for the ten low-energy partial wave amplitudes is carried out in Sec. IV.

In Sec. V we outline the numerical evaluation of the various contributions to the dispersion integrals. In particular we present a method for estimating the integrals over energies above the 33-resonance energy.

Our results are presented graphically and discussed in Sec. VI.

## II. FORM OF THE DISPERSION RELATIONS

We consider the scattering of a meson of four momentum $q_0$ by a nucleon of momentum $p_0$ into a meson $q_f$ and nucleon $p_f$. It is easily seen that there are only two independent scalar invariants which may be constructed from these four-vectors. We choose (with $M$ equal to the nucleon mass)

$$\nu=-(p_0+p_f)\cdot(q_0+q_f)/4M,\quad \Delta^2=(q_0-q_f)^2/4. \tag{2.1}$$

$\Delta^2$ is one quarter of the invariant momentum transfer squared and

$$\Delta^2=\tfrac{1}{2}q^2(1-\cos\theta), \tag{2.2}$$

where $q$ is the magnitude of the three-momentum in the center-of-mass system and $\theta$ is the scattering angle. Also

$$\nu=\nu_L-\Delta^2/M, \tag{2.3}$$

where $\nu_L$ is the energy of the incident meson in the lab system.

The only further independent invariant which can be formed by introducing the Dirac matrices is

$$i\gamma\cdot Q \quad\text{with}\quad Q=\tfrac{1}{2}(q_0+q_f). \tag{2.4}$$

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[1] R. Oehme, Phys. Rev. **100**, 1503 (1955); **102**, 1174 (1956).

[2] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[3] D. Greenberger, Phys. Rev. **117**, 1378 (1960).

The $S$ matrix can be written

$$S=\delta_{f0}-i(2\pi)^4\delta(p_f+q_f-p_0-q_0)\times\left(\frac{M^2}{4\omega_0\omega_f E_0 E_f}\right)^{\frac{1}{2}}\bar{u}_f T u_0, \quad (2.5)$$

where $E_0$ and $E_f$ are the initial and final nucleon energies and $\omega_0$ and $\omega_f$ are the initial and final meson energies. The spinor normalization is

$$\bar{u}_f u_f=\bar{u}_0 u_0=1. \quad (2.6)$$

We may now write

$$T=-A(\nu,\Delta^2)+i\gamma\cdot QB(\nu,\Delta^2). \quad (2.7)$$

$A$ and $B$ are invariant functions of $\nu$ and $\Delta^2$ as well as matrices in charge space. We let $\beta$ represent the charge state of the final meson ($\beta=1, 2, 3$) and $\alpha$ that of the initial meson. Then, charge independence allows us to write

$$A_{\beta\alpha}=\delta_{\beta\alpha}A^{(+)}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]A^{(-)}, \quad (2.8)$$

and

$$B_{\beta\alpha}=\delta_{\beta\alpha}B^{(+)}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]B^{(-)}. \quad (2.9)$$

Expressing these in terms of eigenstates of the total isotopic spin or the amplitudes for $\pi^\pm$ mesons on protons we find

$$A^{(+)}=\tfrac{1}{3}[A^{(\frac{1}{2})}+2A^{(\frac{3}{2})}]=\tfrac{1}{2}[A(\pi^- - p)+A(\pi^+ - p)], \quad (2.10)$$

$$A^{(-)}=\tfrac{1}{3}[A^{(\frac{1}{2})}-A^{(\frac{3}{2})}]=\tfrac{1}{2}[A(\pi^- - p)-A(\pi^+ - p)]. \quad (2.11)$$

In the following we use the system of units in which $\hbar=c=\mu=1$ ($\mu$ is the meson mass).

Subject to the convergence of the integrals the following dispersion relations have been shown to exist[4,5] provided

$$\Delta^2\leq\frac{8}{3}\frac{2M+\mu}{2M-\mu}\cdot\mu^2\approx 3\mu^2,$$

$$\mathrm{Re}A^{(\pm)}(\nu,\Delta^2)=\frac{P}{\pi}\int_{1-\Delta^2/M}^{\infty}d\nu'\,\mathrm{Im}A^{(\pm)}(\nu',\Delta^2)\left[\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right]+\sum_{n=0}a_n^{(\pm)}(\Delta^2)\nu^{(2n+\frac{1}{2}\mp\frac{1}{2})}, \quad (2.12)$$

$$\mathrm{Re}B^{(\pm)}(\nu,\Delta^2)=-4\pi\cdot 2Mf^2\times\left[\frac{1}{\nu+(1/2M)+(\Delta^2/M)}\pm\frac{1}{\nu-(1/2M)-(\Delta^2/M)}\right]+\frac{P}{\pi}\int_{1-\Delta^2/M}^{\infty}d\nu'\,\mathrm{Im}B^{(\pm)}(\nu',\Delta^2)\left[\frac{1}{\nu'-\nu}\mp\frac{1}{\nu'+\nu}\right]+\sum_{n=0}b_n^{(\pm)}(\Delta^2)\nu^{(2n+\frac{1}{2}\pm\frac{1}{2})}, \quad (2.13)$$

[4] H. J. Bremermann, R. Oehme, and J. G. Taylor, Phys. Rev. **109**, 2178 (1958).

[5] H. Lehmann, Nuovo cimento **10**, 579 (1958).

where $f^2=g_r^2/4\pi(2M)^2$ and $g_r^2$ is the rationalized renormalized pseudoscalar coupling constant and $a_n^{(\pm)}(\Delta^2)$, $b_n^{(\pm)}(\Delta^2)$ are arbitrary undetermined functions of the momentum transfer.

If the above dispersion integrals do not converge then we must use the so-called subtracted dispersion reltions. For example one subtraction gives

$$\mathrm{Re}A^{(+)}(\nu,\Delta^2)=\mathrm{Re}A^{(+)}(\nu_0,\Delta^2)+\frac{P}{\pi}\int_{1-\Delta^2/M}^{\infty}d\nu'\,\mathrm{Im}A^{(+)}(\nu',\Delta^2)\times\left[\frac{1}{\nu'-\nu}-\frac{1}{\nu'-\nu_0}+\frac{1}{\nu'+\nu}-\frac{1}{\nu'+\nu_0}\right]+\sum_{n=1}a_n^{(+)}(\Delta^2)(\nu^{2n}-\nu_0^{2n}), \quad (2.14)$$

$$\mathrm{Re}A^{(-)}(\nu,\Delta^2)=\frac{\nu}{\nu_0}\mathrm{Re}A^{(-)}(\nu_0,\Delta^2)+\frac{P}{\pi}\int_{1-\Delta^2/M}^{\infty}d\nu'\,\mathrm{Im}A^{(-)}(\nu',\Delta^2)\times\left[\frac{1}{\nu'-\nu}-\frac{1}{\nu'+\nu}-\frac{\nu}{\nu_0}\left(\frac{1}{\nu'-\nu_0}-\frac{1}{\nu'+\nu_0}\right)\right]+\sum_{n=1}a_n^{(-)}(\Delta^2)\left(\nu^{2n+1}-\frac{\nu}{\nu_0}\nu_0^{2n+1}\right), \quad (2.15)$$

and analogous equations for $B^{(\pm)}$. The remainder of this section will be devoted to determining the number of subtractions necessary and the restrictions on the $a_n^{(\pm)}(\Delta^2)$ and $b_n^{(\pm)}(\Delta^2)$.

It will be necessary to know the relation between the $A$'s and $B$'s and the conventional scattering amplitudes in states of definite parity and angular momentum. We introduce the center-of-mass variables: $W$ = total energy, $E$ = nucleon energy, $z=\cos\theta$, and $\omega=W-M$. Then it can be shown that

$$\frac{A^{(\pm)}}{4\pi}=\frac{W+M}{E+M}f_1^{(\pm)}-\frac{W-M}{E-M}f_2^{(\pm)}, \quad (2.16)$$

$$\frac{B^{(\pm)}}{4\pi}=\frac{1}{E+M}f_1^{(\pm)}+\frac{1}{E-M}f_2^{(\pm)}, \quad (2.17)$$

where $f_1$ and $f_2$ are related to the scattering cross section in the center of mass by

$$\frac{d\sigma}{d\Omega}=\sum|f|^2;\quad f=f_1+\frac{\boldsymbol{\sigma}\cdot\mathbf{q}_f\boldsymbol{\sigma}\cdot\mathbf{q}_0}{q_f q_0}f_2. \quad (2.18)$$

Here $\sum$ represents a sum over final and an average over initial spin states. In addition

$$f_1(\theta)=\sum_{l=0}^{\infty}f_{l+}P_{l+1}'(z)-\sum_{l=2}^{\infty}f_{l-}P_{l-1}'(z), \quad (2.19)$$

$$f_2(\theta)=\sum_{l=1}^{\infty}(f_{l-}-f_{l+})P_l'(z), \quad (2.20)$$

where $f_{l\pm}$ are the scattering amplitudes in states of total angular momentum $j=l\pm\frac{1}{2}$ and parity $-(-1)^l$. $P_l'(z)$ is the derivative of the usual Legendre polynomial with respect to $z$.

The $f_{l\pm}$ are normalized such that

$$(j+\tfrac{1}{2})\,\mathrm{Im}f_{l\pm}=(q/4\pi)\sigma_{l\pm}, \quad (2.21)$$

where $\sigma_{l\pm}$ is the total cross section in the state $j=l\pm\frac{1}{2}$ and parity $-(-1)^l$. Below the two-meson threshold

$$f_{l\pm}=\frac{e^{i\delta_{l\pm}}\sin\delta_{l\pm}}{q}, \quad (2.22)$$

where $\delta_{l\pm}$ are the usual phase shifts.

In the later calculations we will find it more convenient to consider

$$h_{l\pm}=f_{l\pm}/q^{2l}. \quad (2.23)$$

We now discuss the asymptotic energy dependence of the dispersion relation amplitudes as implied by the assumption of a finite range of interaction. Quantitatively, we assume that for fixed momentum $q$ there is a maximum $l$ value, $l_{\max}=qR$ in each of the partial wave sums with $R$ independent of $q$. $R$ is effectively the range of interaction.

It can be shown that $f_{l\pm}$ can be written in the form

$$f_{l\pm}(q)=(i/2q)[1-e^{-\alpha_{l\pm}(q)}e^{2i\delta_{l\pm}(q)}], \quad (2.24)$$

with $\alpha_l$ and $\delta_l$ real and $\alpha_l\geq 0$. Introducing the change of variables

$$\rho=(l+\tfrac{1}{2})/q, \quad (2.25)$$

we have

$$\begin{aligned}\mathrm{Re}f_{l\pm}(q)&=(1/2q)g_{\pm}(\rho,q)\\ &=(1/2q)e^{-\alpha_{\pm}(\rho,q)}\sin 2\delta_{\pm}(\rho,q), \quad (2.26)\end{aligned}$$

$$\begin{aligned}\mathrm{Im}f_{l\pm}(q)&=(1/2q)k_{\pm}(\rho,q)\\ &=(1/2q)[1-e^{-\alpha_{\pm}(\rho,q)}\cos 2\delta_{\pm}(\rho,q)]. \quad (2.27)\end{aligned}$$

We will make use of the following important properties of $g_{\pm}$ and $k_{\pm}$

$$-1\leq g_{\pm}(\rho,q)\leq 1, \quad (2.28)$$

$$0\leq k_{\pm}(\rho,q)\leq 1, \quad (2.29)$$

and

$$k_{\pm}(\rho,q)\geq\tfrac{1}{4}g_{\pm}^2(\rho,q), \quad (2.30)$$

which hold for all $\rho$ and $q$.

In order to calculate the asymptotic form of $A$ and $B$ we note that

$$P_l(z)=J_0(2\rho\Delta)+O(1/q^2), \quad (2.31)$$

$$P_l'(z)=-(q^2\rho/2\Delta)J_1(2\rho\Delta)+O(1). \quad (2.32)$$

Substituting these and Eqs. (2.26) and (2.27) into (2.19) and (2.20) allows us to evaluate $f_1$ and $f_2(\theta)$. We simplify the asymptotic $q$ dependence by replacing the discrete sums over $l$ by integrals over $\rho$. Then, using Eqs. (2.16), (2.17), (2.28), and (2.29) we obtain the following bounds on $A^{(\pm)}$ and $B^{(\pm)}$

$$A^{(\pm)}(\nu_L,\Delta^2)=O(q^2), \quad (2.33)$$

$$B^{(\pm)}(\nu_L,\Delta^2)=O(1), \quad (2.34)$$

$$f^{(\pm)}(0)=O(q). \quad (2.35)$$

We also note that as $\nu_L$ approaches infinity $\nu\simeq\nu_L\simeq 2q^2/M$.

In more detail we have

$$\begin{aligned}\lim_{q\to\infty}\mathrm{Re}f^{(+)}(0)/q
&=\frac{1}{6}\bigg(\int_0^{R+(\frac{1}{2})}\rho d\rho g_{+}{}^{\frac{1}{2}}J_0+\int_0^{R-(\frac{1}{2})}\rho d\rho g_{-}{}^{\frac{1}{2}}J_0\\
&\quad+2\int_0^{R+(\frac{3}{2})}\rho d\rho g_{+}{}^{\frac{3}{2}}J_0+2\int_0^{R-(\frac{3}{2})}\rho d\rho g_{-}{}^{\frac{3}{2}}J_0\bigg), \quad (2.36)\end{aligned}$$

and

$$\begin{aligned}\lim_{q\to\infty}\mathrm{Re}f^{(-)}(0)/q
&=\frac{1}{6}\bigg(\int_0^{R+(\frac{1}{2})}\rho d\rho g_{+}{}^{\frac{1}{2}}J_0+\int_0^{R-(\frac{1}{2})}\rho d\rho g_{-}{}^{\frac{1}{2}}J_0\\
&\quad-\int_0^{R+(\frac{3}{2})}\rho d\rho g_{+}{}^{\frac{3}{2}}J_0-\int_0^{R-(\frac{3}{2})}\rho d\rho g_{-}{}^{\frac{3}{2}}J_0\bigg). \quad (2.37)\end{aligned}$$

The corresponding imaginary parts are obtained by replacing $g$ by $2k$. We also note that the forward scattering amplitude is related to the dispersion amplitudes by

$$f^{(\pm)}(0)=\frac{M}{W}\frac{1}{4\pi}[A^{(\pm)}(\nu_L,0)+\nu_L B^{(\pm)}(\nu_L,0)]. \quad (2.38)$$

The bounds in (2.33) and (2.34) restrict the form of the dispersion relations. For $A^{(+)}$ the condition (2.33) means that all the $a_n^{(+)}(\Delta^2)$ $n\geq 1$ in Eq. (2.12) must vanish. However, this condition is not strong enough to insure the convergence of the unsubtracted dispersion integral for $A^{(+)}$ but instead one with one subtraction.

For $B^{(+)}$ the condition (2.34) implies the existence of the following dispersion relation with one subtraction and all $b_n^{(+)}(\Delta^2)=0$ for $n\geq 1$

$$\begin{aligned}\mathrm{Re}B^{(+)}(\nu,\Delta^2)&=\frac{\nu}{\nu_0}\mathrm{Re}B^{(+)}(\nu_0,\Delta^2)\\
&\quad+2\nu(\nu^2-\nu_0^2)\frac{P}{\pi}\int_0^\infty d\nu'\frac{\mathrm{Im}B^{(+)}(\nu',\Delta^2)}{(\nu'^2-\nu^2)(\nu'^2-\nu_0^2)}. \quad (2.39)\end{aligned}$$

From this it follows that

$$\begin{aligned}\lim_{\nu\to\infty}\frac{\mathrm{Re}B^{(+)}(\nu,\Delta^2)}{\nu}&=\frac{\mathrm{Re}B^{(+)}(\nu_0,\Delta^2)}{\nu_0}\\
&\quad-2\frac{P}{\pi}\int_0^\infty d\nu'\frac{\mathrm{Im}B^{(+)}(\nu',\Delta^2)}{\nu'^2-\nu_0^2}. \quad (2.40)\end{aligned}$$

But this must be zero by Eq. (2.34) so that we have proven the existence of the unsubtracted dispersion relation in Eq. (2.13) with all $b_n^{(+)}(\Delta^2)=0$ for $n\geq 0$.

The condition (2.33) for $A^{(-)}$ implies the existence of the following dispersion relation with one subtraction and all $a_n^{(-)}(\Delta^2)=0$ for $n\geq 1$

$$\mathrm{Re}A^{(-)}(\nu,\Delta^2)=\frac{\nu}{\nu_0}\mathrm{Re}A^{(-)}(\nu_0,\Delta^2)+2\nu(\nu^2-\nu_0^2)\frac{P}{\pi}\int_{1-\Delta^2/M}^{\infty}d\nu'\frac{\mathrm{Im}A^{(-)}(\nu',\Delta^2)}{(\nu'^2-\nu^2)(\nu'^2-\nu_0^2)}. \quad (2.41)$$

We have found that for all simple asymptotic energy dependences of $\mathrm{Im}A^{(-)}(\nu',\Delta^2)$ allowed by Eq. (2.33) the above equation is consistent with $\mathrm{Re}A^{(-)}(\nu,\Delta^2)/\nu$ being bounded by $\nu$ approaches infinity if and only if $\mathrm{Im}A^{(-)}(\nu',\Delta^2)$ has an asymptotic $\nu$ dependence such that $P\int_0^\infty d\nu'\,\mathrm{Im}A^{(-)}(\nu',\Delta^2)/(\nu'^2-\nu^2)$ exists. [For example, if $\lim_{\nu\to\infty}\mathrm{Im}A^{(-)}(\nu',\Delta^2)=H_A^{(-)}\nu'$, then Eq. (2.41) leads to $\lim_{\nu\to\infty}\mathrm{Re}A^{(-)}(\nu,\Delta^2)=H_A^{(-)}\nu\ln\nu$ which violates condition (2.33) unless $H_A^{(-)}=0$.]

These conditions thus imply the existence of the unsubtracted dispersion relation for $A^{(-)}$ in Eq. (2.12) with all $a_n^{(-)}(\Delta^2)=0$ except $a_0^{(-)}(\Delta^2)$. In addition $\lim_{\nu\to\infty}[\mathrm{Re}A^{(-)}(\nu,\Delta^2)/\nu]=a_0^{(-)}(\Delta^2)$ if $a_0^{(-)}(\Delta^2)\neq 0$.

The condition (2.34) for $B^{(-)}$ implies the existence of the following dispersion relation with one subtraction and all $b_n^{(-)}(\Delta^2)=0$ for $n\geq 1$

$$\mathrm{Re}B^{(-)}(\nu,\Delta^2)=\mathrm{Re}B^{(-)}(\nu_0,\Delta^2)+2(\nu^2-\nu_0^2)\frac{P}{\pi}\int_0^{\infty}d\nu'\frac{\nu'\,\mathrm{Im}B^{(-)}(\nu',\Delta^2)}{(\nu'^2-\nu^2)(\nu'^2-\nu_0^2)}. \quad (2.42)$$

Just as for $A^{(-)}$ the condition that $\mathrm{Re}B^{(-)}(\nu,\Delta^2)$ be bounded is consistent with the above equation if and only if $P\int_0^\infty d\nu'\nu'\,\mathrm{Im}B^{(-)}(\nu',\Delta^2)/(\nu'^2-\nu^2)$ exists. This then implies the existence of the unsubtracted dispersion relation for $B^{(-)}$ in Eq. (2.13) with all $b_n^{(-)}(\Delta^2)=0$ except $b_0^{(-)}(\Delta^2)$. In addition $\lim_{\nu\to\infty}\mathrm{Re}B^{(-)}(\nu,\Delta^2)=b_0^{(-)}(\Delta^2)$ if $b_0^{(-)}(\Delta^2)\neq 0$.

We note that one consequence of the above results for $A^{(-)}$ and $B^{(-)}$ is that using Eqs. (2.21) and (2.38)

$$\lim_{\nu\to\infty}\frac{\mathrm{Im}f^{(-)}(0)}{q}=\lim_{q\to\infty}\frac{1}{4\pi}[\sigma_T(\pi^--p)-\sigma_T(\pi^+-p)]$$

$$=\lim_{q\to\infty}\frac{M}{Wq}\frac{\mathrm{Im}A^{(-)}(\nu_L,0)+\nu_L\,\mathrm{Im}B^{(-)}(\nu_L,0)}{4\pi}=0, \quad (2.43)$$

so that $\lim_{q\to\infty}\sigma_T(\pi^--p)=\lim_{q\to\infty}\sigma_T(\pi^+-p)$. This has been proven earlier by Pomeranchuk.[6]

We obtain further information in the case when $(a_0^{(-)}+b_0^{(-)})\neq 0$. Using Eq. (2.38) and our previous limits on $\mathrm{Re}A^{(-)}(\nu,\Delta^2)$ and $\mathrm{Re}B^{(-)}(\nu,\Delta^2)$ we have

$$\lim_{q\to\infty}\frac{\mathrm{Re}f^{(-)}(0)}{q}=\lim_{q\to\infty}\frac{M\nu_L}{Wq}[a_0^{(-)}(0)+b_0^{(-)}(0)]$$

$$=[a_0^{(-)}(0)+b_0^{(-)}(0)]\neq 0. \quad (2.44)$$

The information contained in Eq. (2.30) combined with Schwarz's inequality implies that

$$\int_0^{R_\pm(T)}\rho d\rho\, k_\pm{}^T(\rho,q)\geq\frac{1}{4}\int_0^{R_\pm(T)}\rho d\rho[g_\pm{}^T(\rho,q)]^2$$

$$\geq\frac{1}{2R_\pm(T)}\left[\int_0^{R_\pm(T)}\rho d\rho\, g_\pm{}^T(\rho,q)\right]. \quad (2.45)$$

We now make use of Eq. (2.37) and the corresponding equation for $\mathrm{Im}f^{(-)}(0)$. Since we have assumed that $\lim_{q\to\infty}\mathrm{Re}f^{(-)}(0)/q=(a_0^{(-)}+b_0^{(-)})\neq 0$ one of the terms in Eq. (2.37) must approach a constant different from zero. The above inequality then implies that the corresponding term in $\lim_{q\to\infty}\mathrm{Im}f^{(-)}(0)/q$ must approach a constant different from zero. However, we have shown earlier that $\lim_{q\to\infty}\mathrm{Im}f^{(-)}(0)/q=0$. Because $k_\pm{}^T(\rho,q)\geq 0$ and because of the structure of Eq. (2.37) this can only be possible if

$$\lim_{q\to\infty}\sigma_T{}^{T=\frac{3}{2}}=\lim_{q\to\infty}\sigma_T{}^{T=\frac{1}{2}}=\text{constant}\neq 0.$$

That is, both cross sections must approach the same constant value.

On the other hand if $[a_0^{(-)}(0)+b_0^{(-)}(0)]=0$ then we can only prove the weaker condition

$$\lim_{q\to\infty}[\sigma_T{}^{T=\frac{3}{2}}-\sigma_T{}^{T=\frac{1}{2}}]=0.$$

In summary, the assumption of a finite range of interaction tells us that for $A^{(+)}$ we have the dispersion relation (2.14) with all $a_n^{(+)}(\Delta^2)=0$. The dispersion relations for $A^{(-)}$, $B^{(+)}$, and $B^{(-)}$ are given by Eqs. (2.12) and (2.13) with all $a_n$ and $b_n=0$ except $a_0^{(-)}(\Delta^2)$ and $b_0^{(-)}(\Delta^2)$.

We can find further information about $a_0^{(-)}(\Delta^2)$ and $b_0^{(-)}(\Delta^2)$ from entirely different considerations. One can easily show using the methods of Lehmann *et al.*[7] that $\mathrm{Re}\bar{u}_fT_{\beta\alpha}u_0$

$$=\frac{i}{2}\frac{E_\Delta}{M}\int d^4x\exp[i\omega x_0-i(\omega^2-\mu^2-\Delta^2)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$$

$$\times\epsilon(x_0)\left\langle P_\Delta\left|\left[j_\beta\left(\frac{x}{2}\right),j_\alpha\left(-\frac{x}{2}\right)\right]\right|P_{-\Delta}\right\rangle$$

$$+i\frac{E_\Delta}{M}\int d^4x\delta(x_0)\exp[i\omega x_0-i(\omega^2-\mu^2-\Delta^2)\mathbf{e}\cdot\mathbf{x}]$$

$$\times\left\langle P_\Delta\left|\left[\dot\varphi_\beta\left(\frac{x}{2}\right)+i\omega\varphi_\beta\left(\frac{x}{2}\right),j_\alpha\left(-\frac{x}{2}\right)\right]\right|P_{-\Delta}\right\rangle, \quad (2.46)$$

[6] I. Pomeranchuk, Zhur. Eksp. i Teoret. Fiz. 34, 725 (1958) [translation: Soviet Phys.-JETP 34(7), 499 (1958)].

[7] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento 1, 205 (1955).

where we use the usual dispersion theory coordinate system.[4] We now make the assumption that the fields satisfy the usual equal time commutation relations and that

$$j_\alpha = ig\bar{\psi}\gamma_5\tau_\alpha\psi + \lambda\varphi_\alpha\varphi_\beta^2 - \delta\mu^2\varphi_\alpha. \tag{2.47}$$

We then obtain

$$\begin{aligned}\mathrm{Re}\bar{u}_f T_{\beta\alpha} u_0 &= -\frac{i}{2}\frac{E_\Delta}{M}\int d^4x \exp[i\omega x_0 - i(\omega^2-\mu^2-\Delta^2)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}] \\ &\times \epsilon(x_0)\left\langle P_\Delta\left|\left[j_\beta\left(\frac{x}{2}\right), j_\alpha\left(-\frac{x}{2}\right)\right]\right|P_{-\Delta}\right\rangle \\ &+ \frac{E_\Delta}{M}\lambda\langle P_\Delta|[\delta_{\beta\alpha}(\varphi_\gamma^2(0)-\delta\mu^2/\lambda) \\ &\qquad + 2\varphi_\beta(0)\varphi_\alpha(0)]|P_{-\Delta}\rangle. \end{aligned}\tag{2.48}$$

Now the last term in Eq. (2.48) is independent of $\omega$ (as it will be for any $j_\alpha$ not containing derivatives). Therefore, because of the structure of Eq. (2.7) it cannot contribute to the spin-flip amplitudes $B^{(\pm)}(\nu,\Delta^2)$. Furthermore since $A^{(-)}(\nu,\Delta^2)$ must be an odd function of $\omega$ it cannot contribute to this amplitude. Thus the last term in Eq. (2.48) contributes only to the $A^{(+)}(\nu,\Delta^2)$ amplitude.

These results lead to the conclusion that $a_0^{(-)}(\Delta^2)$ and $b_0^{(-)}(\Delta^2)$ are related to the limit as $\omega$ approaches infinity of the first term in Eq. (2.48). To calculate this limit requires detailed knowledge of the singularities of the matrix element. Such knowledge does not exist at present. As an illustration we note that if the matrix element had the same singularities as $\Delta(x)$ then the first term would asymptotically approach a function of $\Delta^2$ only, which would contribute only to $\mathrm{Re}A^{(+)}(\nu,\Delta^2)$. Thus we are led to no conclusive results about $a_0^{(-)}(\Delta^2)$ and $b_0^{(-)}(\Delta^2)$. In the following, we will set them equal to zero.

On the basis of the previous discussion we will use the following dispersion relations. We change variables from $\nu$ to $\nu_L$ so that the limits of integration are independent of $\Delta^2$.

$$\begin{aligned}\mathrm{Re}A^{(+)}(\nu_L,\Delta^2) &= \mathrm{Re}A^{(+)}(\nu_{L0},\Delta^2) + \frac{\mathrm{P}}{\pi}\int_1^\infty d\nu_L'\,\mathrm{Im}A^{(+)}(\nu_L',\Delta^2) \\ &\qquad\times\left[\frac{1}{\nu_L'-\nu_L} - \frac{1}{\nu_L'-\nu_{L0}} + \frac{1}{\nu_L'+\nu_L-2\Delta^2/M} - \frac{1}{\nu_L'+\nu_{L0}-2\Delta^2/M}\right], \\ \mathrm{Re}A^{(-)}(\nu_L,\Delta^2) &= \frac{\mathrm{P}}{\pi}\int_1^\infty d\nu_L'\,\mathrm{Im}A^{(-)}(\nu_L',\Delta^2)\left[\frac{1}{\nu_L'-\nu_L} - \frac{1}{\nu_L'+\nu_L-2\Delta^2/M}\right], \\ \mathrm{Re}B^{(\pm)}(\nu_L,\Delta^2) &= -4\pi\cdot 2Mf^2\left[\frac{1}{\nu_L+1/2M} \pm \frac{1}{\nu_L-1/2M-2\Delta^2/M}\right] \\ &\qquad + \frac{\mathrm{P}}{\pi}\int_1^\infty d\nu_L'\,\mathrm{Im}B^{(\pm)}(\nu_L',\Delta^2)\left[\frac{1}{\nu_L'-\nu_L} \mp \frac{1}{\nu_L'+\nu_L-2\Delta^2/M}\right].\end{aligned}\tag{2.49}$$

The equations for $A^{(+)}$ and $B^{(+)}$ follow from the assumption of a finite range of interaction. The equations for $A^{(-)}$ and $B^{(-)}$ follow from the same assumption and the additional unproven requirement that $a_0^{(-)}(\Delta^2)$ and $b_0^{(-)}(\Delta^2)$ vanish.

### III. PARTIAL WAVE ANALYSIS AND ANALYTICITY IN MOMENTUM TRANSFER

Our goal is to determine the individual partial wave amplitudes in terms of the dispersion amplitudes $A$ and $B$. We first note that if we consider $f_1(z)$ and $f_2(z)$ at the particular values $z=1, 0, -1$ we obtain from Eqs. (2.19) and (2.20)

$$\begin{aligned} f_1(z=0) &= h_s - q^4(h_{d\frac{3}{2}} + \tfrac{3}{2}h_{d\frac{5}{2}}) + (g\text{ waves}) + \cdots, \\ f_2(z=0) &= q^2(h_{p\frac{1}{2}} - h_{p\frac{3}{2}}) + (f\text{ waves}) + \cdots, \\ [f_1(1)-f_1(-1)] &= 6q^2 h_{p\frac{3}{2}} + (f\text{ waves}) + \cdots, \\ [f_2(1)-f_2(-1)] &= 6q^4(h_{d\frac{3}{2}} - h_{d\frac{5}{2}}) + (g\text{ waves}) + \cdots, \\ [f_1(1)+f_1(-1)] &= 2h_s - q^4(2h_{d\frac{3}{2}} - 12h_{d\frac{5}{2}}) \\ &\qquad + (g\text{ waves}) + \cdots, \\ [f_2(1)+f_2(-1)] &= 2q^2(h_{p\frac{1}{2}} - h_{p\frac{3}{2}}) + (f\text{ waves}) + \cdots, \\ [f_1(1)-2f_1(0)+f_1(-1)] &= 15q^4 h_{d\frac{5}{2}} + (g\text{ waves}) + \cdots. \end{aligned}\tag{3.1}$$

For low energies we expect these to be reasonably rapidly convergent series. In addition $z$ has been chosen so that alternate $l$ values drop out making the series more convergent. We now assume that the convergence is rapid enough so that all partial amplitudes with $l \geq 3$ may be neglected. Then Eqs. (3.1) determine the desired $s$-, $p$-, and $d$-wave amplitudes in terms of $f_1$ and $f_2$.

To relate the partial amplitudes to $A^{(\pm)}$ and $B^{(\pm)}$, we invert Eqs. (2.16) and (2.17) in the following forms

$$\left[f_2^{(\pm)} + \frac{E-M}{E+M}f_1^{(\pm)}\right] = (E-M)\frac{B^{(\pm)}}{4\pi}, \tag{3.2}$$

$$f_1^{(\pm)} = \frac{E+M}{2W}\frac{A^{(\pm)}+\omega B^{(\pm)}}{4\pi}. \tag{3.3}$$

These have been chosen so that $A^{(+)}$ with its unknown

function of $\Delta^2$ in it appears in only one of the four equations. This enables us to extract the maximum information contained in the unsubtracted dispersion relations before using the $A^{(+)}$ relation.

We now make our only kinematical approximation

$$\frac{E-M}{E+M} \simeq \frac{q^2}{(2M)^2},$$

where we have neglected corrections of order $q^2/(2M)^2$ compared to one. The largest $q^2$ we will consider is $q^2 \approx 4$ so that $q^2/(2M)^2 \lesssim 2\%$. (We will not expand in powers of $\nu/M$ or $\omega/M$.) The $f_1$ term in Eq. (3.2) is thus a small correction to $f_2$ except in those cases in which it contains a resonant 33 amplitude.

Introducing the notation

$$\begin{aligned} h_p{}^{(\pm)(-)} &= (h_{p\frac{1}{2}}{}^{(\pm)} - h_{p\frac{3}{2}}{}^{(\pm)}), \\ h_d{}^{(\pm)(-)} &= (h_{d\frac{3}{2}}{}^{(\pm)} - h_{d\frac{5}{2}}{}^{(\pm)}), \end{aligned} \tag{3.4}$$

we obtain the desired result

$$\begin{aligned}
&\mathrm{Re}\left[h_p{}^{(\pm)(-)} + \frac{1}{(2M)^2} h_s{}^{(\pm)}\right] = \frac{1}{2M} \frac{\mathrm{Re}B^{(\pm)}(\nu_L, q^2/2)}{4\pi}, \\
&\mathrm{Re}\left[h_d{}^{(\pm)(-)} + \frac{1}{(2M)^2} h_{p\frac{3}{2}}{}^{(\pm)}\right] = \frac{1}{6q^2 \cdot 2M}\left[\frac{\mathrm{Re}B^{(\pm)}(\nu_L,0)}{4\pi} - \frac{\mathrm{Re}B^{(\pm)}(\nu_L,q^2)}{4\pi}\right], \\
&\mathrm{Re}h_{p\frac{3}{2}}{}^{(\pm)} = \frac{1}{6q^2}\frac{M}{W}\left[\frac{\mathrm{Re}A^{(\pm)}(\nu_L,0)+\omega\,\mathrm{Re}B^{(\pm)}(\nu_L,0)}{4\pi} - \frac{\mathrm{Re}A^{(\pm)}(\nu_L,q^2)+\omega\,\mathrm{Re}B^{(\pm)}(\nu_L,q^2)}{4\pi}\right], \\
&\mathrm{Re}h_{d\frac{5}{2}}{}^{(\pm)} = \frac{1}{15q^4}\frac{M}{W}\left[\frac{\mathrm{Re}A^{(\pm)}(\nu_L,0)+\omega\,\mathrm{Re}B^{(\pm)}(\nu_L,0)}{4\pi}\right. \\
&\qquad\qquad \left. -2\frac{\mathrm{Re}A^{(\pm)}(\nu_L,q^2/2)+\omega\,\mathrm{Re}B^{(\pm)}(\nu_L,q^2/2)}{4\pi} + \frac{\mathrm{Re}A^{(\pm)}(\nu_L,q^2)+\omega\,\mathrm{Re}B^{(\pm)}(\nu_L,q^2)}{4\pi}\right], \\
&\mathrm{Re}\left[h_s{}^{(\pm)} - q^4\left(h_{d\frac{3}{2}}{}^{(\pm)} + \tfrac{3}{2} h_{d\frac{5}{2}}{}^{(\pm)}\right)\right] = \frac{M}{W}\left[\frac{\mathrm{Re}A^{(\pm)}(\nu_L,q^2/2)+\omega\,\mathrm{Re}B^{(\pm)}(\nu_L,q^2/2)}{4\pi}\right].
\end{aligned} \tag{3.5}$$

In (3.5) $A$ and $B$ are considered as functions of $\nu_L$ and $\Delta^2$. We have also used [from Eq. (2.2)]

$$\begin{aligned} z &= \cos\theta = 1, & \Delta^2 &= 0, \\ z &= 0, & \Delta^2 &= q^2/2, \\ z &= -1, & \Delta^2 &= q^2. \end{aligned} \tag{3.6}$$

To complete the solution of our problem we must substitute the dispersion relations (2.49) into Eqs. (3.5) and perform the integrations. To carry out the integrations we will expand $\mathrm{Im}A^{(\pm)}(\nu_L',\Delta^2)$ and $\mathrm{Im}B^{(\pm)} \times(\nu_L',\Delta^2)$ in a Legendre series in $\cos\theta'$ at low energies and in a powers series in $\Delta^2$ at high energies. We will also find it necessary to expand the subtraction term $\mathrm{Re}A^{(+)}(1,\Delta^2)$ in a Legendre series.

Before proceeding, however, we must pause and consider more carefully the validity of the various expansions outlined above. The validity of these expansions in $\cos\theta$ and $\Delta^2$ will be determined by the analytic properties of the various amplitudes as a function of $\Delta^2$ or $\cos\theta$ for fixed energy $\nu_L$. That there might be a difficulty is seen by noting that we would like to expand $\mathrm{Im}A^{(+)}(\nu_L',\Delta^2)$ in a Legendre series in $\cos\theta' = 1-2\Delta^2/q'^2$ for fixed $\Delta^2$ different from zero. Now the dispersion integrals extend over all $q'$ down to $q'=0$ and hence introduce angles for which $\cos\theta' < -1$. It is by no means obvious that a Legendre expansion will exist and converge outside the physical region $-1 \leq \cos\theta \leq 1$.

We first note that all our results are subject to the restriction on the validity of the dispersion relations. They have been shown to exist as a consequence of the general principles of field theory only for $\Delta^2 \leq \Delta_{\max}{}^2 \simeq 3.1\mu^2$.[5] In carrying out our projection of partial waves, we have set $\Delta^2 = 0$, $q^2/2$, and $q^2$. Therefore our results are strictly valid only for $q^2 \lesssim 3\mu^2$ or $KE_\pi \lesssim 220$ Mev. We would like to have $q^2$ as large as $q^2 = 4\mu^2$. There is at present no evidence to suggest that the limit on $\Delta^2$ may not be extended to include $\Delta^2 \simeq 4\mu^2$. In fact, Mandelstam[8] has postulated a representation in which the dispersion relations are valid for all momentum transfer. At present there are no counter examples to contradict the Mandelstam conjecture.

We now turn to the analytic properties of the dispersion amplitudes considered as a function of $\cos\theta$ or $\Delta^2$ for fixed $q^2$. All present derivations of analyticity regions do not make use of the spin and charge properties of the scattering amplitude and thus give the same regions of analyticity for $A^{(\pm)}(\nu_L,\Delta^2)$ and $B^{(\pm)} \times(\nu_L,\Delta^2)$. In the following discussion we will denote all of these amplitudes by the single symbol $T(\nu_L,\Delta^2)$. We now discuss successively the four types of expansions to be made.

(1) Legendre expansion of $\mathrm{Im}T(\nu_L',\Delta^2)$ in $\cos\theta' = 1 - 2\Delta^2/q'^2$ for fixed $\Delta^2 = 0$, $q^2/2$, and $q^2$. Lehmann[5] has shown from general principles that $\mathrm{Im}T(\nu_L',\Delta^2)$ is analytic within an ellipse in the complex $\cos\theta'$ plane.

[8] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

This region is such that the Legendre expansion is valid for all $q'$ provided $\Delta^2 \leq \Delta_{\max}^2 \simeq 3.1\mu^2$. This is the same limitation as that on the validity of the dispersion relations. Mandelstam's representation leads to a larger region of analyticity which easily contains our maximum $\Delta^2 \simeq 4$. The speed with which the series converges will be determined by the closeness of the singularities. However, in practice this is not important since $\mathrm{Im}T$ is dominated by the 33-resonance term in the Legendre expansion.

(2) Power series expansion of $\mathrm{Im}T(\nu_L',\Delta^2)$ in $\Delta^2$ for large values of $\nu_L'$. This expansion will be used to simplify the energy dependence of our estimate of the dispersion integrals over high energies. Neither Lehmann's nor Mandelstam's region of analyticity justifies this expansion. However, we shall see later that it is necessary to introduce a model in order to estimate the high-energy integrals. This model will consist of resonances in given angular momentum states superimposed upon a background scattering described by a spin-independent purely absorbing optical model. The model is consistent with all present data and has the property that a power series expansion of the scattering amplitude in $\Delta^2$ has an infinite radius of convergence for all $q'$.

(3) Legendre expansion of $\mathrm{Re}T(\nu_L,\Delta^2)$ in $\cos\theta$ for $\Delta^2=0$, $q^2/2$, and $q^2$. These values of $\Delta^2$ are all in the physical region $-1 \leq z \leq 1$. Lehmann[5] has shown that $\mathrm{Re}T(\nu_L,\Delta^2)$ is an analytic function of $\cos\theta$ in a region which contains the physical region. Thus, the partial wave expansions we have used in projecting out the partial wave amplitudes are valid. There still remains the question of the speed of convergence of the expansion. The result of the general theory is not too encouraging since the proven region of analyticity does not extend much beyond the physical region. Mandelstam's nearest singularities are due to the bound state pole at $\Delta^2 \simeq M\nu_L/2$ or $\cos\theta = 1 - M\nu_L/q^2$ and a branch line at $\Delta^2 = -\mu^2$ or $\cos\theta = 1 + 2\mu^2/q^2$. These lead to a reasonable convergence of the Legendre expansion up to $q^2 \simeq 4$.

The larger analyticity region of Mandelstam can be justified to some extent by looking at the dispersion relations (2.49). These imply the following singularities for $\mathrm{Re}T(\nu_L,\Delta^2)$. (a) Bound-state pole at $\Delta^2 = M(\nu_L - \mu^2/2M)/2$. (b) Branch line from $\Delta^2 = M(\nu_L+1)/2$ to infinity due to the dispersion denominator. (c) Singularities whenever $\mathrm{Im}T(\nu_L',\Delta^2)$ has singularities. (d) Singularities due to the arbitrary functions of $\Delta^2$, $a_n(\Delta^2)$ and $b_n(\Delta^2)$.

If we neglect the singularities due to (d) and use Lehmann's results for the singularities due to $\mathrm{Im}T(\nu_L',\Delta^2)$, then for $q^2 \lesssim 4$ the closest singularities are due to the bound-state pole. If these were the only singularities, the Legendre expansion would converge quite rapidly. However, there still remain the singularities due to the $a_n$ and $b_n$. These are in general unknown. If as we have suggested only $a_0^{(+)}(\Delta^2)$ is different from zero, then only $\mathrm{Re}A^{(+)}(\nu_L,\Delta^2)$ will have these singularities. We have removed $a_0^{(+)}(\Delta^2)$ by considering only a subtracted dispersion relation for $A^{(+)}$ and thus none of these arbitrary singularities appear except in the subtraction term $\mathrm{Re}A^{(+)}(\nu_{L0},\Delta^2)$.

(4) Legendre expansion of $\mathrm{Re}A^{(+)}(\nu_{L0},\Delta^2)$ in the limit $\nu_{L0}$ approaches 1. This reduces to a power series in $\Delta^2$ in this limit. Lehmann's general result gives a zero radius of convergence. Mandelstam gives the nearest singularity at $\Delta^2=-\mu^2$. Thus even in this case the expansion fails for $\Delta^2=q^2\geq 1$. The existence of the singularity at $\Delta^2=-1$ thus makes any result based on the $A^{(+)}$ dispersion relation of doubtful reliability.

## IV. EQUATIONS FOR PARTIAL WAVE AMPLITUDES

We now derive expressions for the low-energy $s$-, $p$-, and $d$-wave amplitudes by inserting the dispersion relations (2.49) into Eqs. (3.5). The essential feature of meson-nucleon scattering which allows an evaluation of the dispersion integrals is the dominance of the 33 resonance in the low-energy region. This suggests that we split the integrals into two parts at $\nu_L=\nu_m=3.88$. This energy is chosen so that the contribution of the integrals below $\nu_m$ is dominated by the 33 resonance. The contribution of the integrals above $\nu_m$ cannot be reliably calculated with existing data but we will be able to estimate the order of magnitude of these contributions. Our philosophy is that only those amplitudes which depend primarily on the 33-resonance integrals and not on the integrals over high energies are determined by dispersion theory.

To evaluate the low-energy dispersion integrals, we must expand $\mathrm{Im}A^{(\pm)}(\nu_L',\Delta^2)$ and $\mathrm{Im}B^{(\pm)}(\nu_L',\Delta^2)$ in a Legendre series in $\cos\theta'=1-2\Delta^2/q'^2$. We then retain only the 33 term and for the $s$-wave equation we keep $\mathrm{Im}h_s^{(\pm)}$ also since the $s$ waves are known to be anomalously small. Thus, combining Eqs. (2.16), (2.17), (2.19), and (2.20) we obtain

$$\begin{aligned}
\frac{\mathrm{Im}A^{(+)}(\nu_L',\Delta^2)}{4\pi} &= \left[\frac{4M\omega'}{3}+2\left(1+\frac{\omega'}{2M}\right)(q'^2-2\Delta^2)\right]\mathrm{Im}h_{33}(\nu_L')+\left(1+\frac{\omega'}{2M}\right)\mathrm{Im}h_s^{(+)}(\nu_L'),\\
\frac{\mathrm{Im}A^{(-)}(\nu_L',\Delta^2)}{4\pi} &= \left[-\frac{2M\omega'}{3}-\left(1+\frac{\omega'}{2M}\right)(q'^2-2\Delta^2)\right]\mathrm{Im}h_{33}(\nu_L')+\left(1+\frac{\omega'}{2M}\right)\mathrm{Im}h_s^{(-)}(\nu_L'),\\
\frac{\mathrm{Im}B^{(+)}(\nu_L',\Delta^2)}{4\pi} &= \left[-\frac{4M}{3}+\frac{q'^2-2\Delta^2}{M}\right]\mathrm{Im}h_{33}(\nu_L')+\frac{1}{2M}\mathrm{Im}h_s^{(+)}(\nu_L'),\\
\frac{\mathrm{Im}B^{(-)}(\nu_L',\Delta^2)}{4\pi} &= \left[\frac{2M}{3}-\frac{q'^2-2\Delta^2}{2M}\right]\mathrm{Im}h_{33}(\nu_L')+\frac{1}{2M}\mathrm{Im}h_s^{(-)}(\nu_L').
\end{aligned} \tag{4.1}$$

Here we have dropped terms of order $q'^2/(2M)^2$ and $\Delta^2/(2M)^2$ compared to unity since these are both always less than 4%. We have also introduced the notation $h_{2T,2J}$ where $T$ is the total isotopic spin and $J$ is the total angular momentum.

We also give the Legendre expansion of the subtraction constant and similar expansions for the other amplitudes.

$$\begin{aligned}\frac{\mathrm{Re}A^{(\pm)}(1,\Delta^2)}{4\pi}&=\left(1+\frac{1}{2M}\right)[\mathrm{Re}h_s^{(\pm)}(1)-6\Delta^2\,\mathrm{Re}h_{p\frac{3}{2}}^{(\pm)}(1)+30\Delta^4\,\mathrm{Re}h_{d\frac{5}{2}}^{(\pm)}(1)+\cdots]\\&\qquad-2M[\mathrm{Re}h_p^{(\pm)(-)}(1)-6\Delta^2\,\mathrm{Re}h_d^{(\pm)(-)}(1)+\cdots],\\\frac{\mathrm{Re}B^{(\pm)}(1,\Delta^2)}{}&=2M[\mathrm{Re}h_p^{(\pm)(-)}(1)-6\Delta^2\,\mathrm{Re}h_d^{(\pm)(-)}(1)+\cdots]\\&\qquad+\frac{1}{2M}[\mathrm{Re}h_s^{(\pm)}(1)-6\Delta^2\,\mathrm{Re}h_{p\frac{3}{2}}^{(\pm)}(1)+30\Delta^4\,\mathrm{Re}h_{d\frac{5}{2}}^{(\pm)}(1)+\cdots].\end{aligned}\tag{4.2}$$

We postpone to the next section the estimation of the high-energy integrals. However, in order to simplify the energy dependence of these contributions we expand the dispersion integrands in powers of $\nu_L/\nu_L'$ and $\Delta^2$. These expansions may be rather poor for $\nu_L\sim 3$ but still give a reasonable estimate of the high-energy contributions.

Instead of evaluating the partial wave amplitudes given in Eqs. (3.5) it is more convenient to consider linear combinations of these amplitudes. For the $p$ waves we consider

$$\begin{aligned}&[h_{11}-h_{13}]=[h_p^{(+)(-)}+2h_p^{(-)(-)}],\\&[(h_{33}-h_{31})+\tfrac{1}{4}(h_{11}-h_{13})]=\tfrac{3}{4}[2h_p^{(-)(-)}-h_p^{(+)(-)}],\\&[h_{13}-h_{31}]=[3h_{p\frac{3}{2}}^{(-)}+h_p^{(-)(-)}-h_p^{(+)(-)}],\\&h_{13}=[h_{p\frac{3}{2}}^{(+)}+2h_{p\frac{3}{2}}^{(-)}],\end{aligned}\tag{4.3}$$

for the $d$ waves

$$\begin{aligned}&[h_{13}-h_{15}]=[h_d^{(+)(-)}+2h_d^{(-)(-)}],\\&[4(h_{35}-h_{33})+(h_{13}-h_{15})]=3[2h_d^{(-)(-)}-h_d^{(+)(-)}],\\&[h_{15}-h_{35}]=3h_{d\frac{5}{2}}^{(-)},\\&[h_{35}-\tfrac{1}{4}h_{15}]=\tfrac{3}{4}[h_{d\frac{5}{2}}^{(+)}-2h_{d\frac{5}{2}}^{(-)}],\end{aligned}\tag{4.4}$$

and for the $s$ waves we still calculate $h_s^{(\pm)}$.

Before stating our results for the low-energy partial wave amplitudes, we outline once again the steps taken in their derivation.

(1) The dispersion relations (2.49) are substituted into Eqs. (3.5) for the partial wave amplitudes.

(2) The dispersion integrals are split into two parts: a low-energy region below $\nu_L=\nu_m$ and a high-energy region above $\nu_m$.

(3) In the low-energy region only the 33 contribution to $\mathrm{Im}A^{(\pm)}$ and $\mathrm{Im}B^{(\pm)}$ given by Eq. (4.1) is kept. The subtraction constant $\mathrm{Re}A^{(+)}(1,\Delta^2)$ is expanded in a Legendre series as in Eq. (4.2).

(4) In the high-energy region we expand the integrands in powers of $\nu_L/\nu_L'$ and $\Delta^2$.

(5) We form the more convenient linear combinations of partial wave amplitudes given by Eqs. (4.3) and (4.4).

The result of these operations is as follows. From the unsubtracted dispersion relations for $A^{(-)}$ and $B^{(\pm)}$ we obtain the $p$ waves

$$\begin{aligned}&\mathrm{Re}\{[h_{11}(\nu_L)-h_{13}(\nu_L)]+[1/(2M)^2][h_s^{(+)}+2h_s^{(-)}]\}\\&\quad=-3\frac{f^2}{\nu_L-1/2M}+\frac{f^2}{\nu_L-1/2M-q^2/M}+\frac{4}{3}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\frac{\mathrm{Im}h_{33}(\nu_L')}{\nu_L'+\nu_L-q^2/M}+\mathrm{Re}[h_p^{(+)(-)}+2h_p^{(-)(-)}]_{\mathrm{HEI}},\end{aligned}\tag{4.5}$$

$$\begin{aligned}&\mathrm{Re}\{[h_{33}(\nu_L)-h_{31}(\nu_L)]+\tfrac{1}{4}[h_{11}(\nu_L)-h_{13}(\nu_L)]+[3/4(2M)^2][2h_s^{(-)}-h_s^{(+)}]\}\\&\quad=-\frac{3}{4}\frac{f^2}{\nu_L+1/2M}+\frac{9}{4}\frac{f^2}{\nu_L-1/2M-q^2/M}+\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\frac{\mathrm{Im}h_{33}(\nu_L')}{\nu_L'-\nu_L}+\tfrac{3}{4}\,\mathrm{Re}[2h_p^{(-)(-)}-h_p^{(+)(-)}]_{\mathrm{HEI}},\end{aligned}\tag{4.6}$$

$$\begin{aligned}&(1+\omega/M)\,\mathrm{Re}\{[h_{13}(\nu_L)-h_{31}(\nu_L)]+[1/(2M)^2][h_s^{(-)}-h_s^{(+)}]\}\\&=-2f^2\frac{\omega}{(\nu_L+1/2M)(\nu_L-1/2M-2q^2/M)}+2f^2\frac{1+\omega/M}{\nu_L-1/2M-q^2/M}-\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_{33}(\nu_L')\\&\times\left\{\frac{(\omega'-\omega)/2M}{\nu_L'-\nu_L}+\frac{1+(\omega'-\omega)/2M}{\nu_L'+\nu_L-2q^2/M}+\frac{1}{3}\frac{1+\omega/M}{\nu_L'+\nu_L-q^2/M}+\frac{2}{3}\frac{(\omega'+\omega)+3(q'^2/2M)[1+(\omega'-\omega)/2M]}{(\nu_L'+\nu_L)(\nu_L'+\nu_L-2q^2/M)}\right\}\\&\qquad+\left(1+\frac{\omega}{M}\right)\mathrm{Re}[3h_{p\frac{3}{2}}^{(-)}+h_p^{(-)(-)}-h_p^{(+)(-)}]_{\mathrm{HEI}},\end{aligned}\tag{4.7}$$

the $d$ waves

$$\mathrm{Re}\{[h_{13}(\nu_L)-h_{15}(\nu_L)]+[1/(2M)^2][h_{13}]_p\}$$

$$=-\frac{f^2}{3M}\frac{1}{(\nu_L-1/2M)(\nu_L-1/2M-2q^2/M)}-\frac{4}{9M}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\frac{\mathrm{Im}h_{33}(\nu_L')}{(\nu_L'+\nu_L)(\nu_L'+\nu_L-2q^2/M)}$$
$$-\frac{4}{3(2M)^2}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\frac{\mathrm{Im}h_{33}(\nu_L')}{\nu_L'+\nu_L}+\mathrm{Re}[h_d^{(+)(-)}+2h_d^{(-)(-)}]_{\mathrm{HEI}},\quad(4.8)$$

$$\mathrm{Re}\{4[h_{35}(\nu_L)-h_{33}(\nu_L)]+[h_{13}(\nu_L)-h_{15}(\nu_L)]+[1/(2M)^2][h_{11}-4h_{31}]_p\}$$

$$=-3\frac{f^2}{M}\frac{1}{(\nu_L-1/2M)(\nu_L-1/2M-2q^2/M)}+3\,\mathrm{Re}[2h_d^{(-)(-)}-h_d^{(+)(-)}]_{\mathrm{HEI}}$$
$$+[3/(2M)^2]\,\mathrm{Re}[2h_p^{(-)(-)}-h_p^{(+)(-)}]_{\mathrm{HEI}},\quad(4.9)$$

$$(1+\omega/M)\,\mathrm{Re}[h_{15}(\nu_L)-h_{35}(\nu_L)]$$

$$=\frac{3}{5}\frac{f^2}{M}\frac{\omega}{(\nu_L-1/2M)(\nu_L-1/2M-q^2/M)(\nu_L-1/2M-2q^2/M)}-\frac{2}{5M}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_{33}(\nu_L')$$
$$\times\left\{\frac{1+(\omega'-\omega)/2M}{(\nu_L'+\nu_L-q^2/M)(\nu_L'+\nu_L-2q^2/M)}-\frac{2}{3}\frac{(\omega'+\omega)+3(q'^2/2M)[1+(\omega'-\omega)/2M]}{(\nu_L'+\nu_L)(\nu_L'+\nu_L-q^2/M)(\nu_L'+\nu_L-2q^2/M)}\right\}$$
$$+3[\mathrm{Re}h_{d\frac{5}{2}}^{(-)}(\nu_L)]_{\mathrm{HEI}},\quad(4.10)$$

and the $s$ wave

$$(1+\omega/M)\,\mathrm{Re}[h_s^{(-)}(\nu_L)-q^4(h_{d\frac{3}{2}}^{(-)}+\tfrac{3}{2}h_{d\frac{5}{2}}^{(-)})]$$

$$=-2Mf^2\left[\frac{1}{\nu_L+1/2M}-\frac{1}{\nu_L-1/2M-q^2/M}\right]-\frac{2M}{3}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_{33}(\nu_L')$$
$$\times\left[\frac{(\omega'-\omega)+3[(q'^2-q^2)/2M][1+(\omega'+\omega)/2M]}{\nu_L'-\nu_L}-\frac{(\omega'+\omega)+3[(q'^2-q^2)/2M][1+(\omega'-\omega)/2M]}{\nu_L'+\nu_L-q^2/M}\right]$$
$$+\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_s^{(-)}(\nu_L')\left[\frac{1+(\omega'+\omega)/2M}{\nu_L'-\nu_L}-\frac{1+(\omega'-\omega)/2M}{\nu_L'+\nu_L-q^2/M}\right]+(1+\omega/M)[\mathrm{Re}h_s^{(-)}(\nu_L)]_{\mathrm{HEI}}.\quad(4.11)$$

Now making use of the $A^{(+)}$ dispersion relation we obtain the $p$-wave equation

$$[(1+\omega/M)\,\mathrm{Re}h_{13}(\nu_L)-(1+1/M)\,\mathrm{Re}h_{13}(1)]$$

$$=-\tfrac{2}{3}f^2\left[\frac{\omega}{(\nu_L-1/2M)(\nu_L-1/2M-2q^2/M)}-\frac{1}{(1-1/2M)(1-1/2M)}\right]$$
$$+\frac{4}{3}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_{33}(\nu_L')\left\{\frac{1+(\omega'-\omega)/2M}{\nu_L'+\nu_L-2q^2/M}-\frac{1+(\omega'-1)/2M}{\nu_L'+1}\right.$$
$$\left.-\frac{2}{3}\frac{(\omega'+\omega)+3(q'^2/2M)[1+(\omega'-\omega)/2M]}{(\nu_L'+\nu_L)(\nu_L'+\nu_L-2q^2/M)}+\frac{2}{3}\frac{(\omega'+1)+3(q'^2/2M)[1+(\omega'-1)/2M]}{(\nu_L'+1)(\nu_L'+1)}\right\}$$
$$-5q^2\left\{\left(1+\frac{1}{M}\right)\mathrm{Re}h_{d\frac{5}{2}}^{(+)}(1)+\frac{4}{15}\frac{f^2}{M}\frac{1}{(1-1/2M)(1-1/2M)(1-1/2M-2q^2/M)}\right.$$
$$\left.+\frac{4}{15M}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_{33}(\nu_L')\left[\frac{1+(\omega'-1)/2M}{(\nu_L'+1)(\nu_L'+1-2q^2/M)}-\frac{2}{3}\frac{(\omega'+1)+3(q'^2/2M)[1+(\omega'-1)/2M]}{(\nu_L'+1)(\nu_L'+1)(\nu_L'+1-2q^2/M)}\right]\right\}$$
$$+\left\{\left(1+\frac{\omega}{M}\right)\mathrm{Re}[h_{p\frac{3}{2}}^{(+)}(\nu_L)+2h_{p\frac{3}{2}}^{(-)}(\nu_L)]_{\mathrm{HEI}}-2\left(1+\frac{1}{M}\right)\mathrm{Re}[h_{p\frac{3}{2}}^{(-)}(1)]_{\mathrm{HEI}}\right\},\quad(4.12)$$

the $d$-wave equation

$$\{(1+\omega/M)\,\mathrm{Re}[h_{35}(\nu_L)-\tfrac{1}{4}h_{15}(\nu_L)]-(1+1/M)\,\mathrm{Re}[h_{35}(1)-\tfrac{1}{4}h_{15}(1)]\}$$
$$=-\frac{3}{5}\frac{f^2}{M}\left[\frac{\omega}{(\nu_L-1/2M)(\nu_L-1/2M-q^2/M)(\nu_L-1/2M-2q^2/M)}\right.$$
$$\left.-\frac{1}{(1-1/2M)(1-1/2M-q^2/M)(1-1/2M-2q^2/M)}\right]$$
$$+\tfrac{3}{4}\{(1+\omega/M)\,\mathrm{Re}[hd_{\frac{5}{2}}^{(+)}(\nu_L)-2hd_{\frac{5}{2}}^{(-)}(\nu_L)]_{\mathrm{HEI}}+2(1+1/M)\,\mathrm{Re}[hd_{\frac{5}{2}}^{(-)}(1)]_{\mathrm{HEI}}\},\quad(4.13)$$

and the $s$-wave equation

$$\{(1+\omega/M)\,\mathrm{Re}[h_s^{(+)}(\nu_L)-q^4(hd_{\frac{3}{2}}^{(+)}+\tfrac{3}{2}hd_{\frac{5}{2}}^{(+)})]-(1+1/M)\,\mathrm{Re}h_s^{(+)}(1)\}$$
$$=-2Mf^2\left[\frac{\omega}{\nu_L+1/2M}+\frac{\omega}{\nu_L-1/2M-q^2/M}-\frac{1}{1+1/2M}-\frac{1}{1-1/2M}\right]+\frac{4M}{3}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_{33}(\nu_L')$$
$$\times\left[\frac{(\omega'-\omega)+3[(q'^2-q^2)/2M][1+(\omega'+\omega)/2M]}{\nu_L'-\nu_L}+\frac{(\omega'+\omega)+3[(q'^2-q^2)/2M][1+(\omega'-\omega)/2M]}{\nu_L'+\nu_L-q^2/M}\right.$$
$$\left.-\frac{(\omega'-1)+3(q'^2/2M)[1+(\omega'+1)/2M]}{\nu_L'-1}-\frac{(\omega'+1)+3(q'^2/2M)[1+(\omega'-1)/2M]}{\nu_L'+1}\right]$$
$$+\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_s^{(+)}(\nu_L')\left[\frac{1+(\omega'+\omega)/2M}{\nu_L'-\nu_L}+\frac{1+(\omega'-\omega)/2M}{\nu_L'+\nu_L-q^2/M}-\frac{1+(\omega'+1)/2M}{\nu_L'-1}-\frac{1+(\omega'-1)/2M}{\nu_L'+1-q^2/M}\right]$$
$$-3q^2\left\{\left(1+\frac{1}{M}\right)\mathrm{Re}hp_{\frac{3}{2}}^{(+)}(1)-\tfrac{2}{3}f^2\frac{1}{(1-1/2M)(1-1/2M)}-\frac{2}{3}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_{33}(\nu_L')\right.$$
$$\left.\times\left[\frac{1+(\omega'+1)/2M}{\nu_L'-1}+\frac{1+(\omega'-1)/2M}{\nu_L'+1}-\frac{2}{3}\frac{(\omega'+1)+3(q'^2/2M)[1+(\omega'-1)/2M]}{(\nu_L'+1)(\nu_L'+1)}\right]\right\}$$
$$+\frac{15}{2}q^4\left\{\left(1+\frac{1}{M}\right)\mathrm{Re}hd_{\frac{5}{2}}^{(+)}(1)+\frac{4}{15}\frac{f^2}{M}\frac{1}{(1-1/2M)(1-1/2M)(1-1/2M-q^2/M)}\right.$$
$$\left.+\frac{4}{15M}\frac{\mathrm{P}}{\pi}\int_1^{\nu_m}d\nu_L'\,\mathrm{Im}h_{33}(\nu_L')\left[\frac{1+(\omega'-1)/2M}{(\nu_L'+1)(\nu_L'+1-q^2/M)}-\frac{2}{3}\frac{(\omega'+1)+3(q'^2/2M)[1+(\omega'-1)/2M]}{(\nu_L'+1)(\nu_L'+1)(\nu_L'+1-q^2/M)}\right]\right\}$$
$$+\{(1+\omega/M)\,\mathrm{Re}[h_s^{(+)}(\nu_L)]_{\mathrm{HEI}}-(1+1/M)[\mathrm{Re}h_s^{(+)}(1)]_{\mathrm{HEI}}\}.\quad(4.14)$$

In the last three equations we have used Eqs. (4.2) to express $\mathrm{Re}A^{(+)}(1,\Delta^2)$ and $\mathrm{Re}B^{(+)}(1,\Delta^2)$ in terms of partial wave amplitudes at threshold. The subtraction constants have been so arranged [for example, see the fifth and sixth lines of Eq. (4.12)] so that the bracket multiplying the $q^2$ dependence would vanish if the unsubtracted dispersion relation were valid. If $\mathrm{Re}A^{(+)}(1,\Delta^2)$ is not expanded in a Legendre series, then when $\Delta^2$ is set equal to 0, $q^2/2$, and $q^2$, we will have terms with arbitrary $q$ dependence and hence will obtain almost no knowledge at all about the desired partial wave amplitudes which depend on the $A^{(+)}$ dispersion relation.

In Eqs. (4.5) through (4.14) $\mathrm{Re}[\cdots]_{\mathrm{HEI}}$ means the contribution to the specified amplitude from the dispersion integrals over energies greater than $\nu_L'=\nu_m$. The expressions for these terms can be obtained directly by substituting the dispersion integrals (2.49) into Eqs. (3.5), keeping only that part from energies greater than $\nu_m$ in the integrals, and expanding the integrands in powers of $\nu_L/\nu_L'$ and $\Delta^2$. We will not write out all the amplitudes but rather we will give two illustrative examples

$$[\mathrm{Re}h_p^{(+)(-)}]_{\mathrm{HEI}}\approx\frac{\omega}{M}\frac{1}{4\pi^2}\int_{\nu_m}^{\infty}d\nu_L'\left[\frac{\mathrm{Im}B^{(+)}(\nu_L',0)}{\nu_L'}+\frac{q^2}{2}\frac{\mathrm{Im}B'^{(+)}(\nu_L',0)}{\nu_L'^2}+\cdots\right],\quad(4.15)$$

and

$$\left(1+\frac{\omega}{M}\right)[\mathrm{Re}h_s^{(-)}(\nu_L)]_{\mathrm{HEI}} \simeq \frac{2\omega}{4\pi^2}\int_{\nu_m}^{\infty} d\nu_L' \left[\frac{\mathrm{Im}A^{(-)}(\nu_L',0)+\nu_L'\,\mathrm{Im}B^{(-)}(\nu_L',0)}{\nu_L'^2}+\frac{q^2}{2}\frac{\mathrm{Im}A'^{(-)}(\nu_L',0)+\nu_L'\,\mathrm{Im}B'^{(-)}(\nu_L',0)}{\nu_L'^2}+\cdots\right]. \quad (4.16)$$

The primes on the $A$'s and $B$'s mean derivatives with respect to $\Delta^2$. The other high-energy integral contributions are of similar form.

## V. NUMERICAL CALCULATIONS

### A. Low-Energy Contributions

We will outline in this section the numerical evaluation of the bound state terms and the values of the dispersion integrals in the region up to $\nu_L=\nu_m=3.88$. All partial wave amplitudes will be calculated for energy values in the range $\nu_L=1$ to $\nu_L=3$ which corresponds to a maximum lab kinetic energy of 280 Mev for the meson.

The bound-state terms are evaluated directly for each of the partial wave amplitudes listed in Eqs. (4.5) through (4.14).

The energy region below $\nu_L=\nu_m$ is dominated by the resonance in the $T=\frac{3}{2}$, $J=\frac{3}{2}$ state of even parity. The basis of our calculation is the assumption that the partial cross section in the 33 state essentially exhausts the scattering and is thus equal to the total cross section for the scattering of $\pi^+$ mesons by protons and equal to three times the total cross section for the scattering of $\pi^-$ mesons by protons. Using this assumption and the data of Lindenbaum and Yuan[9] and Cool *et al.*[10] we can calculate $\delta_{33}$ from Eqs. (2.21) and (2.22). Instead of using these values of $\delta_{33}$ directly we fit the data by

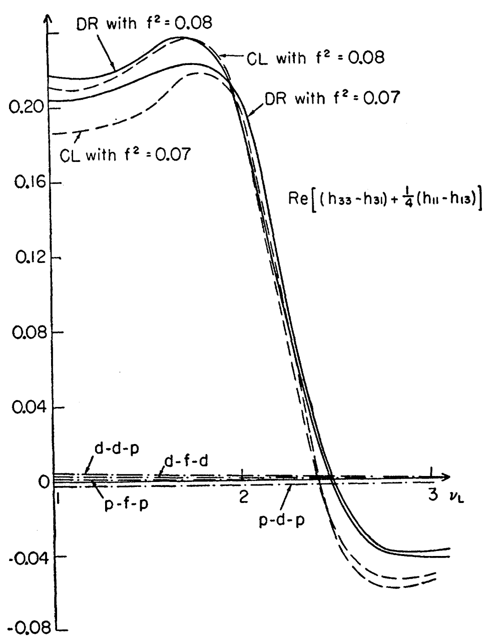


FIG. 1. Contributions to $\mathrm{Re}[(h_{33}-h_{31})+(h_{11}-h_{13})/4]$. The following notation is used. Solid line, DR: Contribution to the dispersion relations from the bound-state term plus the 33 integral. Dashed line, CL: Approximations given by Eqs. (6.1), (6.2), and (6.3). Computations of DR and CL are done with $f^2=0.08$ in all figures except Fig. 1 in which the results for $f^2=0.08$ and $f^2=0.07$ are both shown. Dotted line, SA: Contribution from the integral over the small $s$- and $p$-wave amplitudes. Alternate dashes and dots, HEI: Contribution of the various high-energy integrals. The symbols $p$-$d$-$p$, $p$-$f$-$p$, $d$-$d$-$p$, and $d$-$f$-$d$ are defined in Eq. (5.6).

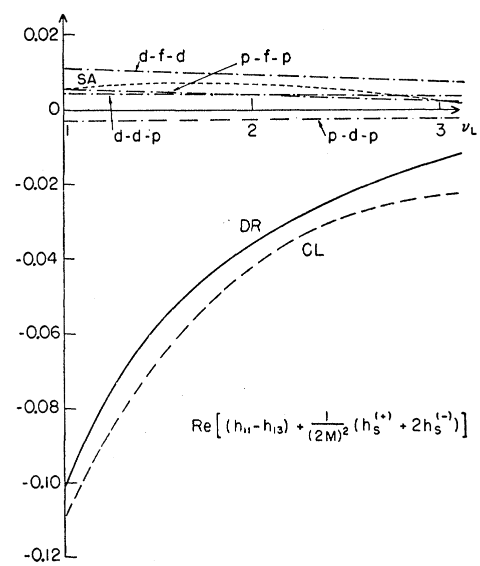


FIG. 2. Contributions to $\mathrm{Re}[(h_{11}-h_{13})+(h_s^{(+)}+2h_s^{(-)})/(2M)^2]$. The notation in this figure and Figs. 3 through 10 is the same as in Fig. 1.

the standard effective range formula given by Chew *et al.*[2]

$$\cot\delta_{33}=\frac{\omega(1-\omega/\omega_r)}{\frac{4}{3}f^2q^3}, \quad (5.1)$$

and use these values to compute $\mathrm{Im}h_{33}$. This smooths the data and allows a more reliable evaluation of the principal value integrals. In addition, through Eq. (4.6) it gives us a rough check on the form of the effective range formula and our choice of the parameters $f^2$ and $\omega_r$ in Eq. (5.1).

[9] S. J. Lindenbaum and L. C. L. Yuan, Phys. Rev. **100**, 30 (1955).

[10] R. Cool, O. Piccioni, and D. Clark, Phys. Rev. **103**, 1082 (1956).

As to the parameters we fix the resonance energy at

$$KE_\pi(\mathrm{res})=194.6 \text{ Mev},$$
$$\nu_L(\mathrm{res})=2.40, \qquad (5.2)$$
$$\omega_r=2.12.$$

Now Haber-Schaim[11] has determined the coupling constant to be $f^2=0.082$. To investigate the sensitivity to changes in the coupling constant we choose $f^2=0.08$ and $f^2=0.07$. Both choices of the coupling constant fit the cross-section data equally well within the limits of experimental error. We will use both choices to evaluate the contribution of the low-energy integrals to Eq. (4.6) for Re$h_{33}$. However, for all the other amplitudes we will only use the Im$h_{33}$ computed with $f^2=0.08$.

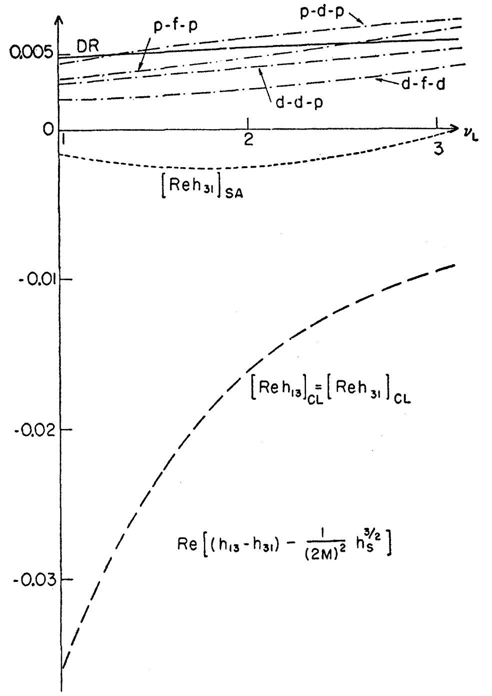


FIG. 3. Contributions to Re$[(h_{13}-h_{31})-h_s^{\frac{3}{2}}/(2M)^2]$.

With these values of Im$h_{33}$ the integrals are evaluated numerically. The result of the integral contributions to Eqs. (4.5) through (4.14) is added to the bound-state terms, and the sums are plotted for each amplitude in Figs. 1 through 10.

Thus far we have neglected the contribution to the low-energy integrals of all partial waves other than the 33 amplitude. To estimate the contribution of the other amplitudes we assume that the Legendre expansion for the imaginary part of the scattering amplitude converges rapidly enough so that only $s$ and $p$ waves contribute significant amounts. For the $s$ waves we use the

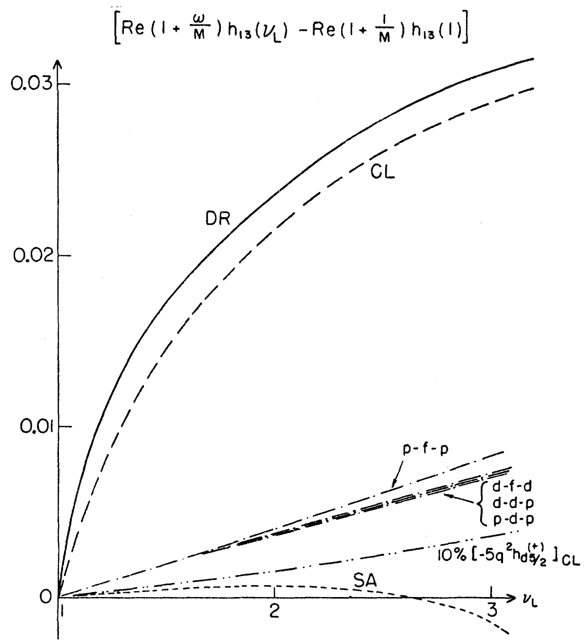


FIG. 4. Contributions to Re$[(1+\omega/M)h_{13}(\nu_L)-(1+1/M)h_{13}(1)]$.

data of the CERN Conference[12] and for the $p$ waves the approximations given by Chew *et al.*[2] These $p$-wave phase shifts are at least as large as any given in the CERN Conference Report. The various contributions of Im$h_s$ and Im$h_p$ are then evaluated numerically and plotted in Figs. 1 through 10 when they are significantly large.

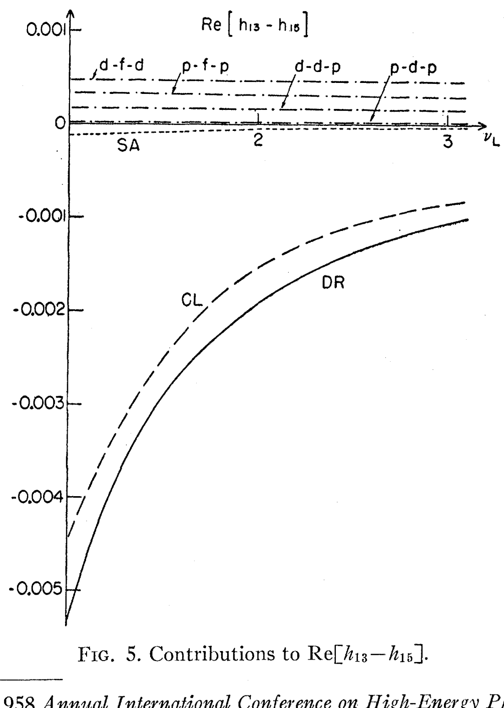


FIG. 5. Contributions to Re$[h_{13}-h_{15}]$.

[11] U. Haber-Schaim, Phys. Rev. **104**, 1113 (1956).

[12] 1958 *Annual International Conference on High-Energy Physics* (CERN Scientific Information Service, Geneva, 1958).

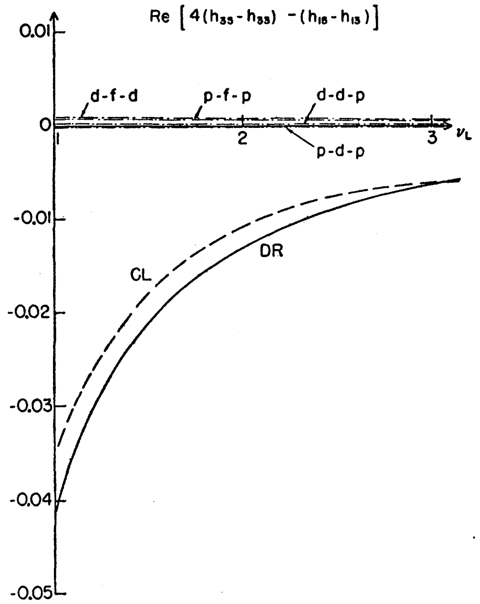


FIG. 6. Contributions to Re[$4(h_{35}-h_{33})-(h_{15}-h_{13})$].

### B. High-Energy Integrals

We now estimate the contribution of the high-energy integrals to the partial wave amplitudes given in Eqs. (4.5) through (4.14). We must evaluate expressions like Eqs. (4.15) and (4.16). To do this we must estimate $\mathrm{Im}A^{(\pm)}(\nu_L,\Delta^2)$ and $\mathrm{Im}B^{(\pm)}(\nu_L,\Delta^2)$ in the energy region $\nu_L \geq \nu_m$. Expanding these amplitudes in partial waves reduces the problem to that of determining $\mathrm{Im}f_{l\pm}(\nu_L)$.

We must turn to the experimental data[12] in order to determine $\mathrm{Im}f_{l\pm}$. The experimental data for energies greater than 400-Mev lab pion energy has the following

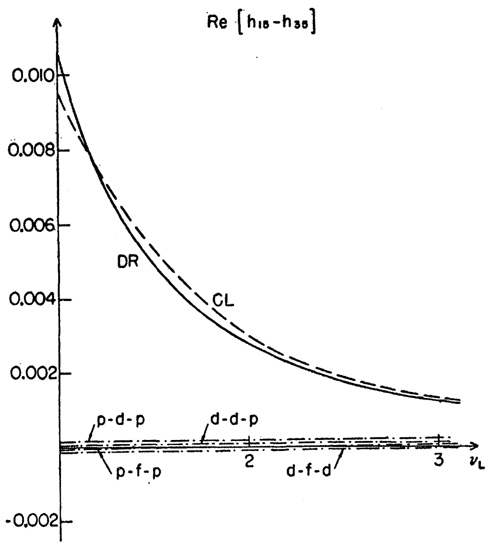


FIG. 7. Contributions to Re[$h_{15}-h_{35}$].

qualitative features: (a) The total cross sections vary rather smoothly with energy except for two resonances in the $T=\frac{1}{2}$ state and one resonance in the $T=\frac{3}{2}$ state. (b) There is a large amount of inelastic scattering. (c) The angular distributions are characterized by a strong forward peaking. (d) For energies greater than 2-Bev lab kinetic energy the total cross sections appear to approach a constant value and the $\pi^-$ and $\pi^+$ on proton cross sections are roughly equal.

If we look in more detail at the experimental results given by Cool *et al.*[10] and by the MIT group[13] we find that it is impossible to interpret the peaks in the total cross sections as due solely to a resonance in a single angular momentum state. With the observed ratio of elastic to inelastic scattering,[12] the maximum total

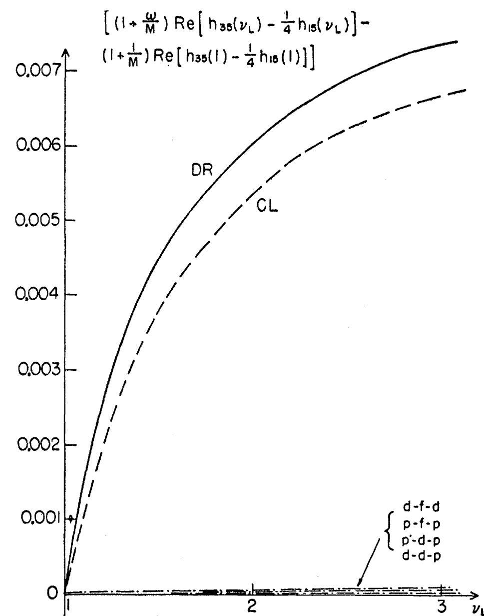


FIG. 8. Contributions to Re{$(1+\omega/M)[h_{35}(\nu_L)-h_{15}(\nu_L)/4]$ $-(1+1/M)[h_{35}(1)-h_{15}(1)/4]$}.

cross section allowed by the unitarity condition

$$\sigma_{\text{total}} \leq \frac{4\pi}{q^2}(J+\tfrac{1}{2})\frac{\sigma_{\text{elastic}}}{\sigma_{\text{inelastic}}}, \tag{5.3}$$

and a reasonable choice of $J$ is only one half the observed total cross section at each of the peaks. We are therefore forced to assume that there is a nonresonant background with resonances in given states superimposed upon it.

The angular distribution data at present is not accurate enough to determine the partial waves involved in the scattering. In addition the high angular momenta involved and the large amount of inelastic scattering make a phase-shift analysis almost meaningless. There-

[13] H. C. Burrowes *et al.*, Phys. Rev. Letters **2**, 119 (1959).

fore, we construct an elementary model which describes the general features of the observed scattering.

The model we choose to correlate the experimental data consists of: (a) a spin-independent purely absorbing optical model to explain the nonresonant scattering, and (b) three resonances in states of fixed total angular momentum superimposed upon the nonresonant background. All possible parities of these states will be considered.

We recall that those amplitudes which have a large high-energy contribution will be assumed not to be determined by dispersion theory. We thus require only an approximate estimate of these high-energy contributions. Hence, in a somewhat arbitrary fashion we assume that the nonresonant scattering is characterized by

$$\sigma_{\text{total}}{}^{T=\frac{1}{2}} \simeq 30 \text{ mb}, \qquad \sigma_{\text{total}}{}^{T=\frac{3}{2}} \simeq 25 \text{ mb}, \tag{5.4}$$

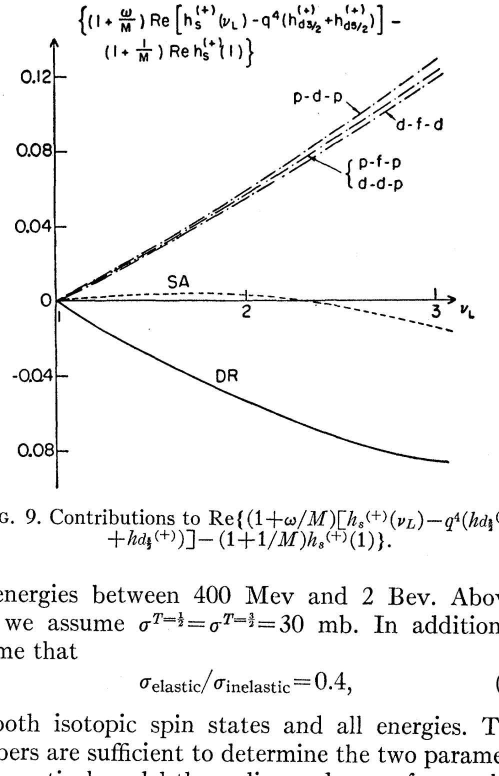


FIG. 9. Contributions to $\mathrm{Re}\{(1+\omega/M)[h_s^{(+)}(\nu_L) - q^4(hd_{\frac{3}{2}}^{(+)} + hd_{\frac{5}{2}}^{(+)})] - (1+1/M)h_s^{(+)}(1)\}$.

for energies between 400 Mev and 2 Bev. Above 2 Bev we assume $\sigma^{T=\frac{1}{2}} = \sigma^{T=\frac{3}{2}} = 30$ mb. In addition we assume that

$$\sigma_{\text{elastic}}/\sigma_{\text{inelastic}} = 0.4, \tag{5.5}$$

for both isotopic spin states and all energies. These numbers are sufficient to determine the two parameters of our optical model the radius and mean free path. In addition this model is consistent with the observed angular distributions peaked strongly in the forward direction and the small ratio of real to imaginary part of the forward scattering amplitude. There is no justification for the assumption that the nonresonant scattering is spin independent but on the other hand this does not contradict any experimental result.

The remainder of the cross section after subtracting out the assumed nonresonant part is interpreted as resonant scattering in single states of angular momentum and parity. These resonances are characterized

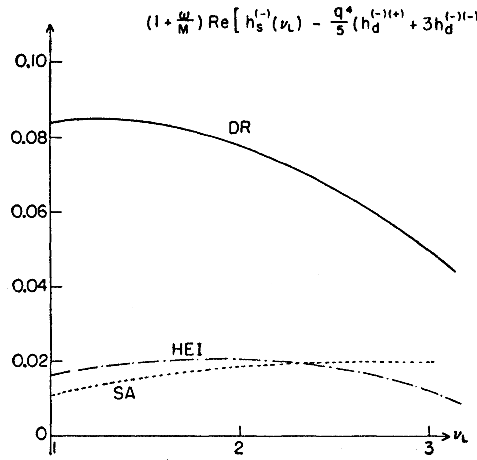


FIG. 10. Contributions to $(1+\omega/M)\,\mathrm{Re}[h_s^{(-)}(\nu_L) - q^4(h_d^{(-)(+)} + 3h_d^{(-)(-)})/5]$.

by the data in Table I. The angular momentum of each of these states is estimated by using Eqs. (5.3) and (5.5). There is very little experimental information on the parity of these resonant states. There are eight possible combinations of parity assignments. By choosing four of these arbitrarily and calculating the contribution of the high-energy integrals in each case we have an estimate of the sensitivity of our calculation to the details of the high-energy scattering. We choose the following combinations of parity assignments

$$\begin{aligned}
&\text{(a)}\quad p_{\frac{3}{2}} - d_{\frac{5}{2}} - p_{\frac{3}{2}},\\
&\text{(b)}\quad p_{\frac{3}{2}} - f_{\frac{5}{2}} - p_{\frac{3}{2}},\\
&\text{(c)}\quad d_{\frac{3}{2}} - d_{\frac{5}{2}} - p_{\frac{3}{2}},\\
&\text{(d)}\quad d_{\frac{3}{2}} - f_{\frac{5}{2}} - d_{\frac{3}{2}},
\end{aligned} \tag{5.6}$$

for the resonances at 650 Mev, 920 Mev, and 1.35 Bev.

The quantitative calculations of $\mathrm{Im}A^{(\pm)}$ and $\mathrm{Im}B^{(\pm)}$ are now quite straightforward. The only possible difficult point is the $\Delta^2$ dependence of these amplitudes. The resonant contributions are trivial since they give a unique Legendre polynomial contribution. In our model the nonresonant scattering leads only to a contribution from $\mathrm{Im}f_1(\nu_L', \cos\theta')$ of the form

$$\mathrm{Im}\, f_1{}^{nr}(\nu_L', \cos\theta') = q'(1-e^{-KR})\frac{R^2}{2}\left[1 - \frac{(\Delta^2R^2)}{2} + \frac{(\Delta^2R^2)^2}{12} - \cdots (-1)^n \frac{(\Delta^2R^2)^n}{n!(n+1)!} + \cdots\right], \tag{5.7}$$

where $R$ is the radius and $K$ the inverse mean free path of our optical model. Earlier we raised the question of the convergence of a power series expansion of $\mathrm{Im}f$ in $\Delta^2$. Equation (5.7) shows that with our model the expansion has an infinite radius of convergence. Of more

TABLE I. Summary of isotopic spins, estimated maximum total cross sections, resonance widths, and proposed total angular momenta for the resonant parts of the total scattering cross section.

| Kinetic energy (lab) Mev | Isotopic spin $T$ | $\sigma$ total at maximum (mb) | Half-width (Mev) | Proposed total ang. mom. $J$ |
|---|---|---|---|---|
| 650 | $\frac{1}{2}$ | 24 | 200 | $\frac{3}{2}$ |
| 920 | $\frac{1}{2}$ | 32 | 200 | $\frac{5}{2}$ |
| 1350 | $\frac{3}{2}$ | 12.5 | 300 | $\frac{3}{2}$ |

practical interest is the fact that this series actually converges quite rapidly since the maximum $\Delta^2$ considered is $\Delta^2\simeq 4$ and $R^2$ is of the order of magnitude of $R^2\simeq 0.3$.

The necessary high-energy integrals can now be evaluated numerically. The contributions of the high-energy integrals to the $s$-, $p$-, and $d$-wave amplitudes in Eqs. (4.5) through (4.14) are plotted in Figs. 1 through 10. For each amplitude we plot four possible high-energy contributions obtained by considering the various combinations of parities listed in Eq. (5.6).

## VI. RESULTS

In Sec. II we have investigated the consequences of assuming a finite range of interaction. This assumption has led to upper bounds on the asymptotic energy dependence of the dispersion amplitudes $A^{(\pm)}(\nu,\Delta^2)$ and $B^{(\pm)}(\nu,\Delta^2)$ as $\nu$ approaches infinity. These bounds imply that unsubtracted dispersion relations exist for $A^{(-)}(\nu,\Delta^2)$ and $B^{(\pm)}(\nu,\Delta^2)$ with $A^{(-)}$ and $B^{(-)}$ having one arbitrary function of momentum transfer each and $B^{(+)}$ none. $A^{(+)}(\nu,\Delta^2)$ satisfies a dispersion relation with one subtraction and no additional arbitrary functions of momentum transfer. We have further shown that the difference of the $\pi^+-p$ and $\pi^--p$ cross sections must vanish for infinite energies and that if $[a_0^{(-)}(0)+b_0^{(-)}(0)]$ does not vanish then the $\pi^+-p$ and $\pi^--p$ cross sections must approach the same constant value.

We have also indicated in Sec. II that the assumption of equal time commutation relations and a particular form for the meson current suggests that $a_0^{(-)}(\Delta^2)$ and $b_0^{(\pm)}(\Delta^2)$ may vanish. We have made this additional assumption.

In Sec. III we have discussed the relation between a partial wave analysis of the dispersion relations and the analytic properties of the scattering amplitude as a function of momentum transfer. In particular we have found that the region of analyticity found by the general theory to date is not large enough to justify all the Legendre expansions necessary to extract equations determining the low-energy partial wave amplitudes. The expansion of the subtraction function of momentum transfer $\mathrm{Re}A^{(+)}(1,\Delta^2)$ appears to present the most serious difficulty and is in all likelyhood not valid.

The information we have obtained about the low-energy $s$-, $p$-, and $d$-wave phase shifts is presented graphically in Figs. 1 through 12. Figures 1 through 10 show the various contributions to the low-energy partial wave amplitudes. In these graphs we have plotted: (a) the bound-state term plus the integral over the 33 resonance, (b) the contributions of the high-energy integrals for various choices of parities for the resonances, (c) the contributions of the integrals over the low-energy $s$- and $p$-wave amplitudes when they are large enough to be plotted, and (d) for comparison the approximate estimates of these amplitudes given by Chew *et al.*[2]

Chew *et al.* give the following approximations for the $p$ waves

$$h_{13}\simeq h_{31}\simeq\tfrac{1}{4}h_{11}\simeq -\frac{2}{3}\frac{f^2}{\omega}\frac{1}{1+\omega/\omega_r}, \tag{6.1}$$

$$\cot\delta_{33}=\frac{\omega(1-\omega/\omega_r)}{\frac{4}{3}f^2q^3}, \tag{6.2}$$

with $\omega_r=2.12$, and for the $d$ waves

$$\begin{aligned}\delta_{13}&=-\lambda_d\left[1+\frac{28}{3}\left(\frac{\omega}{\omega+\omega_r}\right)^2\right],\\ \delta_{33}&=\lambda_d\left[2-\frac{7}{3}\left(\frac{\omega}{\omega+\omega_r}\right)^2\right],\\ \delta_{15}&=\lambda_d\left[4-\frac{8}{3}\left(\frac{\omega}{\omega+\omega_r}\right)^2\right],\\ \delta_{35}&=-\lambda_d\left[8+\frac{2}{3}\left(\frac{\omega}{\omega+\omega_r}\right)^2\right],\end{aligned} \tag{6.3}$$

where

$$\lambda_d=\frac{1}{15}\frac{f^2}{M}\frac{q^5}{\omega^2}.$$

If we impose the criterion that those amplitudes whose principle contribution comes from the bound-state term plus the 33 integral are the only ones said

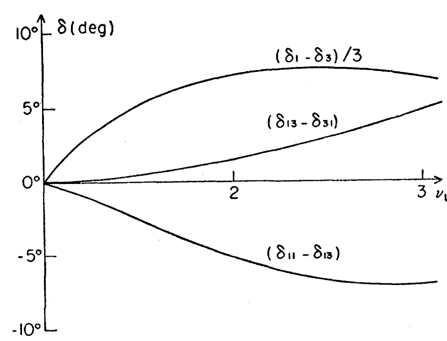


FIG. 11. Predictions for the low-energy $s$- and $p$-wave phase shifts determined by dispersion theory. The estimate for $(\delta_{13}-\delta_{31})$ is given by $0.01q^3$ and is an upper bound obtained by adding all the contributions in Fig. 3.

to be determined by dispersion theory, then it seems reasonable to say that we have determined seven of the ten low-energy amplitudes. The amplitudes $h_s^{(+)}$, $h_{13}$, and $h_{35}$ either have large contributions from the high-energy integrals or have subtraction terms which are at present indeterminate. It is interesting to note that these are just the amplitudes which depend on the $A^{(+)}$ dispersion relation and hence contain the subtraction function $\mathrm{Re}A^{(+)}(1,\Delta^2)$. The doubt cast by our previous discussion on the validity of a Legendre expansion of $\mathrm{Re}A^{(+)}(1,\Delta^2)$ makes any calculation of $h_s^{(+)}$, $h_{13}$, and $h_{35}$ appear even more unreliable.

The degree of accuracy to which the other seven amplitudes have been calculated can be judged by an inspection of the relative magnitude of the various contributions to Figs. 1 through 10. In general as the energy increases the results appear more unreliable. It is important to note that the contributions of the high-energy integrals are quite sensitive to the parities chosen for the resonances and hence sensitive to the details of the high-energy scattering. Thus any serious calculation of these high-energy integrals will require a considerable improvement in theory and/or experiment. The predicted values of the low-energy phase shifts are plotted in Figs. 11 and 12.

We now turn to a more detailed discussion of the individual $s$-, $p$-, and $d$-wave amplitudes. It must be remembered that all our results are subject to the assumption of charge independence.

## (1) *p* Waves

(a) The 33 amplitude is determined by Eq. (4.6) and is plotted in Fig. 1. We recall that $\mathrm{Im}h_{33}$ has been calculated using the effective range relation (5.1). We have also plotted the values of $\mathrm{Re}h_{33}$ obtained from the same effective range formula. Comparing this curve with the prediction of dispersion theory shows that closer agreement is obtained by choosing $f^2=0.08$. We conclude that dispersion theory predictions are consistent with

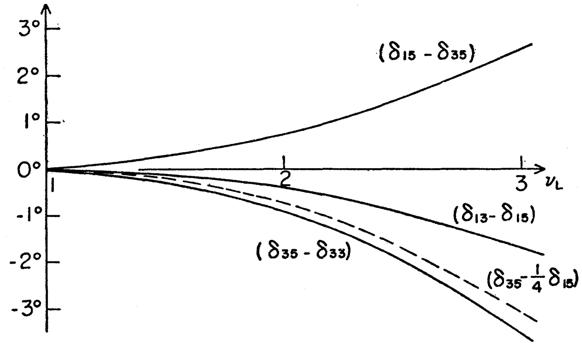


FIG. 12. Predictions for the low-energy $d$-wave phase shifts determined by dispersion theory. The curve for $(\delta_{35}-\delta_{15}/4)$ is the estimate of Chew *et al.* given in Eq. (6.3).

the effective range equation

$$\tfrac{4}{3}f^2\frac{q^3}{\omega}\cot\delta_{33}=1-\frac{\omega}{\omega_r},$$

with $\omega_r=2.12$ and $f^2=0.08$. However, we are unable to determine the resonance energy from first principles since an attempt to calculate it depends on either the cutoff energy $\nu_m$ or on the detailed shape of the resonance. See also similar arguments by Chew *et al.*[2]

(b) $\mathrm{Re}(h_{11}-h_{13})$ is determined quite well by dispersion theory (see Fig. 2). The $s$-wave contribution is completely negligible. The corresponding phase shift is plotted in Fig. 11.

(c) Figure 3 leads to the prediction that $\mathrm{Re}(h_{13}-h_{31})\simeq 0$. To obtain an idea of the deviation of $\mathrm{Re}(h_{13}-h_{31})$ from zero, we have plotted the approximate result predicted by Eq. (6.1) for $\mathrm{Re}h_{13}\simeq\mathrm{Re}h_{31}$ in Fig. 3. The high-energy integral contributions are of the same order of magnitude as the 33 integral plus bound-state contribution so that our estimate of $(\delta_{13}-\delta_{31})$ shown in Fig. 11 is only an order of magnitude upper bound obtained by adding all the contributions in Fig. 3.

(d) In Fig. 4 we have plotted our results for $\mathrm{Re}[(1+\omega/M)h_{13}(\nu_L)-(1+1/M)h_{13}(1)]$. This result is of little practical value since it contains unknown subtraction constants. First, we must know $\mathrm{Re}h_{13}(1)$ which is quite difficult to determine experimentally. Second, we have the more serious problem of requiring the knowledge of $\mathrm{Re}h_{d\frac{1}{2}}^{(+)}(1)$ to a high degree of accuracy. This contributes to the coefficient of $5q^2$ in Eq. (4.12). We have plotted an estimate of this contribution by assuming that there is a ninety percent cancellation within the bracket multiplying $5q^2$ so that the bracket is ten percent of $h_{d\frac{1}{2}}^{(+)}(1)$. We estimate $h_{d\frac{1}{2}}^{(+)}(1)$ by using the prediction of Eq. (6.3).

Some of the above $p$-wave predictions are in conflict with recent analyses of experimental data.[12,14] The experimental results for $(\delta_{13}-\delta_{31})$ appear to exceed our rough upper bound by a factor of two above 150 Mev while below this energy the data is in agreement with our calculation. For $(\delta_{11}-\delta_{13})$ there is rough agreement above 150 Mev and disagreement below this energy. In all cases the data is not sufficiently accurate to take any disagreement too seriously. In addition, all reductions of the data have used charge independent analyses and hence are subject to possibly large charge dependent corrections.

## (2) *d* Waves

The $d$-wave phase shifts predicted by dispersion theory are plotted in Fig. 12. The accuracy to which they have been evaluated can be seen in Figs. 5 through 8. $(\delta_{35}-\frac{1}{4}\delta_{15})$ involves the unknown subtraction constant $\mathrm{Re}[h_{35}(1)-\frac{1}{4}h_{15}(1)]$ which must be determined

[14] H. Y. Chiu and E. L. Lomon, Ann. Phys. 6, 50 (1959).

experimentally before $(\delta_{35}-\frac{1}{4}\delta_{15})$ is determined by dispersion theory.

The only $d$-wave data which can be compared with our predictions is from polarization data at 307 Mev giving $(\delta_{35}-\delta_{33})\simeq -4°\pm 5°$.[15] This agrees with our calculations.

### (3) s Waves

(a) The $s$-wave amplitude $\mathrm{Re}h_s^{(+)}(\nu_L)$ is not determined by dispersion theory. First, the high-energy integral contribution is of the same magnitude as the contribution from the resonant integral plus bound-state term. Second, the equation for $h_s^{(+)}(\nu_L)$ depends on $\mathrm{Re}A^{(+)}(1,q^2/2)$ which in principle is an unknown function of $q^2$. The Legendre expansion of this leads to the practical difficulty that the expansion coefficients of the series in $q^2$ must be known to an unreasonable degree of accuracy. For example, the $p$-wave amplitudes appearing in Eq. (4.14) must be known to one percent to determine $h_s^{(+)}$ to ten percent. This is probably an indication of the lack of convergence of this expansion.

(b) The prediction for $\delta_s^{(-)}=(\delta_1-\delta_3)/3$ is plotted in Fig. 11 by adding all the contributions in Fig. 10. The high-energy integral contributions are independent of the parity of the scattering states and hence our estimate of them may be slightly more reliable.

The threshold $s$-wave scattering length $h_s^{(-)}(1)$ gives us a check on the accuracy of our calculations since it is given by an integral over total cross sections only. If we add up all the contributions in Fig. 10, we predict that $(1+1/M)\,\mathrm{Re}h_s^{(-)}(1)=0.110$ compared with the experimental value of Orear[16] which is $(1+1/M)\times\mathrm{Re}h_s^{(-)}(1)=0.103$. This seven percent difference indicates the limits of accuracy of our numerical integrations and our high-energy integral approximations.

Our result for the energy dependence of $h_s^{(-)}(\nu_L)$ differs from that of earlier calculations[1,2] which predicted that $(1+\omega/M)\,\mathrm{Re}h_s^{(-)}(\nu_L)$ be proportional to $\omega$. This is due to our more careful treatment of the kinematical factors. For example, Chew *et al.* give for the bound-state term $2f^2\omega$ while we find

$$2f^2\omega\left[\frac{(1+q^2)}{\nu_L(\nu_L-q^2/M)}\right].$$

The bracket deviates from unity by ten percent at 40 Mev and by twenty percent at 150 Mev. The other contributions tend to increase this deviation from a linear $\omega$ dependence. Note that the integrals over the small amplitudes and over high energies contribute significantly to the energy dependence of $h_s^{(-)}(\nu_L)$. This indicates that the actual energy dependence shown in Fig. 10 is unreliable. This result is important in any attempt to extrapolate the charge exchange scattering amplitude to threshold.

The general shape and magnitude of our prediction for $(\delta_1-\delta_3)$ especially the flattening off above 150 Mev, is in agreement with the analysis of experimental data by Chiu and Lomon.[14]

**ACKNOWLEDGMENT**

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[15] Ninth Annual International Conference on High-Energy Physics, Kiev, 1959 (to be published).

[16] J. Orear, Phys. Rev. **100**, 288 (1955).