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Strong Interactions and a Model for Hyperons*

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Using a 4-dimensional approach, the couplings of the strongly interacting particles are restricted in a simple way which is not inconsistent with experiment. This leads to the consideration of a Goldhaber-type model. The gross properties of the hyperons are calculated in the intermediate-coupling approximation for this model.

STRONG INTERACTIONS AND A MODEL FOR HYPERONS

SINCE all of the strange particles are strongly interacting, a theory of strange particles faces not only the problems associated with a field theory, but the additional problem that the strong interactions are not susceptible to the perturbation theory approach, which has been successful for calculations involving weak and electromagnetic couplings. However, one can still attempt to apply the general approach of field theory to the "strange" particles and attempt to use experiment to deduce information about the interactions.

Many theories[1–5] have been advanced to deal with the strongly interacting particles and can best be classified in terms of the number of coupling constants which are introduced. These range from one to eight, for theories linear in boson fields, with special titles such as global symmetry, cosmic symmetry, etc., given to theories containing few coupling constants. The gamut runs from a single coupling constant for all strong interactions to separate constants for each interaction involving a different isotopic spin multiplet. It would seem desirable, particularly in view of the successes of the universal Fermi interaction, to limit the number of coupling constants as far as possible consistent with experiment. The procedure carried out in this paper indicates that it is possible to restrict the couplings of strongly interacting particles in a simple way. The particular approach leads to considera-tion of a Goldhaber-type model[6–8] which may be an aid in understanding the properties of baryons.

The first limitation which might be attempted is to equate the Σ and Λ coupling constants:

$$\begin{aligned}[][\Sigma NK]&=[\Lambda NK],\\ [\Sigma\Xi K]&=[\Lambda\Xi K],\\ [\Sigma\Sigma\pi]&=[\Sigma\Lambda\pi].\end{aligned}\tag{1}$$

We are led to an extremely simple way of formulat-ing the K interactions by observing that insofar as isotopic spin states are concerned there exist three spinors: the nucleon, cascade, and K; and a singlet and a triplet, the Λ and Σ, respectively. Suppose we consider the nucleon as a spinor in a space I with basis vectors

$$\begin{pmatrix}I_1\\ I_2\end{pmatrix}$$

representing p, n, respectively, and the $\bar{K}$ a spinor in a space K, basis vectors

$$\begin{pmatrix}K_1\\ K_2\end{pmatrix}$$

representing the $\bar{K}^0K^-$ doublet. If the product space, $I\times K$, is formed, the possible basis vectors are given by

$$\begin{bmatrix}I_1K_1\\ I_2K_1\\ I_1K_2\\ I_2K_2\end{bmatrix}\rightarrow\begin{bmatrix}p\bar{K}^0\\ n\bar{K}^0\\ p\bar{K}^-\\ nK^-\end{bmatrix}.\tag{2}$$

This set of quantities, which transforms as a spinor under independent rotations in the spaces I and K, will be denoted by η_1. If we examine the terms of this spinor, we notice that the quantum numbers of the first component are baryon number 1, strangeness -1, z component of isotopic spin $+1$. These are also the characteristics of Σ^+. Thus, we could form a second spinor such that

$$\eta_2=\begin{bmatrix}\Sigma^+\\ Y^0\\ Z^0\\ \Sigma^-\end{bmatrix}\rightarrow\begin{bmatrix}I_1K_1\\ I_2K_1\\ I_1K_2\\ I_2K_2\end{bmatrix},\tag{3}$$

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1 M. Gell-Mann, Phys. Rev. **106**, 1296 (1957).
2 A. Pais, Phys. Rev. **110**, 574 (1958).
3 J. Schwinger, Ann. Phys. **2**, 407 (1957).
4 A. Pais, Phys. Rev. **110**, 1480 (1958).
5 A. Pais, Phys. Rev. **112**, 624 (1958).
6 M. Goldhaber, Phys. Rev. **92**, 1279 (1953).
7 M. Goldhaber, Phys. Rev. **101**, 433 (1956).
8 G. Gyorgi, Nuclear Phys. **10**, 197 (1959).

insofar as the properties by which one describes the particles are concerned. We can carry out a similar process using the cascade and the charge conjugate of the K^0, K^- doublet and form a third spinor η_3 where

$$\eta_3=\begin{bmatrix}\Xi^0 K^+\\ \Xi^- K^+\\ -\Xi^0 K^0\\ -\Xi^- K^0\end{bmatrix}. \tag{4}$$

Then interactions between hyperons, nucleons, and K's may be written very simply as spinor products. The simplest terms we can write are

$$a_1\bar{\eta}_1 O\eta_2+b_1\bar{\eta}_3 O\eta_2+\text{Hermitian conjugate}, \tag{5}$$

where the symbol O indicates 1 or γ^5. This set of interactions is identical with the one proposed by Pais. The S_1, S_2 symmetry is present and clearly inconsistent with experiment. However, these are not the only possible terms, since any scalar in the space $I\times K$ is allowable.[9] There is one other scalar which can be formed, namely

$$a_2\bar{\eta}_1 O\boldsymbol{\tau}_I\cdot\boldsymbol{\tau}_K\eta_2+b_2\bar{\eta}_3 O\boldsymbol{\tau}_I\cdot\boldsymbol{\tau}_K\eta_2+\text{H.c.}, \tag{6}$$

where $\boldsymbol{\tau}_I$ and $\boldsymbol{\tau}_K$ are isotopic spin operators in the spaces I and K, respectively. Then the most general charge-independent interaction linear in the K field with equal coupling for Σ and Λ is given by

$$\bar{\eta}_1 O(a_1+a_2\boldsymbol{\tau}_I\cdot\boldsymbol{\tau}_K)\eta_2+\bar{\eta}_3 O(b_1+b_2\boldsymbol{\tau}_I\cdot\boldsymbol{\tau}_K)\eta_2+\text{H.c.} \tag{7}$$

This differs from the Pais interactions[2] in that the S_1, S_2 symmetry does not exist unless $a_2=b_2=0$. We can demand the so-called cosmic symmetry by setting $a_1=b_1$, $a_2=b_2$ with no restrictions on the π interactions. These, of course, are not the most general charge-independent interactions, since there might be terms of higher order in the K field, derivative couplings, etc. In this simplified notation the various symmetries are obvious. For example, invariance under G conjugation rests on the fact that $\boldsymbol{\tau}_I\cdot\boldsymbol{\tau}_K$ commutes with the G operator and invariance under interchange of nucleon and casade K and K^c are obvious from the manner in which the spinors were formed. However, the N_2, N_3 symmetry of Pais is not present.

The π interactions of Σ and Λ may also be written in terms of the spinor η_2; for example,

$$c_1\boldsymbol{\pi}\cdot\bar{\eta}_2\gamma_5\boldsymbol{\tau}_I\eta_2+c_2\boldsymbol{\pi}\cdot\bar{\eta}_2\gamma^5\boldsymbol{\tau}_K\eta_2, \tag{8}$$

is the most general π interaction involving the Σ, Λ system. The first term is identical with the Σ, Λ, π interaction of Pais and others. The second term differs from the first only in that Λ is replaced by $-\Lambda$ in every term. In particular, the interaction $\boldsymbol{\pi}\cdot\bar{\eta}_2\gamma_5\boldsymbol{\tau}\eta_2$ where $\boldsymbol{\tau}=\boldsymbol{\tau}_I+\boldsymbol{\tau}_K$, the total isotopic spin, corresponds to a choice $c_1=c_2$ and eliminates the direct Λ, π interaction.

It is perhaps worth noting that the nucleon and cascade could be combined to form a four-component spinor and the π interaction written

$$c\boldsymbol{\pi}\cdot\bar{\eta}_4\boldsymbol{\tau}_I\gamma_5\eta_4, \tag{9}$$

if

$$\eta_4 \text{ is } \begin{bmatrix}p\\ n\\ \Xi^0\\ \Xi^-\end{bmatrix}=\begin{bmatrix}I_1\\ I_2\\ I_1\\ I_2\end{bmatrix}.$$

This corresponds to the assumption that the Ξ and N have the same coupling with the π's, and is merely presented to indicate that all baryon interactions may be written in terms of "simple" spinor products.

The K-baryon interactions in the form given by Eq. (7) are suggestive of a Goldhaber-type model for the baryons. This is, of course, due to the fact that Σ and Λ have, insofar as isotopic spin space is concerned, been written in terms of combinations of nucleons and K's. If we adopt the philosophy of Goldhaber, the logical course is to try to write all hyperons as combinations of nucleons and K's. Then we could formally consider a product space of strangeness -2. This corresponds to $I\times K\times K$. If this is carried out, two different sets of basis vectors occur. One set is a simple two-component spinor in the space I:

$$K_1K_2\begin{pmatrix}I_1\\ I_2\end{pmatrix}.$$

The K_1K_2 coefficient is present only to indicate that the basis vector correspond to strangeness -2. The second set corresponds to a total isotopic spin $\frac{3}{2}$, and it might be argued that this would correspond to a scattering state rather than to additional particles. In order to get some feeling for the Goldhaber-type model in terms of the interactions given by Eq. (7), we can calculate in fixed-source approximation the energies corresponding to mass splittings of the hyperons. The deficiencies of this type of model are obvious, since it is clear that a fixed-source approximation is unreasonable. However, this type of calculation, which can be carried out exactly with intermediate coupling, can give us some qualitative feeling for the situation.

GOLDHABER MODEL

To take as simple a Hamiltonian as possible, we consider a fixed square source which interacts with pairs of K mesons by means of an isotopic vector interaction. (To obtain a completely relativistic Hamiltonian for the Goldhaber model consistent with the considerations outlined in the previous section, our interaction would be

$$\begin{aligned}\mathcal{H}&=g_1\bar{\eta}_1\eta_1+g_2\bar{\eta}_1\boldsymbol{\tau}_I\cdot\boldsymbol{\tau}_K\eta_1\\ &=g_1\bar{\psi}_N\psi_N\bar{\phi}\phi+g_2\bar{\psi}_N\boldsymbol{\tau}_N\psi_N\cdot\bar{\phi}\boldsymbol{\tau}\phi.\end{aligned}$$

As usual, the fixed source approximation is obtained by replacing $\bar{\psi}\psi$ by U: then the first term simply contributes to the K mass and we will consider it no

[9] Y. Shimamoto, Phys. Rev. Letters **1**, 463 (1958).

further.) To simplify further, we will consider a separable interaction

$$H=\int(\bar{\pi}\pi+\bar{\phi}\omega^2\phi)d^3x+\frac{g^2}{M}\boldsymbol{\tau}^N\cdot\int U\bar{\phi}d^3x\,\boldsymbol{\tau}\int U\phi d^3x, \quad (10)$$

where ϕ is a spinor in isotopic spin space. The ϕ, π satisfy the commutation relations

$$[\phi_\alpha(x),\pi_\beta(x)]=[\bar{\phi}_\alpha(x),\bar{\pi}_\beta(x')]=i\delta_{\alpha\beta}\delta(x-x'), \quad (11)$$

with all other commutators 0.

Because of the pair interaction, the dominant interaction is an S-wave one between the K meson and nucleon. The fixed-source assumptions restricts us exclusively to S waves.

The constants of the motion are, aside from energy, the total isotopic spin, $\mathbf{T}$,

$$\mathbf{T}=-\tfrac{1}{2}i\int(\bar{\pi}\boldsymbol{\tau}\phi-\bar{\phi}\boldsymbol{\tau}\pi)d^3x, \quad (12)$$

and strangeness, S, given by

$$S=-i\int(\bar{\pi}\phi-\bar{\phi}\pi)d^3x. \quad (13)$$

We are principally interested in obtaining the bound states of this system. We will calculate in intermediate-coupling theory:[10,11] i.e., in any normal product of creation and annihilation operators, any annihilation operator $a(k)$ is replaced by $f(k)a$ and $a^\dagger(k)$ by $f(k)a^\dagger$ and the variational principle is applied to obtain the best wave function $f(k)$. This is equivalent to a Hartree-Fock procedure. (That this procedure is valid for a pair Hamiltonian is not at all obvious: however, it has been investigated by Drachman[12] for the scalar pair theory and found to give excellent results.) The Hamiltonian so obtained will then be exactly diagonalized.

The reduced space Hamiltonian we must consider is, thus,

$$H_R=\Omega(\bar{\pi}^0\pi^0+\bar{\phi}^0\phi^0)+(g^2/2M)U^2\boldsymbol{\tau}^N\cdot\bar{\phi}^0\boldsymbol{\tau}\phi^0, \quad (14)$$

where

$$\Omega=\frac{1}{(2\pi)^3}\int d^3k\,\omega_kf^2(k), \quad (15)$$

$$U=\frac{1}{(2\pi)^3}\int d^3k\frac{u(k)}{\omega_k^{\frac{1}{2}}}f(k), \quad (16)$$

and $f(k)$ satisfies

$$f(k)=-\frac{g^2}{M}\frac{u(k)}{\omega_k^{\frac{1}{2}}}\frac{U}{\omega_k+\lambda}\frac{\langle\boldsymbol{\tau}^N\cdot\bar{\phi}^0\boldsymbol{\tau}\phi^0\rangle}{\langle\bar{\pi}^0\pi^0+\bar{\phi}^0\phi^0\rangle}. \quad (17)$$

Following Wentzel,[13] we introduce four real variables instead of the two complex variables:

$$\phi_+{}^0=(1/\sqrt{2})(x_1+ix_2), \quad \phi_-{}^0=(1/\sqrt{2})(x_3+ix_4), \quad (18)$$

with

$$\left.\begin{aligned}x_1&=r\cos\varphi\cos\theta_1, & x_2&=-r\cos\varphi\sin\theta_1,\\ x_3&=r\sin\varphi\cos\theta_2, & x_4&=-r\sin\varphi\sin\theta_2.\end{aligned}\right\} \quad (19)$$

Then,

$$H_R{}^0=\tfrac{1}{2}\Omega[\sum p_i{}^2+\sum x_1{}^2], \quad (20)$$

and the bound K field behaves like a 4-dimensional harmonic oscillator.

In Appendix A, we calculate the 4-dimensional Laplacian in these coordinates and find that

$$\sum p_1{}^2=-\frac{1}{r^3}\frac{\partial}{\partial r}r^3\frac{\partial}{\partial r}-\frac{1}{r^2}\left[\frac{\partial^2}{\partial\phi^2}+2\cot2\varphi\frac{\partial}{\partial\varphi}+\frac{1}{\cos^2\varphi}\frac{\partial^2}{\partial\theta_1{}^2}+\frac{1}{\sin^2\varphi}\frac{\partial^2}{d\theta_2{}^2}\right]. \quad (21)$$

The meaning of the angular part becomes more obvious if we change variables:

$$\begin{gathered}\theta=2\varphi,\\ \varphi_+=\theta_1+\theta_2, \quad \varphi_-=\theta_1-\theta_2.\end{gathered} \quad (22)$$

Then,

$$\sum p_1{}^2=-\frac{1}{r^3}\frac{\partial}{\partial r}r^3\frac{\partial}{\partial r}-\frac{4}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\left\{\frac{\partial^2}{\partial\varphi_+{}^2}-2\cos\theta\frac{\partial^2}{\partial\varphi_+\partial\varphi_-}\right\}\right]. \quad (23)$$

As shown in Appendix B, since

$$\mathbf{T}_K=-i\left[\bar{\pi}^0\frac{\boldsymbol{\tau}}{2}\phi^0-\bar{\phi}^0\frac{\boldsymbol{\tau}}{2}\pi^0\right], \quad (24)$$

$T_K{}^2$ is given by

$$T_K{}^2=-\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\times\left[\frac{\partial^2}{\partial\varphi_+{}^2}+\frac{\partial^2}{\partial\varphi_-{}^2}-2\cos\theta\frac{\partial^2}{\partial\varphi_+\partial\varphi_-}\right]\right\}, \quad (25)$$

and

$$T_3=\frac{1}{i}\frac{\partial}{\partial\varphi_-}, \quad (26)$$

$$S=\frac{2}{i}\frac{\partial}{\partial\varphi_+}. \quad (27)$$

[10] S. Tomanaga, Progr. Theoret. Phys. (Kyoto) **2**, 6 (1947).
[11] T. D. Lee and R. Christian, Phys. Rev. **94**, 1760 (1954).
[12] R. J. Drachman, Phys. Rev. **109**, 996 (1958).
[13] G. Wentzel, Helv. Phys. Acta **30**, 135 (1957).

Hence, the angular part is the isotopic spin of the bound K-meson field: it behaves like a symmetric top whose two 3-components are the two conserved quantities, the 3-component of isotopic spin and strangeness. From Edmonds,[14] the normalized angular wave functions are

$$\psi_{j,m,n}=\left(\frac{2j+1}{8\pi^2}\right)^{\frac{1}{2}}\mathfrak{D}_{m,n}{}^{j}(\varphi_+,\theta,\varphi_-). \quad (28)$$

We can write ϕ^0 in a form from which its matrix elements can be immediately determined:

$$\begin{aligned}\phi^0&=\frac{r}{\sqrt{2}}\begin{pmatrix}e^{-\frac{1}{2}i\varphi_+}\cos\frac{1}{2}\theta e^{-\frac{1}{2}i\varphi_-}\\ e^{-\frac{1}{2}i\varphi_+}\sin\frac{1}{2}\theta e^{-\frac{1}{2}i\varphi_-}\end{pmatrix}\\ &=\frac{r}{\sqrt{2}}\begin{pmatrix}\mathfrak{D}_{-\frac{1}{2},-\frac{1}{2}}{}^{\frac{1}{2}}(\varphi_+,\theta,\varphi_-)\\ -\mathfrak{D}_{-\frac{1}{2},\frac{1}{2}}{}^{\frac{1}{2}}(\varphi_+,\theta,\varphi_-)\end{pmatrix}.\end{aligned} \quad (29)$$

Because of the vector nature of the interaction, we can write the wave function in the following form:

$$\Psi_{i,s,i_3}=\frac{u_+(r)}{r^{\frac{3}{2}}}\mathcal{Y}^i{}_{i+1,\frac{1}{2},i_3,s}+\frac{u_-(r)}{r^{\frac{3}{2}}}\mathcal{Y}^i{}_{i-1,\frac{1}{2},i_3,s}, \quad (30)$$

where

$$\mathcal{Y}^i{}_{i\pm1,\frac{1}{2},i_3,s}=\sum_n\langle i,\,i_3,\,i\pm1,\,\tfrac{1}{2}\,|\,i\pm1,\,i_3-n,\,\tfrac{1}{2},\,n\rangle\times\psi_{i,s/2,i_3-n}\chi_n. \quad (31)$$

The eigenvalue equations are

$$H_R\Psi_{i,s,i_3}=E_{i,s}\Psi_{i,s,i_3}, \quad (32)$$

or

$$\begin{aligned}-\frac{1}{2}\frac{d^2u_+}{dr^2}&+\frac{2}{r^2}[(i+\tfrac{1}{2})(i+\tfrac{3}{2})+\tfrac{3}{16}]u_+\\ &+\frac{1}{2}\left[1-\frac{g^2}{M}\frac{U^2}{\Omega}\frac{s}{2i+1}\right]r^2u_+\\ &-\frac{1}{2}\frac{g^2}{M}\frac{U^2}{\Omega}\frac{[(2i+1)^2-s^2]^{\frac{1}{2}}}{2i+1}r^2u_-=\frac{E_{i,s}}{\Omega}u_+,\end{aligned} \quad (33)$$

$$\begin{aligned}-\frac{1}{2}\frac{d^2u_-}{dr^2}&+\frac{2}{r^2}[(i+\tfrac{1}{2})(i-\tfrac{1}{2})+\tfrac{3}{16}]u_-\\ &+\frac{1}{2}\left[1+\frac{g^2}{M}\frac{U^2}{\Omega}\frac{s}{2i+1}\right]r^2u_-\\ &-\frac{1}{2}\frac{g^2}{M}\frac{U^2}{\Omega}\frac{[(2i+1)^2-s^2]^{\frac{1}{2}}}{2i+1}r^2u_+=\frac{E_{i,s}}{\Omega}u_-.\end{aligned} \quad (34)$$

Note that s varies between $-(2i+1)\leqq s\leqq(2i+1)$ by steps of two and that for the maximum value of s consistent with a given i, i.e., $s=\pm(2i+1)$, the equations decouple since only $T_K=i+\frac{1}{2}$ can give this value of s.

There are no solutions satisfying the criterion of square integrability for $s\neq\pm(2i+1)$. For $s=\pm(2i+1)$, if g is sufficiently large, i.e.,

$$\frac{g^2}{M}\frac{U^2}{\Omega}>1,$$

there are no solutions for $s=(2i+1)$. Hence

$$E_{i,-(2i+1)}=(2i+3)[\Omega^2+(g^2/M)U^2]^{\frac{1}{2}}. \quad (35)$$

Estimating the value of g necessary for the disappearance of positive-strangeness states, we find (with M equal to the K mass)

$$g^2\approx10/3.$$

The Λ corresponds to an $s=-1$, $i=0$ state, the Ξ to the $s=-2$, $i=\frac{1}{2}$ state. Then, the $\Xi-\Lambda$ mass difference is

$$m_\Xi-m_\Lambda=[\Omega^2+(g^2/M)U^2]^{\frac{1}{2}}\cong300\text{ Mev}.$$

The above result is obviously much too large; however, the π-mesonic effects have not been taken into account and these would be expected to increase the energy of the Λ (since the interaction with π's is quadratic) whereas it would depress the Ξ.

We can now estimate the magnetic moment of the Λ^0. The K mesons will not contribute since they are in S states relative to the nucleon. Thus, the magnetic moment operator is

$$\mu=\mu_p\left(\frac{1+\tau_3}{2}\right)+\mu_n\left(\frac{1-\tau_3}{2}\right).$$

Then,

$$\mu=\tfrac{1}{2}(\mu_p+\mu_n)\cong\tfrac{1}{2}\text{nm},$$

which is quite large compared with most previous estimates.

CONCLUSION

The work presented here is intended to point out that there is in fact no valid reason for assuming that the $S=-1$ baryons cannot be coupled identically to all other particles without introducing an additional symmetry, and to examine the possibility of introducing only a single strange field. The previous section indicates that we can qualitatively understand the fact that only hyperons with negative strangeness exist. The model chosen here gives rise to to two particles corresponding to Ξ and Λ. However, if the neglected π's are bound, an $s=-1$, $T=1$ multiplet corresponding to the Σ might be introduced. The presence of recoil, of course, will shift the levels and be responsible for the existence of the unbound states, the free nucleon, and K. At present we are trying to formulate quantitatively the effect of the pion field and of recoil. This is, of

[14] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), Chap. 4.

course, difficult. One possible approach is to introduce the pion effects in strong-coupling approximation and to treat recoil as a perturbation.

It is worth repeating that this model is a qualitative one. There is no guarantee that the K and π effects can be discussed separately. In fact, in the exact treatment the bound states must arise from the combined fields. No direct $K\pi$ interaction has been introduced so the $K-\pi$ coupling takes place solely through virtual baryons. If the predictions of this model agree with experiment, it should still not be regarded as a true description of the hyperons; rather we must attempt to understand the factors which lead to the success of the model. In a subsequent paper a more quantitative treatment will be carried out. In particular, the effects of the pion field and of recoil will be introduced.

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APPENDIX A

The line element in the 4-dimensional space is given by

$$dS^2=dr^2+r^2d\varphi^2+r^2\cos^2\varphi d\theta_1^2+r^2\sin^2\varphi d\theta_2^2 \equiv \sum h_i^2 d\xi_i^2 \quad \text{(A.1)}$$

and, thus,

$$h_1=1,\quad h_2=r,\quad h_3=r\cos\varphi,\quad h_3=r\sin\varphi. \quad \text{(A2)}$$

In general,[15]

$$\sum_i \frac{\partial^2}{\partial x_i^2}=\frac{1}{\prod_l h_l}\sum_j \frac{\partial}{\partial\xi_j}\frac{\prod_k h_k}{h_j^2}\frac{\partial}{\partial\xi_j}$$
$$=\frac{1}{r^3}\frac{\partial}{\partial r}r^3\frac{\partial}{\partial r}+\frac{1}{r^2}\left\{\frac{1}{\sin 2\varphi}\frac{\partial}{\partial\varphi}\sin 2\varphi\frac{\partial}{\partial\varphi}+\frac{1}{\cos^2\varphi}\frac{\partial^2}{\partial\theta_1^2}+\frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta_1^2}\right\}. \quad \text{(A3)}$$

[15] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, Chap. 5.

APPENDIX B

We now derive the expressions for the strangeness and isotopic spin of the K field in terms of the angular variables,

$$S=-i(\bar{\pi}^0\phi^0-\bar{\phi}^0\pi^0). \quad \text{(B1)}$$

Then

$$S=-\tfrac{1}{2}[(p_1-ip_2)(x_1+ix_2)+(p_3-ip_4)(x_3+ix_4)-(p_1+ip_2)(x_1-ix_2)-(p_3+ip_4)(x_3-ix_4)]$$
$$=-[(x_1p_2-x_2p_1)+(x_3p_4-x_4p_3)]$$
$$=\frac{1}{i}\left(\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}\right). \quad \text{(B2)}$$

Introducing $\varphi_\pm$, then

$$S=\frac{2}{i}\frac{\partial}{\partial\varphi_+}, \quad \text{(B3)}$$

Now,

$$\mathbf{T}=-\tfrac{1}{2}i\left[\bar{\pi}^0\boldsymbol{\tau}\phi^0-\bar{\phi}^0\boldsymbol{\tau}\pi^0\right], \quad \text{(B4)}$$

and introducing angular variables, we have

$$T_3=\frac{1}{2i}\left(\frac{\partial}{\partial\theta_1}-\frac{\partial}{\partial\theta_2}\right)=\frac{1}{i}\frac{\partial}{\partial\varphi_-} \quad \text{(B5)}$$

$$T_1=\frac{1}{2i}\left[-\sin(\theta_1-\theta_2)\frac{\partial}{\partial\varphi}+\cos(\theta_1-\theta_2)\times\left(\tan\varphi\frac{\partial}{\partial\theta_1}+\cot\varphi\frac{\partial}{\partial\theta_2}\right)\right]$$
$$=\frac{1}{i}\left[-\sin\varphi_-\frac{\partial}{\partial\theta}+\cos\varphi_-\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi_+}-\cot\theta\frac{\partial}{\partial\varphi_-}\right)\right] \quad \text{(B6)}$$

$$T_2=\frac{1}{i}\left[\cos\varphi_-\frac{\partial}{\partial\theta}+\sin\varphi_-\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi_+}-\cot\theta\frac{\partial}{\partial\varphi_-}\right)\right]. \quad \text{(B7)}$$

Comparing with Edmonds,[14] we see that the three Euler angles for rigid rotator correspond exactly to θ, φ_-, φ_+:

$$\alpha=\varphi_-,\quad \beta=\theta,\quad \gamma=\varphi_+. \quad \text{(B8)}$$

Hence,

$$T_K^2=-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi_+^2}+\frac{\partial^2}{\partial\varphi_-^2}-2\cos\theta\frac{\partial^2}{\partial\varphi_+\partial\varphi_-}\right)\right]. \quad \text{(B9)}$$