Meson-Meson Scattering Term and Low-Energy Pion-Nucleon Scattering*

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The modified Chew-Low integral equation for the pion-nucleon P-wave scattering amplitudes was derived by the present authors using the Chew-Low-Wick formalism and assuming a general static interaction Hamiltonian plus the meson-meson scattering term. Essentially the same result is shown to follow from dispersion relations in the no-nucleon-recoil approximation if we assume that the almost-forward elastic scattering amplitude becomes a finite real number at large incident pion energy. If we further presume that the meson-meson scattering term alone is responsible for this asymptotic behavior, we can show that the extra term to be introduced into the Chew-Low integral equation is just the zero-energy limit of the corresponding term, which was energy dependent in our previous derivation. The modified effective-range expansion of the P-wave phase shift is compared with the data. The S-wave integral equations are also given; they are formally much simpler than those obtained previously in the static model calculation.

DISPERSION RELATIONS

FOLLOWING Oehme,[1] we introduce the no-nucleon-recoil approximation to the pion-nucleon scattering T matrix:

$$T(\omega,\cos\theta)=-i\int e^{i\mathbf{q}\cdot\mathbf{x}}e^{-i\mathbf{q}'\cdot\mathbf{y}}\int_{-\infty}^{0}e^{-i\omega t}d\mathbf{x}d\mathbf{y}dt \times(\Psi_0,[j_\beta(\mathbf{y},0),j_\alpha(\mathbf{x},t)]_-\Psi_0) +\lambda\int e^{i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}}d\mathbf{x}(\Psi_0,\{\delta_{\alpha\beta}\phi^2(\mathbf{x},0) +2\phi_\alpha(\mathbf{x},0)\phi_\beta(\mathbf{x},0)\}\Psi_0). \quad (1)$$

This is exact if Ψ_0, the static physical nucleon state, is replaced appropriately by the initial and final physical nucleon states. $(\mathbf{q},\alpha)$ and $(\mathbf{q}',\beta)$ are the momenta and charges of the initial and final pions, respectively. The Heisenberg operators $\phi_\alpha(\mathbf{x},t)$ and $j_\alpha(\mathbf{x},t)$ are, respectively, the pion field variable and the pion source function. The second term of (1) is due to the meson-meson scattering term, $(\lambda/4)\int[\phi^2]^2d\mathbf{x}$. The scattering angle θ is the angle between $\mathbf{q}$ and $\mathbf{q}'$.

As usual, we define non-spin-flip and spin-flip amplitudes by

$$T(\omega,\cos\theta)=A(\omega,\cos\theta)+i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{q}'B(\omega,\cos\theta) \quad (2)$$

and, further, non-isospin-flip and isospin-flip amplitudes by

$$A(\omega,\cos\theta)=A^+(\omega,\cos\theta)\delta_{\alpha\beta}+A^-(\omega,\cos\theta)\tfrac{1}{2}[\tau_\beta,\tau_\alpha]_-, \quad (3)$$

etc. The exact dispersion relations for $A^\pm(\omega)\equiv A^\pm(\omega,\theta=0)$ and $B^\pm(\omega)\equiv B^\pm(\omega,\theta=0)$ are known.[2]

We define additional amplitudes $M^\pm(\omega)$ by

$$M^\pm(\omega)=\left[\frac{\partial}{\mathbf{q}^2\partial\cos\theta}A^\pm(\omega,\cos\theta)\right]_{\theta=0}. \quad (4)$$

The dispersion relations for $M^\pm(\omega)$ in the no-nucleon-recoil approximation (1) have been derived by Oehme.[1] Summarizing all these, $A^+(\omega)$, $B^-(\omega)$, $M^+(\omega)$ are even functions of real ω in the sense that $A^+(\omega)=[A^+(-\omega)]^*$, etc., while $A^-(\omega)$, $B^+(\omega)$, $M^-(\omega)$ are odd, so that $A^-(\omega)=-[A^-(-\omega)]^*$, etc. All amplitudes are analytic on the upper half complex ω plane except for two branch cuts, $(+\mu,+\infty)$ and $(-\mu,-\infty)$, and a pole at $\omega=0$. The residues are $2f^2$ for $A^-(\omega)$, $2f^2/\mu^2$ for both $B^+(\omega)$ and $M^-(\omega)$, and zero for the rest, f being the renormalized equivalent *ps-pv* coupling constant.

Regarding the behavior at infinity in the ω plane, we know practically nothing except that no essential singularities occur. As for $A^\pm(\omega)$, we assume one subtraction motivated by the experimental possibility that the total cross section stays finite at large ω. As for the other amplitudes, we have no such clue from the empirical viewpoint since these amplitudes don't satisfy optical theorems. We here presume the simplest situation to be expected from (1): We assume that $M^+(\omega)$ alone becomes a real (finite) constant at infinity, while $B^\pm(\omega)$ and $M^-(\omega)$ become zero asymptotically. We make this assumption simply because $M^+(\omega)$ cannot go to zero at large ω if the λ term is included, while the other amplitudes, unaffected by the second term of (1), could go to zero.

With these assumptions we can write down the dispersion relations as follows:

$$A^+(\omega)=A^+(\mu)+\frac{q^2}{\pi}\int_\mu^\infty\frac{2\omega'\,\mathrm{Im}A^+(\omega')d\omega'}{q'^2(\omega'^2-\omega^2-i\epsilon)},$$
$$A^-(\omega)=\frac{\omega}{\mu}A^-(\mu)+\frac{2f^2q^2}{\mu^2\omega}+\frac{q^2}{\pi}\int_\mu^\infty\frac{2\omega\,\mathrm{Im}A^-(\omega')d\omega'}{q'^2(\omega'^2-\omega^2-i\epsilon)}, \quad (5)$$

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[1] R. Oehme, Phys. Rev. **102**, 1174 (1956). The derivation of (1) is due to F. E. Low, Phys. Rev. **97**, 1392 (1955).
[2] M. L. Goldberger, Phys. Rev. **99**, 979 (1955); M. L. Goldberger, H. Miyazawa, and R. Oehme, Phys. Rev. **99**, 986 (1955).

$$B^+(\omega)=\frac{2f^2}{\mu^2\omega}+\frac{1}{\pi}\int_\mu^\infty \frac{2\omega\,\mathrm{Im}B^+(\omega')d\omega'}{\omega'^2-\omega^2-i\epsilon},$$
$$B^-(\omega)=\frac{1}{\pi}\int_\mu^\infty \frac{2\omega'\,\mathrm{Im}B^-(\omega')d\omega'}{\omega'^2-\omega^2-i\epsilon}, \tag{6}$$

$$M^+(\omega)=M^+(\infty)+\frac{1}{\pi}\int_\mu^\infty \frac{2\omega'\,\mathrm{Im}M^+(\omega')d\omega'}{\omega'^2-\omega^2-i\epsilon},$$
$$M^-(\omega)=\frac{2f^2}{\mu^2\omega}+\frac{1}{\pi}\int_\mu^\infty \frac{2\omega\,\mathrm{Im}M^-(\omega')d\omega'}{\omega'^2-\omega^2-i\epsilon}, \tag{7}$$

where $A^\pm(\mu)$ are subtraction constants and $M^+(\infty)$ is some real number.

P-WAVE INTEGRAL EQUATIONS

To derive the integral equations for partial wave amplitudes from (5), (6), and (7), we make this further approximation: We retain only S- and P-wave amplitudes in $T(\omega,\cos\theta)$. We remark that this is no worse than the one-meson approximation in the Chew-Low-Wick formalism,[3] as seen, for example, by the fact that the one-meson approximation modifies the T matrix even below the inelastic threshold, while we do not, except for dropping higher partial elastic amplitudes. We therefore use the partial wave expansion of $T(\omega,\cos\theta)$ in the static model calculation.[3] It is then shown that $B^\pm(\omega)$ and $M^\pm(\omega)$ are four linearly independent combinations of the four P-wave amplitudes. Dispersion relations (6) and (7) are finally rewritten as

$$t_\alpha(\omega)=\frac{C_\alpha f^2}{6\pi\mu^2\omega}-\frac{M^+(\infty)}{12\pi}+\frac{1}{\pi}\int_\mu^\infty q'^3d\omega'\left\{\frac{|t_\alpha(\omega')|^2}{\omega'-\omega-i\epsilon}+\sum_\beta A_{\alpha\beta}\frac{|t_\beta(\omega')|^2}{\omega'+\omega}\right\}, \tag{8}$$

where $t_\alpha(\omega)=\sin\delta_\alpha\exp(i\delta_\alpha)/q^3$, δ_α being the phase shifts of the three P-wave channels with total isospin and angular momentum quantum numbers $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{3}{2})=(\frac{3}{2},\frac{1}{2})$, and $(\frac{3}{2},\frac{3}{2})$. The $A_{\alpha\beta}$ are the same as those given by Chew and Low,[3] and $C_\alpha=(-4,-1,2)$.

Equation (8) is just the Chew-Low equation[3] if a term with $M^+(\infty)$ is dropped, and is identical to that derived previously by the present authors from a general static model supplemented by the λ term.[4]

One remarkable improvement in the present derivation is that the new term in (8) is exactly energy independent, while in our previous derivation it was an energy-dependent term. This difference is considered to be solely due to the difference between our retention of only elastic S- and P-wave amplitudes in $T(\omega,\cos\theta)$ and the conventional one-meson approximation in the Chew-Low-Wick formalism.

[3] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).
[4] M. Sugawara and A. Kanazawa, Phys. Rev. **115**, 1310 (1959).

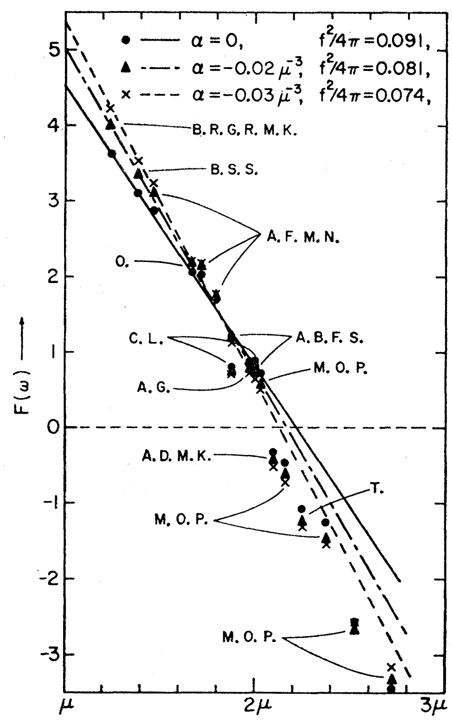

FIG. 1. The modified effective-range formula (9) is plotted against ω, where the experimental values of δ are those of reference 5. Three values of α are assumed as shown on the figure. These then determine three values of $f^2/4\pi$, which are also shown on the figure. Here ω has been replaced by the sum of the incident pion energy and the nucleon kinetic energy in the c.m. frame of reference, and μ is the rest mass of the pion.

MODIFIED EFFECTIVE-RANGE EXPANSION OF $(\frac{3}{2},\frac{3}{2})$ PHASE SHIFT

The modified effective-range expansion of the $(\frac{3}{2},\frac{3}{2})$ phase shift was derived in our previous paper.[4] However, only the low-energy limit was compared with data, since (8) was considered as a zero-energy approximation. Since we have now shown that (8) is the exact modification under the assumptions made, we here present a comparison with data of our (complete) modified effective-range expansion.

The modified effective-range formula is

$$F(\omega)=\frac{q^3\cot\delta(1+\alpha q^3\cot\delta)+\alpha q^6}{\omega\mu^2[(1+\alpha q^3\cot\delta)^2+\alpha^2q^6]}=\frac{3\pi}{f^2}[1-\mathrm{const}\,\omega+\cdots], \tag{9}$$

where δ is the $(\frac{3}{2},\frac{3}{2})$ phase shift and $\alpha=M^+(\infty)/12\pi$. Data[5] are plotted, assuming $\alpha=0$, $-0.02\mu^{-3}$, and

[5] S. W. Barnes, B. Rose, G. Giacomelli, J. Ring, K. Miyake, and K. Kinsey, Phys. Rev. Letters **3**, 592 (1959); D. Bodansky, A. M. Sachs, and J. Steinberger, Phys. Rev. **93**, 1367 (1954); J. Ashkin, J. P. Blaser, F. Feiner, and M. O. Stern, Phys. Rev. **105**, 724 (1957); H. L. Anderson, E. Fermi, R. Martin, and D. E. Nagle, Phys. Rev. **91**, 155 (1953); J. Orear, Phys. Rev. **96**,

$-0.03\mu^{-3}$, in Fig. 1, which also includes the corresponding values of $f^2/4\pi$ determined by the curves. It is seen that $f^2/4\pi=0.08$ is attainable for $\alpha=-0.02\mu^{-3}$ or $M^+(\infty)=-0.75\mu^{-3}$. This value of α is somewhat smaller than our previous determination[4] based upon the low-energy limit of (9).

IDENTIFICATION OF $M^+(\omega)$

So far $M^+(\omega)$ is an unknown real constant. Suppose, however, that the (finite) meson-meson scattering term contained in the second term of (1) alone survives at large ω. This is consistent with our assumption that $M^+(\omega)$ becomes a real, finite number at large ω, while $M^-(\omega)$ and $B^\pm(\omega)$ become zero asymptotically. Since the pion variable $\phi_\alpha(\mathbf{x},0)$ in the second term of (1) can be identified as a Schrödinger operator, we can apply the same technique as we used previously regarding the λ term.[4] We can show that

$$T(\omega,\cos\theta)\underset{\omega\to\infty}{\longrightarrow}\delta_{\alpha\beta}\left\{\frac{(\mathbf{q}\cdot\mathbf{q}')}{\mu^3}\eta(\mathbf{q}'^2,\mathbf{q}^2)+2\zeta(\mathbf{q}'^2,\mathbf{q}^2)\right\}, \quad (10)$$

where η and ζ were defined before.[4] From definition (4) it then follows that

$$M^+(\infty)=\eta(0,0)/\mu^3, \quad (11)$$

where we have used the fact that the contribution from the second term of (1) to $M^+(\infty)$ is an energy-independent constant.

Equation (8) with (11) is just our previous result [Eq. (17) of reference 4]. The present determination of α or $M^+(\infty)$ implies, according to our previous analysis,[4] that the meson-meson scattering term constant $\lambda\approx 12$ (note the difference in definitions of λ here and in the previous paper[4]).

S-WAVE INTEGRAL EQUATIONS

The S-wave integral equations are derived by combining (5) and (7). They are as follows:

$$\begin{aligned} t_1(\omega)+2t_3(\omega)&=-3A^+(\mu)/4\pi+3q^2M^+(\infty)/4\pi\\ &\quad+\frac{q^2}{\pi}\int_0^\infty\frac{\mathrm{Im}[t_1(\omega')+2t_3(\omega')]dq'^2}{q'^2(q'^2-q^2-i\epsilon)},\\ \frac{\mu}{\omega}[t_1(\omega)-t_3(\omega)]&=-\frac{3A^-(\mu)}{4\pi}+\frac{q^2}{\pi}\\ &\quad\times\int_0^\infty\frac{\mathrm{Im}\{(\mu/\omega')[t_1(\omega')-t_3(\omega')]\}dq'^2}{q'^2(q'^2-q^2-i\epsilon)}, \end{aligned} \quad (12)$$

where $t_\alpha(\omega)=\sin\delta_\alpha\exp(i\delta_\alpha)/q$, δ_α being S-wave phase shifts for total isospin quantum numbers $\frac{1}{2}$ and $\frac{3}{2}$. It would be interesting to check whether the term with $M^+(\infty)$ is important or not in achieving the fit of (12) to data. It is added that Eqs. (12) are formally much simpler than the S-wave equations derived from the static model calculation.[4]

1417 (1954); H. L. Anderson and M. Glicksman, Phys. Rev. **100**, 268 (1955); H. L. Anderson, W. C. Davidon, M. Glicksman, and U. E. Kruse, Phys. Rev. **100**, 279 (1955); H. D. Taft, Phys. Rev. **101**, 1116 (1956); A. I. Mukhin, E. B. Ozerov, and B. Pontekorvo, J. Exptl. Theoret. Phys. (U.S.S.R.) **31**, 371 (1957) [translation: Soviet Phys.—JETP **4**, 273 (1957)].