Ground and Excited States of a Many-Body System with Singular Interaction

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(Received December 14, 1959)

The procedure to get the ground-state energy and excitation spectrum by looking at the normal modes for the simple excitations of a many-body system is extended and applied to a system with singular interaction. It is shown that the singular two-body interaction can be consistently replaced by the so-called reaction matrix in the equations of motion and in the expressions for the energies of ground and excited states.

INTRODUCTION

A PROCEDURE to investigate directly the equation of motion of an operator which represents a simple excitation of a many-body system to obtain knowledge about the ground-state energy and the character of a simple excitation of the system[1–3] is developed further to show that the "bare" two-body interaction potential in the equation of motion can be replaced by the "reaction" matrix without a perturbation expansion in the coupling constant. For a system of Bose particles, we obtain a phonon spectrum and ground-state energy similar to that given in an earlier work,[4] and for a system of Fermi particles, we derive the ground-state energy[5] and the equation of motion for pair excitation[6] in terms of the reaction matrix. In both cases our equations become sets of self-consistent (nonlinear) equations.

MANY-FERMION SYSTEM

The system of many fermions we consider in this section is composed of N fermions with two-body interactions with its total Hamiltonian given by[7]

$$H_T=H_{\text{total}}=\sum_p \epsilon_p C_p^* C_p + \sum_{p,q,r,s} C_p^* C_q^* \tfrac{1}{2} V_{p,q;s,r} C_r C_s, \tag{1}$$

$$V_{p,q;s,r}=\int\int \psi_p^*(\mathbf{x})\psi_q^*(\mathbf{x}') \times V(|\mathbf{x}-\mathbf{x}'|)\psi_s(\mathbf{x})\psi_r(\mathbf{x}')d^3x d^3x',$$

where $\epsilon_p=p^2/2m$ is the kinetic energy and C_p^*, $\psi_p(x)$ represent a creation operator and a plane wave, respectively, in a large volume (Ω) with momentum, spin, etc., described by p. The usual separation into "particle" and "hole" operators for C_p^* is defined as

$$\begin{aligned} C_p^* &= a_p^*, \quad p>p_F, \\ &= b_p, \quad p<p_F, \end{aligned} \tag{2}$$

where a^*, b^* are the creation operators for particles and holes, and $p_F \equiv$ Fermi momentum. However, it is more convenient to separate C_p^* by the following definition:

$$\begin{aligned} C_p^* &= a_p^*, \quad \omega_p>\mu, \\ &= b_p, \quad \omega_p<\mu, \end{aligned} \tag{3}$$

where ω_p is the "true" one-particle energy [defined in (4)], and $\mu=dE_N/dN$. ($E_N \equiv$ lowest energy of the N-particle system, and $N \equiv$ total number of particles.) Here we have introduced two unknowns, ω_p and μ, the equation for which we shall establish later. The reason we use (3) in this section is as follows: The "true" one-particle energy is the energy of the one-particle eigenmode which is defined by the following equation of motion;

$$[A_p^*, H_T]_- = [(a_p^*+\cdots), H_T]_- = -\omega_p(a_p^*+\cdots) = -\omega_p A_p^*, \tag{4}$$

where $\cdots$ represents an ordered operator (a combination of operators a^*, a, b^*, b, and the other normal mode operators defined as in (7) below so arranged that the creation operator stands always to the left of the annihilation operator) which is supplied so as to satisfy (4), and ω_p is the so-called "true" one-particle energy. From (4), we get, denoting the N particle *lowest energy state* and *energy* by $|N\rangle$ and E_N, respectively,

$$H_T A_p^* |N\rangle = (\omega_p + E_N) A_p^* |N\rangle. \tag{5}$$

From the definition of μ and the manner of separation of C into a and b in (3), $\omega_p > dE_N/dN$; hence $\omega_p + E_N > (dE_N/dN) + E_N = E_{N+1}$ (for very large N), and $A_p^*|N\rangle$ represents an excited state with energy

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[2] K. Sawada, Phys. Rev. **106**, 372 (1957); K. Sawada, K. A. Brueckner, N. Fukuda, and R. Brout, Phys. Rev. **108**, 507 (1957).

[3] P. W. Anderson, Phys. Rev. **110**, 985 (1958); **112**, 1900 (1958).

[4] K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117 and 1128 (1957).

[5] The expression we obtained here is analogous to that of K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958). Expressions (20) and (24) contain effects of hole-hole scattering but do not contain the effect of simultaneous excitations of many particles [which appears in the expression of the R matrix as ΔE in the denominator (see this reference)].

[6] The equations of motion we got here correspond to the one used by A. E. Glassgold, W. Heckrotte, and K. M. Watson, Ann. Phys. **6**, 1 (1959). See in this connection K. Sawada and R. M. Rockmore, Phys. Rev. **116**, 1618 (1959).

[7] To avoid complications, we assume the potential V to be such that it does not produce the effect of recombinations of particles or holes such as discussed by J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957); and N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity, 1958* (Consultant's Bureau, Inc., New York, 1959); in other words, we assume that the interaction does not cause a drastic change in the Fermi sphere. Throughout this paper, we assume a spherically symmetric potential and use the momentum conservation law. [In (1), $\mathbf{p}+\mathbf{q}=\mathbf{r}+\mathbf{s}$.]

ω_p+E_N, (particle number $N+1$). For $A_p|N\rangle$ we have similarly

$$H_T A_p|N\rangle=(-\omega_p+E_N)A_p|N\rangle, \tag{6}$$

and $E_N-\omega_p<E_N-(dE_N/dN)=E_{N-1}$; hence $A_p|N\rangle=0$ because $A_p|N\rangle$ belongs to the state with particle number $N-1$ and so its energy cannot be smaller than E_{N-1}. In this sense $A_p^*(A_p)$ is a creation (annihilation) operator.

The definition (3) has merit in that $a_p^*(b_p)$ is the zeroth order operator for $A_p^*(B_p)$ with exact energy and we can work with true excitations; of course, the equation for ω_p will become an intrinsic equation and some complication arises from this aspect. Similar to (4), we can define two, three, $\cdots$-particle normal modes if we can find an operator α_n^* which satisfies

$$[\alpha_n^*,H_T]_-=-\omega_n\alpha_n^*. \tag{7}$$

If α_n^* contains products of two, three, $\cdots C^*$'s [in (3)] (and an equal number of C^*'s and C's) we consider α_n^* as referring to two, three, $\cdots$-particle normal modes. Some of these modes can, of course, be equivalant to products of one-particle modes (the author has not tried to enter into the question of the independence of the modes defined in this way[8]; the independence problem does not affect the following argument). If $\omega_n>m\mu$, $(\mu=dE_N/dN)$ for an m-particle mode, a discussion similar to that given above leads to the conclusion that $\alpha_n^*|N\rangle$ represents an excited state with particle number $m+N$, and if $\omega_n<m\mu$, $\alpha_n^*|N\rangle=0$. If α_n^* contains only the same number of C^*'s and C's (in combination with a, b), for $\omega_n>0$, $\alpha_n^*|N\rangle$ represents an excited state of the N-particle system and, for $\omega_n<0$, $\alpha_n^*|N\rangle=0$. (For $\alpha_n|N\rangle$ all arguments are reversed.)

Having found sufficient normal modes, we can expand the operator representing the total number of particles into normal modes;

$$\begin{aligned}N&=\sum_p C_p^*C_p=\sum_p(a_p^*+b_p)(a_p+b_p^*)\\&=\sum_p[(A_p^*+B_p)(A_p+B_p^*)+\cdots],\end{aligned} \tag{8}$$

where $\cdots$ represents combinations of normal-mode operators. Since the total number operator commutes with H_T, if we operate with (8) on the wave function $|N\rangle$ we get the expression

$$N=\sum_{(\omega_p<\mu)}1+\cdots, \tag{9}$$

where $\cdots$ is the c-number expression obtained by shifting the normal mode from left to right or vice versa ("ordering") in order to use the equation $\alpha_n^*|N\rangle=0$ for $\omega_n<m\mu$ (m: m-particle mode). To reach (9), we have assumed that a or b in (8) can be expanded into normal modes. This assumption is completely reasonable since Eqs. (4) and (7) are more numerous than the equation of motion for the free particle (which can form a complete set).[8]

The separation (3) together with (9) provides the fundamental equation with which to proceed. The definition $\mu=dE_N/dN$, when combined with (9), essentially determines E_N as a function of particle number N, and this is a direct consequence of the fact that we have used in "ordering" the normal mode the assumption that E_N is the lowest energy state of the N-particle system.

Our Hamiltonian can now be split into the following form:

$$\begin{aligned}&H_T=H_0+H_I,\\&H_0=\sum_p\omega_p a_p^*a_p-\sum_p\omega_p b_p^*b_p+\sum_{(\omega_p<\mu)}\epsilon_p,\\&H_I=\sum_{p,q,r,s}(a_p^*+b_p)(a_q^*+b_q)\\&\qquad\times\tfrac{1}{2}V_{p,q;s,r}(a_r+b_r^*)(a_s+b_s^*)\\&\qquad+\sum_p(\epsilon_p-\omega_p)a_p^*a_p-\sum_p(\epsilon_p-\omega_p)b_p^*b_p.\end{aligned} \tag{10}$$

The case where the potential V is given by the Coulomb repulsion between particles was discussed earlier[9,1,2] [starting from definition (2)] and it was found that at high density the pair mode (a^*b^*, ba in its lowest representation) plays an important role. In this section, we shall examine the case where V contains hard-sphere interactions. Obviously, in this case, the zeroth order two-particle mode a^*a^* can hardly represent the creation of particles in this mode because the correlation due to scattering with hard-sphere interaction is too big, and we should first of all set up equations for normal modes of two-particle (or two-hole, bb in zeroth order) states and solve them without perturbation expansions. Of course, since we cannot set up exact eigenmodes, this being simply too difficult, we should content ourselves with approximate eigenmodes of the form (denoting such an approximate mode by X^*)

$$[X^*,H_T]_-=-\omega X^*+Y.$$

If $Y=0$, we get $H_TX^*|N\rangle=(\omega+E_N)X^*|N\rangle$ ($|N\rangle$: N-particle ground state with energy E_N); $X^*|N\rangle$ represents the state with energy $\omega+E_N$. Naturally, the eigenmode X^* contains a^*a^*, bb for scattering and also would contain other types of excitations attached to it (just as the electron interacting with a photon accompanies an attached cloud of photons). We should expect the approximate eigenmode X^* to be "good" if the "interaction" Y contains small interaction. In the present case, we show, in the course of constructing X^*, that it contains only the "reaction" matrix which is less singular than the original V.

(a) Scattering Eigenmodes

To find the approximate scattering eigenmode, we first construct the equation for it; using (10) we have,

[8] K. Nishijima, Progr. Theoret. Phys. **10**, 549 (1953), contains a discussion on this point in field theory.

[9] M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106**, 364 (1957) and reference 2.

$$[\sum_{p,q}(\psi_{p,q}{}^n a_p{}^* a_q{}^* + \chi_{p,q}{}^n b_p b_q + 2\xi_{p,q}{}^n a_p{}^* b_q),\ H_T]_-$$
$$= \sum_{p,q} a_p{}^* a_q{}^* \{ -(\omega_p+\omega_q)\psi_{p,q}{}^n - \sum_{p',q'} v_{p,q;p',q'}(\psi_{p',q'}{}^n - \chi_{p',q'}{}^n)\} + \sum_{p,q} b_p b_q \{ -(\omega_p+\omega_q)\chi_{p,q}{}^n$$
$$- \sum_{p',q'} v_{p,q;p',q'}(\psi_{p',q'}{}^n - \chi_{p',q'}{}^n)\} + 2\sum_{p,q} a_p{}^* b_q \{ -(\omega_p+\omega_q)\xi_{p,q}{}^n - \sum_{p',q'} v_{p,q;p',q'}(\psi_{p',q'}{}^n - \chi_{p',q'}{}^n)\}$$
$$+ [\sum_{p,q}(\omega_p - \tilde{\epsilon}_p + \omega_q - \tilde{\epsilon}_q)(a_p{}^* a_q{}^* \psi_{p,q}{}^n + b_p b_q \chi_{p,q}{}^n + 2a_p{}^* b_q \xi_{p,q}{}^n) - 2\sum_{p,q}(\psi_{p,q}{}^n + \xi_{p,q}{}^n)$$
$$\times \sum_{q'}(F_{p,q'} a_p{}^* a_{q'} + a_p{}^* b_{q'}{}^* F_{q,q'}) - 2\sum_{p,q}(\xi_{p,q}{}^n + \chi_{p,q}{}^n)\sum_{q'}(F_{p,q'} a_{q'} b_q + b_{q'}{}^* b_q F_{p,q'})], \quad (11)$$

where

$$\psi_{p,q}{}^n = -\psi_{q,p}{}^n, \quad \chi_{p,q}{}^n = -\chi_{q,p}{}^n,$$
$$\tilde{\epsilon}_p = \epsilon_p + 2\sum_{(\omega_q<\mu)_q} v_{p,q;p,q},$$
$$F_{r,s} = \sum_{p,q}(a_p{}^* a_q{}^* + 2a_p{}^* b_q + b_p b_q) v_{p,q;r,s},$$
$$v_{i,j;k,l} = \tfrac{1}{4}(V_{i,j;k,l} - V_{i,j;l,k} - V_{j,i;k,l} + V_{j,i;l,k}).$$

By setting (11) equal to the following quantity $(-\omega X^*)$ (representing the quantity in the square bracket by Y^n),

$$(11) = -\omega_n \sum_{p,q}(\psi_{p,q}{}^n a_p{}^* a_q{}^* + \chi_{p,q}{}^n b_p b_q + 2\xi_{p,q}{}^n a_p{}^* b_q) + Y^n, \quad (11')$$

and constructing equations for ψ, χ, and ξ, we get the approximate normal mode $X_n{}^*$ and interaction Y^n.

$$\{\omega_n - (\omega_p+\omega_q)\}\psi_{p,q}{}^n = \sum_{p',q'} v_{p,q;p',q'}(\psi_{p',q'}{}^n - \chi_{p',q'}{}^n),$$
$$\{\omega_n - (\omega_p+\omega_q)\}\chi_{p,q}{}^n = \sum_{p',q'} v_{p,q;p',q'}(\psi_{p',q'}{}^n - \chi_{p',q'}{}^n), \quad (12)$$
$$\{\omega_n - (\omega_p+\omega_q)\}\xi_{p,q}{}^n = \sum_{p',q'} v_{p,q;p',q'}(\psi_{p',q'}{}^n - \chi_{p',q'}{}^n).$$

We can see from these equations that ψ and χ are coupled to each other to give the eigenvalue equation but ξ is determined solely by ψ and χ, and hence ξ (or operator a^*b) represents the "attached" field in our scattering mode. The energy ω_p in the above equations is not determined yet and should be determined later.

The first two equations of (12) can be written compactly as follows; defining

$$\psi_{p,q}{}^n = \tfrac{1}{2}[1+\theta(p,q;\mu)]\Psi_{p,q}{}^n,$$
$$\chi_{p,q}{}^n = \tfrac{1}{2}[1-\theta(p,q;\mu)]\Psi_{p,q}{}^n,$$
$$\theta(i,j;\mu) = \begin{cases} 1 & \text{if } \omega_i \text{ and } \omega_j > \mu, \\ -1 & \text{if } \omega_i \text{ and } \omega_j < \mu, \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$\{\omega_n - (\omega_p+\omega_q)\}\Psi_{p,q}{}^n = \sum_{p',q'} v_{p,q;p',q'}\theta(p',q';\mu)\Psi_{p',q'}{}^n, \quad (13)$$

and the orthonormality condition

$$\sum_{p,q}\Psi_{p,q}{}^{m*}\theta(p,q;\mu)\Psi_{p,q}{}^n = \delta_{m,n}\theta(\omega_n;2\mu),$$
$$\theta(\omega_n;2\mu) = \begin{cases} 1 & \text{if } \omega_n > 2\mu, \\ -1 & \text{if } \omega_n < 2\mu. \end{cases}$$

The solutions of Eq. (13) have both types of solution with $\omega_n > 2\mu$ and $\omega_n < 2\mu$. To proceed, for simplicity we assume V is such that it gives for (13) only the continuous eigenvalue ω_n. [See also footnote for Eq. (1).]

Following the considerations concerning Eq. (7), we should write our approximate scattering eigenmode as follows: (we here do not specify the boundary condition on Ψ^n)

$$(X_n{}^* \equiv) \sum_{p,q}\Psi_{p,q}{}^n(a_p{}^* a_q{}^* + b_p b_q) + 2\sum_{p,q} a_p{}^* b_q \times \sum_{p',q'} v_{p,q;p',q'}\theta(p',q';\mu)\Psi_{p',q'}{}^n / \{\omega_n - (\omega_p+\omega_q)\}$$
$$= \alpha_n{}^* \quad \text{for} \quad \omega_n > 2\mu,$$
$$= \beta_n \quad \text{for} \quad \omega_n < 2\mu, \quad (14)$$

where as usual the starred operator means creation and the nonstarred operator means annihilation of the corresponding eigenmode.

By using a completeness relation for the wave function Ψ^n,

$$\sum_n \theta(p,q;\mu)\Psi_{p,q}{}^n\theta(\omega_n;2\mu)\Psi_{p',q'}{}^{n*} = \tfrac{1}{2}(\delta_{p,p'}\delta_{q,q'} - \delta_{q,p'}\delta_{p,q'}), \quad (15)$$

[note that this relation only applies for pq, $p'q'$ which satisfies ω_p and ω_q (or $\omega_{p'}$ and $\omega_{q'}$) both larger or smaller than μ simultaneously], we can obtain the expansion of a^*a^* and bb into (approximate) eigenmodes $X_n{}^*$;

$$\left.\begin{matrix} a_p{}^* a_q{}^* \\ b_p b_q \end{matrix}\right\} = \sum_n \Psi_{p,q}{}^{n*}(\alpha_n{}^* - \beta_n)\theta(p,q;\mu)$$
$$- 2\sum_{p',q'} a_{p'}{}^* b_{q'} \times \sum_n \bar{R}_{p'q',}{}^n \Psi_{pq}{}^n \theta(p,q;\mu)\theta(\omega_n;2\mu)/ \{\omega_n - (\omega_{p'}+\omega_{q'})\}, \quad (16)$$

where Eq. (16) gives a^*a^* for ω_p and $\omega_q > \mu$, and bb for ω_p and $\omega_q < \mu$, and in $\alpha_n(\beta_n)$ the subscript n means states $\omega_n > 2\mu (< 2\mu)$, and

$$\bar{R}_{pq,}{}^n = \sum_{p',q'} v_{pq;p'q'}\theta(p',q';\mu)\Psi_{p',q'}{}^n = (\bar{R}_{,pq}{}^n)^*; \quad (17)$$

$\bar{R}_{pq,}{}^n$ is a reaction matrix merging into the outside of its defining space [the defining space is ω_p and ω_q both $>\mu$ or both $<\mu$, but in (16) $\bar{R}_{p'q',}{}^n$ has $\omega_{p'} > \mu$ and $\omega_{q'} < \mu$; hence we put a bar on R]. Substituting (16) into $F_{r,s}$ as defined in (11), we have

$$F_{r,s} = \sum_n \bar{R}_{,rs}{}^n(\alpha_n{}^* - \beta_n) + 2\sum_{p,q} a_p{}^* b_q \bar{R}_{,rs}{}^{pq}, \quad (16')$$

where

$$\tilde{R}_{,rs}{}^{pq}=v_{pq;rs}-\sum_n \bar{R}_{pq,}{}^n\theta(\omega_n;2\mu)\bar{R}_{,rs}{}^n/\{\omega_n-(\omega_p+\omega_q)\},\quad(18)$$

(this reduces to the ordinary two-particle scattering R matrix if $\mu\to 0$; one can see this by using completeness for Ψ^n), is the reaction matrix for which both legs may be out of its defining space. Since (16′) shows that $F_{r,s}$ contains α^* and β, we now shift α^* to the extreme left and β to the extreme right in the "interaction" term (Y^n) in (11); the reason for doing so is that our operators α^* and β are much better operators than a^*, a or b^*, b, etc. [namely, when written down by (approximate) one-particle normal modes, a (or a^*) can contain operators which represent creation (or absorption) of particles as well] and since $\alpha^*(\beta)$ represents creation (annihilation) of scattering modes the latter position of the operator guarantees a minimal effect of the scattering mode on the equation.

We obtain in this way;

$$\begin{aligned}\left[\begin{pmatrix}\alpha_n^*\\ \beta_n\end{pmatrix},H_T\right]_-&=-\omega_n\begin{pmatrix}\alpha_n^*\\ \beta_n\end{pmatrix}-2\sum_{p,q}(\psi_{p,q}{}^n+\xi_{p,q}{}^n)\\&\times\sum_{q'}\{\sum_m\alpha_m^*\bar{R}_{,qq'}{}^m(a_p^*a_{q'}+a_p^*b_{q'}^*)\\&-(a_p^*a_{q'}+a_p^*b_{q'}^*)\sum_m\beta_m\bar{R}_{,qq'}{}^m\\&+2\sum_{p'',q''}a_{p''}{}^*a_p^*(a_{q'}b_{q''}+b_{q'}^*b_{q''})\tilde{R}_{,qq'}{}^{p''q''}\}\\&-2\sum_{p,q}(\xi_{p,q}{}^n+\chi_{p,q}{}^n)\\&\times\sum_{q'}\{\sum_m\alpha_m^*\bar{R}_{,pq'}{}^m(a_{q'}b_q+b_{q'}^*b_q)\\&-(a_{q'}b_q+b_{q'}^*b_q)\sum_m\beta_m\bar{R}_{,pq'}{}^m\\&+2\sum_{p'',q''}(a_{p''}{}^*a_{q'}+a_{p''}{}^*b_{q'}^*)b_qb_{q''}\tilde{R}_{,pq'}{}^{p''q''}\},\end{aligned}\quad(19)$$

with an equation for ω_p,[5]

$$\omega_p=\epsilon_p+2\Big[\sum_{(\omega_q<\mu)}{}_q\, v_{p,q;p,q}-\sum_n\sum_q\Big\{\begin{pmatrix}\omega_n>2\mu\\ \omega_q<\mu\end{pmatrix}+\begin{pmatrix}\omega_n<2\mu\\ \omega_q>\mu\end{pmatrix}\Big\}\frac{\bar{R}_{pq,}{}^n\bar{R}_{,pq}{}^n}{\omega_n-(\omega_p+\omega_q)}\Big],\quad(20)$$

or

$$\left.\begin{matrix}\omega_p(>\mu)=\\ \omega_p(<\mu)=\end{matrix}\right\}\epsilon_p+2\Big[\sum_{(\omega_q<\mu)}{}_q\begin{Bmatrix}\tilde{R}_{,pq}{}^{pq}\\ -\bar{R}_{,pq}{}^{pq}\end{Bmatrix}-\sum_n\sum_{(\omega_n<2\mu)}{}_q\frac{\bar{R}_{pq,}{}^n\bar{R}_{,pq}{}^n}{\omega_n-(\omega_p+\omega_q)}\Big].\quad(20')$$

To get (20′) we have used Eq. (18) together with the relation derivable from the completeness relation (15) [multiplying on the right of (15) with $\theta\cdot v$ and using the expression $\Psi^n=\phi^n+(\omega_n-\omega)^{-1}v\cdot\theta\Psi^n$]

$$\bar{R}_{,rs}{}^{pq}=\theta(p,q;\mu)\times\Big(v_{pq;rs}-\sum_n\frac{\bar{R}_{pq,}{}^n\theta(\omega_n;2\mu)\bar{R}_{,rs}{}^n}{\omega_n-(\omega_p+\omega_q)}\Big).\quad(18')$$

(This expression also gives the ordinary reaction matrix for two-body scattering in the limit $\mu\to 0$.) The last two terms of (19) represent the "interaction" between (approximate) normal modes α, β, and other modes (including collective eigenmodes which we shall discuss in the next section). At this point, we should point out that Eq. (19) is exact and can be used to find a more exact eigenmode.

Now, turning to the one-particle mode, we have its equation of motion given by the following:

$$\left[\begin{pmatrix}b_p\\ a_p^*\end{pmatrix},H_T\right]_-=-\epsilon_p\begin{pmatrix}b_p\\ a_p^*\end{pmatrix}-\sum_q F_{p,q}(a_q+b_q^*),\quad(21)$$

with F defined in (11); using (16′) for F and shifting α^* to the extreme left and β to the right, we have for (21),

$$\begin{aligned}\left[\begin{pmatrix}b_p\\ a_p^*\end{pmatrix},H_T\right]_-&=-\omega_p\begin{pmatrix}b_p\\ a_p^*\end{pmatrix}\\&-\sum_q\{\sum_n\alpha_n^*\bar{R}_{,pq}{}^n(a_q+b_q^*)\\&-(a_q+b_q^*)\sum_n\bar{R}_{,pq}{}^n\beta_n\\&+2\sum_{k,l}a_k^*(b_la_q-b_q^*b_l)\tilde{R}_{,pq}{}^{kl}\},\end{aligned}\quad(22)$$

with ω_p given by (20) or (20′). In the approximation which neglects the interaction term in (22), a_p^* (or b_p) represents the normal mode for one particle and hence

$$N|N\rangle=\sum_p(a_p^*+b_p)(a_p+b_p^*)|N\rangle\doteqdot\sum_{(\omega_p<\mu)}{}_p\,1\cdot|N\rangle,$$

or

$$N\doteqdot\sum_{(\omega_p<\mu)}{}_p\,1,\quad(23)$$

which determines μ as a function of N. [Usually this equation gives $\mu=\omega_{p_F}$ where p_F is determined by

$$N=\sum_{(p<p_F)}{}_p\,1.]$$

The ground-state energy in this approximation can be obtained by integrating the relation $dE_N/dN=\mu$, but since this cannot be performed explicitly, we content ourselves here with the result obtained by the following procedure: Writing down the total Hamiltonian (10) using (16′) and shifting the operators α^* and β (and β^*, α) to the left and right, respectively, we get a constant term and sums of operators which have zero expectation values in the ground state (in the approxi-

mation we are considering). The constant term is our ground-state energy,[5]

$$E_N=\sum_{(\omega_p<\mu)}\epsilon_p-\sum_{(\omega_p,\omega_q<\mu)}\bar{R}_{,pq}{}^{pq}+\sum_{p,q}\theta(p,q;\mu)\sum_{(\omega_n<2\mu)}\bar{R}_{pq,}{}^n\bar{R}_{,pq}{}^n/\{\omega_n-(\omega_p+\omega_q)\}. \quad (24)$$

Unfortunately, we cannot find the exact relation $dE_N/dN=\mu$ between (24) and (23), but if one restricts d/dN to operate only on the limitation on momentum sums [namely, $\omega_p<\mu$, ω_p, $\omega_q<\mu$ and $\theta(p,q;\mu)$] one gets $dE_N/dN\doteqdot\mu$; to this extent (24) is the approximation to E_N obtained by integration of equation $dE_N/dN=\mu$ [namely, at low density; there is no guarantee that (24) approximates E_N obtained by integration at moderate or high density].

Equation (20) together with (13) is the self-consistency equation and (24) approximates the ground-state energy.

(b) Pair-Scattering Mode

Turning to the eigenmode which represents pair scattering (particle-hole correlation), we have the equations of motion derived from (10):

$$[a_p{}^*b_q{}^*,H_T]_-=-(\tilde{\epsilon}_p-\tilde{\epsilon}_q)a_p{}^*b_q{}^*-A_{q,p}+\sum_{q'}(a_p{}^*a_{q'}{}^*F_{q,q'}{}^\dagger+a_p{}^*F_{q,q'}{}^\dagger b_{q'}+b_q{}^*b_{q'}{}^*F_{p,q'}+b_q{}^*F_{p,q'}a_{q'}), \quad (25a)$$

$$[a_p{}^*a_q,H_T]_-=-(\tilde{\epsilon}_p-\tilde{\epsilon}_q)a_p{}^*a_q+\sum_{q'}(a_p{}^*a_{q'}{}^*F_{q,q'}{}^\dagger+a_p{}^*F_{q,q'}{}^\dagger b_{q'}-F_{p,q'}a_{q'}a_q-b_{q'}{}^*F_{p,q'}a_q), \quad (25b)$$

$$[b_q{}^*b_p,H_T]_-=-(\tilde{\epsilon}_p-\tilde{\epsilon}_q)b_q{}^*b_p+\sum_{q'}(F_{q,q'}{}^\dagger b_{q'}b_p+a_{q'}{}^*F_{q,q'}{}^\dagger b_p-b_q{}^*b_{q'}{}^*F_{p,q'}-b_q{}^*F_{p,q'}a_{q'}), \quad (25c)$$

where $\tilde{\epsilon}$ and F are defined in (11) and the expansion of F into the approximate scattering eigenmode is given by (16′);

$$A_{q,p}=2\sum_{p',q'}v_{p',q;q',p}(a_{p'}{}^*b_{q'}{}^*+a_{p'}{}^*a_{q'}-b_{q'}{}^*b_{p'}+b_{p'}a_{q'}), \quad (26)$$

and the dagger means complex-conjugated and transposed operator.

After shifting α^*, β^* to left and α, β to right in (25), we arrive at the following equations;

$$[a_p{}^*b_q{}^*,H_T]_-=-(\omega_p-\omega_q)a_p{}^*b_q{}^*-\mathfrak{A}_{q,p}+(M_{q,p}+N_{q,p}), \quad (27a)$$

$$[a_p{}^*a_q,H_T]_-=-(\omega_p-\omega_q)a_p{}^*a_q-\mathfrak{B}_{q,p}+(M_{q,p}-M_{p,q}{}^\dagger), \quad (27b)$$

$$[b_q{}^*b_p,H_T]_-=-(\omega_p-\omega_q)b_q{}^*b_p-\mathfrak{C}_{q,p}+(N_{p,q}{}^\dagger-N_{q,p}), \quad (27c)$$

with

$$\mathfrak{A}_{q,p}=2\sum_{p',q'}\Bigg(v_{p',q;\,q',p}-\sum_{(\omega_n>2\mu)}\frac{\bar{R}_{p'q,}{}^n\bar{R}_{,q'p}{}^n}{\omega_n-(\omega_q+\omega_{p'})}+\sum_{(\omega_n<2\mu)}\frac{\bar{R}_{p'q,}{}^n\bar{R}_{,q'p}{}^n}{\omega_n-(\omega_p+\omega_{q'})}\Bigg)\times(a_{p'}{}^*b_{q'}{}^*+a_{p'}{}^*a_{q'}-b_{q'}{}^*b_{p'}+b_{p'}a_{q'}), \quad (28a)$$

$$\mathfrak{B}_{q,p}=-2\sum_{p',q'}\times\sum_{(\omega_n<2\mu)}\Bigg(\frac{\bar{R}_{p'q,}{}^n\bar{R}_{,q'p}{}^n}{\omega_n-(\omega_q+\omega_{p'})}-\frac{\bar{R}_{p'q,}{}^n\bar{R}_{,q'p}{}^n}{\omega_n-(\omega_p+\omega_{q'})}\Bigg)\times(a_{p'}{}^*b_{q'}{}^*+a_{p'}{}^*a_{q'}-b_{q'}{}^*b_{p'}+b_{p'}a_{q'}), \quad (28b)$$

$$\mathfrak{C}_{q,p}=2\sum_{p',q'}\sum_{(\omega_n>2\mu)}n\ (\text{same as for }\mathfrak{B}), \quad (28c)$$

$$M_{q,p}=\sum_{q'}\{(a_p{}^*a_{q'}{}^*+a_p{}^*b_{q'})\sum_n\bar{R}_{qq',}{}^n\alpha_n-\sum_n\bar{R}_{qq',}{}^n\beta_n{}^*(a_p{}^*a_{q'}{}^*+a_p{}^*b_{q'})-2\sum_{p'',q''}a_p{}^*b_{q''}{}^*(a_{q'}{}^*a_{p''}+b_{q'}a_{p''})\bar{R}_{qq',}{}^{p''q''}\}, \quad (29a)$$

$$N_{q,p}=\sum_{q'}\{\sum_n\bar{R}_{,pq'}{}^n\alpha_n{}^*(b_q{}^*b_{q'}{}^*+b_q{}^*a_{q'})-(b_q{}^*b_{q'}{}^*+b_q{}^*a_{q'})\sum_n\bar{R}_{,pq'}{}^n\beta_n-2\sum_{p'',q''}a_{p''}{}^*b_q{}^*(b_{q''}a_{q'}-b_{q'}{}^*b_{q''})\bar{R}_{,pq'}{}^{p''q''}\}; \quad (29b)$$

ω_p, ω_q are given by Eq. (20). We can see by comparing (26) and (28a) that the interaction v is replaced by a reaction matrix

$$v_{p',q;q',p}-\sum_{(\omega_n>2\mu)}\frac{\bar{R}_{p'q,}{}^n\bar{R}_{,q'p}{}^n}{\omega_n-(\omega_q+\omega_{p'})}+\sum_{(\omega_n<2\mu)}\frac{\bar{R}_{p'q,}{}^n\bar{R}_{,q'p}{}^n}{\omega_n-(\omega_p+\omega_{q'})},$$

(which is in fact the ordinary reaction matrix for scattering if $\mu\to 0$ as can be seen by using the completeness relation for Ψ^n); $\mathfrak{B}$ and $\mathfrak{C}$ represent the rather weak interaction induced by the scattering; in fact $\mathfrak{B}\to 0$ for $\mu\to 0$ and $\mathfrak{C}$ goes into $R_{,q'p}{}^{p'q}-R_{p'q,}{}^{q'p}$ in this limit, namely the difference between two reaction matrices. M and N represent "interaction" and contain as an interaction kernel the reaction matrix only.

Equations (27), (28), and (29) provide sets of equations to discuss the collective oscillations of the system when the interaction contains a hard-sphere repulsion.[6] The eigenmode can be approximately represented by a packet,

$$X_n{}'=\sum_{p,q}(\Phi_{p,q}{}^n a_p{}^*b_q{}^*+\mathrm{X}_{p,q}{}^n a_p{}^*a_q+\Xi_{p,q}{}^n b_q{}^*b_p+\Theta_{p,q}{}^n b_p a_q), \quad (30)$$

and from (27) we can see that, if we regard $\mathfrak{B}$ and $\mathfrak{C}$ as small, essential couplings exist only for Φ and Θ (which determines the eigenvalue of state n) and hence the fields X and Ξ can be regarded as attached fields. We

will not go into the details of this calculation although we may proceed in the same way as in section (a): Namely, in setting up the equation for (30), we leave the one-particle energy ω_p undetermined and determine ω_p as a sum of the right-hand side of (20), and $\Delta\omega_p$ which arises from the ordering of operator X_n' in the interaction term (M and N). Similarly for (22), we can expand ba, in the last term of the curly bracket into X_n'; making a shift of creation and annihilation operators, we get the residual change in ω_p in the first term of (22). The same effect would arise in (19) also. Corresponding to the $V\rightarrow R$ replacement, we find a "shielding" of the interaction R for small-momentum transfer occurring in the "interaction" term due to the partial attachment of the fields a^*a, b^*b in (30).

In concluding the case of Fermi systems, we should emphasize that all equations written down in Secs. (a) and (b) are still exact, except the energy of the ground state (24), and hence we can use these equations to get more correct normal modes, excitation energies, and ground-state energy; moreover, from these equations we can get the lifetime of the collective excitations [or of the packet we made in (30)] as well.

II. MANY-BOSON SYSTEM

The system of Bose particles can be described (near its ground state) by the procedure originally due to Beliaev[10–12] in the following manner; in the Hamiltonian, first replace operators which represent the creation and annihilation of zero-momentum particles by $(\bar{N})^{\frac{1}{2}}$, where $\bar{N}$ is a c number, and add to the Hamiltonian a term

$$-\sum_{p(\neq 0)} C_p^* C_p \mu, \tag{31}$$

μ being some constant, C_p representing an annihilation operator for a particle with momentum p. Then, after the evaluation of energies with the Hamiltonian $H_T(\bar{N})$ obtained in this way, $\bar{N}$ and μ are determined by

$$\bar{N}=N-(\Psi_{\bar{N}}, \sum_{p(\neq 0)} C_p^* C_p \Psi_{\bar{N}}), \tag{32}$$

$$\mu=\frac{\partial}{\partial \bar{N}}(\Psi_{\bar{N}}, H_T(\bar{N})\Psi_{\bar{N}}), \tag{33}$$

with $\Psi_{\bar{N}}$ the (ground state) wave function belonging to the Hamiltonian $H_T(\bar{N})$, with N the total number of particles. The Hamiltonian $H_T(\bar{N})$ has the following form:

$$\begin{aligned} H_T(\bar{N})=&\sum_p C_p^* C_p(\epsilon_p+\bar{N}V_{0,p(0,p)})+\tfrac{1}{2}\bar{N}^2 V_{0,0;0,0} \\ &+\sum_{p,q,r,s} C_p^* C_q^* \tfrac{1}{2} V_{p,q;s,r} C_r C_s \\ &+\sum_{p,r,s} C_p^* \tfrac{1}{2} V_{0,p(s,r)} C_r C_s (\bar{N})^{\frac{1}{2}} \\ &+\sum_{p,q,s} C_p^* C_q^* \tfrac{1}{2} V_{(p,q)0,s} C_s (\bar{N})^{\frac{1}{2}} \\ &+\sum_{r,s} \tfrac{1}{2} V_{0,0;s,r} C_r C_s \bar{N} \\ &+\sum_{p,q} C_p^* C_q^* \tfrac{1}{2} V_{p,q;0,0} \bar{N}, \end{aligned} \tag{34}$$

where $\epsilon_p=p^2/2m-\mu$ and $V_{p,q;s,r}$ is the same as defined in (1), $V_{0,p(s,r)}=V_{0,p;s,r}+V_{0,p;r,s}$, and all summations exclude the zero value of momentum.

The energy of the system is then given by

$$E=E_{\bar{N}}+\mu(\Psi_{\bar{N}}, \sum_{p(\neq 0)} C_p^* C_p \Psi_{\bar{N}}), \tag{35}$$

where $E_{\bar{N}}$ is $(\Psi_{\bar{N}}, H_T(\bar{N})\Psi_{\bar{N}})$.

Now, when the interaction between particles involves hard-sphere repulsion, the correlations between a pair of particles are so strong that we should first of all construct an (approximate) eigenmode which takes into account the major part of these correlations. For this purpose, let us construct the simplest equation for the (approximate) particle-particle scattering eigenmode $(\psi_{kl}{}^n=\psi_{lk}{}^n)$:

$$\begin{aligned} &[\sum_{k,l}\psi_{k,l}{}^n C_k^* C_l^*+\sum_k \chi_k{}^n C_k^*+\xi^n, H_T]_- \\ &=\sum_{k,l} C_k^* C_l^* \{-(\omega_k+\omega_l)\psi_{k,l}{}^n-\sum_{p,q}\tfrac{1}{2}V_{k,l(p,q)}\psi_{p,q}{}^n\} \\ &\quad+\sum_k C_k^*[-\omega_k\chi_k{}^n-2\sum_{p,q}\tfrac{1}{2}V_{0.k(p,q)}(\bar{N})^{\frac{1}{2}}\psi_{p,q}{}^n] \\ &\quad-\sum_{p,q}\tfrac{1}{2}V_{0,0(p,q)}\psi_{p,q}{}^n\bar{N}+[\sum_k(\sum_l 2\psi_{k,l}{}^n C_l^*+\chi_k{}^n) \\ &\quad\times\{(\omega_k-\epsilon_k-\bar{N}V_{0,k(0,k)})C_k^* \\ &\quad-\sum_{p,q,r} C_p^* C_q^* \tfrac{1}{2} V_{(p,q)r,k} C_r \\ &\quad-2\sum_{p,r} C_p^* \tfrac{1}{2} V_{0,p(r,k)} C_r (\bar{N})^{\frac{1}{2}} \\ &\quad-\sum_r \tfrac{1}{2} V_{0,0(r,k)} C_r (\bar{N})^{\frac{1}{2}} \\ &\quad-\sum_{p,q} C_p^* C_q^* \tfrac{1}{2} V_{(p,q)0,k} (\bar{N})^{\frac{1}{2}}\}], \end{aligned} \tag{36}$$

and the equations for ψ, χ, and ξ [obtained from the first two lines of (36)],

$$\begin{aligned} \{\omega_n-(\omega_k+\omega_l)\}\psi_{k,l}{}^n&=\sum_{p,q}\tfrac{1}{2}V_{k,l(p,q)}\psi_{p,q}{}^n, \\ (\omega_n-\omega_k)\chi_k{}^n&=2\sum_{p,q}\tfrac{1}{2}V_{0,k(p,q)}\psi_{p,q}{}^n(\bar{N})^{\frac{1}{2}}, \\ \omega_n\xi^n&=\sum_{p,q}\tfrac{1}{2}V_{0,0(p,q)}\psi_{p,q}{}^n\bar{N}. \end{aligned} \tag{37}$$

For simplicity, we have picked out a packet which is sufficient to show that all the V's can be replaced by the reaction matrix; later we will obtain a result indicating the need to improve our approximate eigenmode.

One more assumption we need in order to proceed is that ω_p, which we must determine later, has the property $\omega_p\rightarrow 0$ for $p\rightarrow 0$ (a condition which should be verified later).

Then we obtain the eigenmode [which gives $H_T(\bar{N})\alpha_n^*\Psi_N\doteqdot(\omega_n+E_N)\alpha_n^*\Psi_N$],

$$\begin{aligned} \alpha_n^*=&\sum_{k,l}\psi_{k,l}{}^n C_k^* C_l^*+2\sum_k C_k^* R_{0k}{}^n(\bar{N})^{\frac{1}{2}}/(\omega_n-\omega_k) \\ &+R_{00}{}^n\bar{N}/\omega_n, \end{aligned} \tag{38}$$

where we can see from (37) that the last two terms play a role of an attached field; we assume that we work with a spherical potential and a boundary condition for (37)

[10] S. T. Beliaev, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 289 (1958) [translation: Soviet Phys. JETP **34**, 417 (1958)].

[11] N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959). This paper contains a proof that $\tilde{\omega}_{p\rightarrow 0}=0$ holds in general if one collects the right combination of quantities. It was suggested by Hugenholtz that, instead of evaluating μ by using (54), we would be able to determine μ by the equation $\tilde{\omega}_{p\rightarrow 0}=0$. In this paper, we do not go into detail on this problem. The author is grateful to Dr. Hugenholtz for a discussion.

[12] K. Sawada, Phys. Rev. **116**, 1344 (1959); T. T. Wu, Phys. Rev. **115**, 1390 (1959).

giving the principal value for the singularity in R. $R_{jk}{}^{i}$ means $\sum_{p,q}\psi_{p,k}{}^{i}\frac{1}{2}V_{(p,q)j,k}=\sum_{p,q}\frac{1}{2}V_{j,k(p,q)}\psi_{p,q}{}^{i}$. The inverse transformation gives

$$C_k^*C_l^*=\sum_n\psi_{k,l}{}^{n*}\alpha_n^* -2(\bar{N})^{\frac{1}{2}}\sum_p C_p^*\sum_n R_{0p}{}^{n}\psi_{kl}{}^{n}/(\omega_n-\omega_p) -\bar{N}\sum_n R_{00}{}^{n}\psi_{kl}{}^{n}/\omega_n, \quad (39)$$

and

$$\sum_{p,q}\tfrac{1}{2}V_{(p,q)i,j}C_p^*C_q^* =\sum_n R_{ij}{}^{n}\alpha_n^*-2(\bar{N})^{\frac{1}{2}}\sum_p C_p^*\sum_n R_{0p}{}^{n}R_{ij}{}^{n}/(\omega_n-\omega_p) -\bar{N}\sum_n R_{00}{}^{n}R_{ij}{}^{n}/\omega_n. \quad (40)$$

Using (39) and (40) in the square bracket on the right-hand side of (36), Eq. (36) becomes after some reduction;

$$[\alpha_n^*,H_T]_- =-\omega_n\alpha_n^*+2\sum_k[\sum_l\psi_{k,l}{}^{n}C_l^*+R_{0k}{}^{n}(\bar{N})^{\frac{1}{2}}/(\omega_n-\omega_k)] \times\{(\omega_k-\epsilon_k-2R_{0k}{}^{0k}\bar{N})C_k^*-R_{k,-k}{}^{00}\bar{N}C_{-k} -\sum_q\sum_m R_{qk}{}^{m}\alpha_m^*C_q-2\sum_{p,q}C_p^*C_qR_{qk}{}^{0p}(\bar{N})^{\frac{1}{2}} -\sum_m R_{0k}{}^{m}\alpha_m^*(\bar{N})^{\frac{1}{2}}\}, \quad (41)$$

To reduce the equation we used the following identity which is derivable from the completeness of ψ^n:

$$R_{ij}{}^{kl}=\tfrac{1}{2}V_{i,j(k,l)}-\sum_n R_{kl}{}^{n}R_{ij}{}^{n}/(\omega_n-\omega_k-\omega_l).$$

This is a representation of the reaction matrix. This equation describes the motion of our packet (38) and the "interaction" term gives an estimate of how "good" a packet it is; the equation itself is still exact; the ω_k's are as yet undetermined.

Let us then examine the single-particle excitation [this turns out to give on the right-hand side of its equation of motion the same quantity which is in the last curly bracket of (41)]:

$$[C_k^*,H_T]_- =-(\epsilon_k+2R_{0k}{}^{0k}\bar{N})C_k^*-R_{k;-k}{}^{00}\bar{N}C_{-k} -\sum_q\sum_m R_{qk}{}^{m}\alpha_m^*C_k-2\sum_{p,q}C_p^*C_qR_{qk}{}^{0p}(\bar{N})^{\frac{1}{2}} -\sum_m R_{0k}{}^{m}\alpha_m^*(\bar{N})^{\frac{1}{2}}. \quad (42)$$

By using the first line of (42), we can assume for the (approximate one-particle) eigenmode a form

$$d_k^*=u_kC_k^*+v_kC_{-k};$$

its equation becomes;

$$[u_kC_k^*+v_kC_{-k},H_T]_- =C_k^*\{-(\epsilon_k+2R_{0k}{}^{0k}\bar{N})u_k+R_{k,-k}{}^{00}\bar{N}v_k\} +C_{-k}\{(\epsilon_k+2R_{0k}{}^{0k}\bar{N})v_k-R_{k,-k}{}^{00}\bar{N}u_k\} -u_k(\sum_q\sum_m R_{qk}{}^{m}\alpha_m^*C_q+2\sum_{p,q}C_p^*C_qR_{qk}{}^{0p}(\bar{N})^{\frac{1}{2}} +\sum_m R_{0k}{}^{m}\alpha_m^*(\bar{N})^{\frac{1}{2}})+v_k(\sum_q\sum_m R_{qk}{}^{m}C_q^*\alpha_m +2\sum_{p,q}C_q^*C_pR_{qk}{}^{0p}(\bar{N})^{\frac{1}{2}}+\sum_m R_{0k}{}^{m}\alpha_m(\bar{N})^{\frac{1}{2}}). \quad (43)$$

Then we get an equation which determines u_k and v_k by setting the first two lines of (43) equal to $-\bar{\omega}_k(u_kC_k^*+v_kC_{-k})=-\bar{\omega}_kd_k^*$.

$$\{\bar{\omega}_k-(\epsilon_k+2R_{0k}{}^{0k}\bar{N})\}u_k+R_{k,-k}{}^{00}\bar{N}v_k=0, \quad -R_{k,-k}{}^{00}\bar{N}u_k+\{\bar{\omega}_k+(\epsilon_k+2R_{0k}{}^{0k}\bar{N})\}v_k=0; \quad (44)$$

from this we have

$$\bar{\omega}_k=[(\epsilon_k+2R_{0k}{}^{0k}\bar{N})^2-(R_{k,-k}{}^{00}\bar{N})^2]^{\frac{1}{2}}, \quad (45)$$

the energy of elementary excitation.

The coefficients u_k and v_k should satisfy $u_k{}^2-v_k{}^2=1$; hence from (44) we get,

$$u_k{}^2=\tfrac{1}{2}[(\tilde{\epsilon}_k/\bar{\omega}_k)+1], \quad v_k{}^2=\tfrac{1}{2}[(\tilde{\epsilon}_k/\bar{\omega}_k)-1], \quad \tilde{\epsilon}_k=\epsilon_k+2R_{0k}{}^{0k}\bar{N}=(k^2/2m)-\mu+2R_{0k}{}^{0k}\bar{N}. \quad (46)$$

The inverse transform of the C's are

$$C_k^*=u_kd_k^*-v_kd_{-k}. \quad (47)$$

To determine ω_k appearing in the particle-particle scattering mode, we note that the first and second term in the curly bracket of (41) can be written as follows by using (46), (47), and (44):

$$(\omega_k-\tilde{\epsilon}_k)(u_kd_k^*-v_kd_{-k})-R_{k,-k}{}^{00}\bar{N}(u_kd_{-k}-v_kd_k^*) =(\omega_k-\bar{\omega}_k)u_kd_k^*-(\omega_k+\bar{\omega}_k)v_kd_{-k},$$

and the first term in the square bracket [on the right-hand side of (41)] is

$$\sum_l\psi_{k,l}{}^{n}C_l^*=\sum_l\psi_{k,l}{}^{n}(u_ld_l^*-v_ld_{-l});$$

hence, if $\omega_k\neq\bar{\omega}_k$, we get a c number when we shift d_k^* to the extreme left because it does not commute with d_{-l} for $-l=k$. Such a c number on the right-hand side of (41) represents the presence of a source which annihilates or creates α; to remove this source it is necessary to shift α_n^* once again.

In order to avoid this, we take (to make the c number vanish)

$$\omega_k=\bar{\omega}_k, \quad (48)$$

and this equation determines ω_k. Then (41) gives,

$$[\alpha_n^*,H_T]_- =-\omega_n\alpha_n^*+2\sum_k[\sum_l\psi_{k,l}{}^{n}C_l^*+R_{0k}{}^{n}(\bar{N})^{\frac{1}{2}}/(\omega_n-\omega_k)] \times[-2\omega_kv_kd_{-k}-\sum_q\sum_m R_{qk}{}^{m}\alpha_m^*C_q -2\sum_{p,q}C_p^*C_qR_{qk}{}^{0p}(\bar{N})^{\frac{1}{2}} -\sum_m R_{0k}{}^{m}\alpha_m^*(\bar{N})^{\frac{1}{2}}], \quad (49)$$

and the equation for the single particle mode (43) becomes

$$[d_k^*,H_T]_- =-\bar{\omega}_kd_k^*-u_k[\sum_r\sum_n R_{rk}{}^{n}d_n^*C_r +2\sum_{p,q}C_p^*C_qR_{qk}{}^{0p}(\bar{N})^{\frac{1}{2}}+\sum_n R_{0k}{}^{n}\alpha_n^*(\bar{N})^{\frac{1}{2}}] +v_k[\sum_r\sum_n R_{rk}{}^{n}C_r^*\alpha_n+2\sum_{p,q}C_q^*C_pR_{qk}{}^{0p}(\bar{N})^{\frac{1}{2}} +\sum_n R_{0k}{}^{n}\alpha_n(\bar{N})^{\frac{1}{2}}]. \quad (50)$$

These equations are still exact, and one can see how V's are replaced by R's.

To determine the ground-state energy in the approximation that α_n and d_k appearing in (49) and (50) represent approximate normal modes, we should put (40) into the Hamiltonian (34) and perform the shift of α^* to left and α to right, then again put (47) in and shift d^* to the left and d to the right. The constant term obtained in this way gives us $(\Psi(\bar{N}),H_T(\bar{N})\Psi(\bar{N}))$ in this approximation:

$$E_{\bar{N}}=\sum_p \tfrac{1}{2}[(\tilde{\epsilon}_p/\tilde{\omega}_p)-1]\{\epsilon_p+2\bar{N}(\psi^{0p}v\psi^{0p})\} +\tfrac{1}{2}\bar{N}^2(\psi^{00}v\psi^{00}), \quad (51)$$

where

$$(\psi^{ij}v\psi^{ij})=\sum_{p,q,r,s}\psi_{pq}{}^{ij}\tfrac{1}{2}V_{pq(rs)}\psi_{rs}{}^{ij},$$

and $\tilde{\epsilon}$, ϵ are defined in (46). By using the relation

$$(\psi^{ij}v\psi^{ij})=R_{ij}{}^{ij}+\sum_{k,l}R_{kl}{}^{ij}R_{kl}{}^{ij}/(\omega_i+\omega_j-\omega_k-\omega_l), \quad (52)$$

[note that we are assuming the solutions ψ of (37) do not belong to the bound state; hence $\psi^n=\phi^n+(\omega_n-\omega)^{-1}v\psi^n$], we get

$$E_{\bar{N}}=\tfrac{1}{2}\sum_p(\tilde{\omega}_p-\tilde{\epsilon}_p)-\tfrac{1}{2}\sum_p(\bar{N}R_{p,-p}{}^{00})^2/(-2\omega_p) +\tfrac{1}{2}\bar{N}^2R_{00}{}^{00}+\sum_p\tfrac{1}{2}[(\tilde{\epsilon}_p/\tilde{\omega}_p)-1] \times 2\sum_{k,l}R_{kl}{}^{0p}R_{kl}{}^{0p}\bar{N}/(\omega_p-\omega_k-\omega_l). \quad (53)$$

The three terms on the right-hand side of (53) represent a generalized form of earlier work[4] (note that the ω's are contained in the energy denominator of the R's which depend on R through (45); hence the equations form a consistent set), and the last term is (part of) a correction to it when the particle density is low.[11-12]

Similarly we can determine μ from (33) by using

$$\mu=\frac{\partial}{\partial\bar{N}}(\Psi_{\bar{N}},H_T(\bar{N})\Psi_{\bar{N}})=\left(\Psi_{\bar{N}},\frac{\partial}{\partial\bar{N}}H_T(\bar{N})\Psi_{\bar{N}}\right). \quad (54)$$

Substituting (40) and (47) into (54) and performing the ordering of α^*, α, and d^*, d, we get

$$\mu=\bar{N}R_{00}{}^{00}+\sum_p\tfrac{1}{2}[(\tilde{\epsilon}_p/\tilde{\omega}_p)-1]2R_{0p}{}^{0p}. \quad (55)$$

Here, we find a discrepancy with our assumption that $\omega_{p\to 0}=0$, because owing to the presence of the second term of (55), ω_p determined by (48) and (45) does not satisfy the assumption used throughout, namely that $\omega_{p\to 0}=0$.[11] This means that to get the right energy spectrum we should improve our (approximate) eigenmode α and d by using (49) and (50).[11] Finally, $\bar{N}$ is given by (32), using (47):

$$\bar{N}=N-\sum_p\tfrac{1}{2}[(\tilde{\epsilon}_p/\tilde{\omega}_p)-1]. \quad (56)$$

Equations (37), (45), (47), (48), (49), and (50), together with (53), (55), and (56) describe the system; among them (55), (45), (48), and (37) form the self-consistent set of equations. The energy of the ground state is given by (35), namely,

$$E=E_{\bar{N}}+\mu\sum_p\tfrac{1}{2}[(\tilde{\epsilon}_p/\tilde{\omega}_p)-1], \quad (57)$$

with μ given by (55).

III. CONCLUSION

Using the equations of motion for operators which represent simple excitations, we constructed a packet which approximates an exact normal mode of oscillation. When the interaction between two particles is very strong, the construction of the approximate scattering eigenmode first and making the effect of this (approximate) mode on the equation minimal (shifting operators representing creation and annihilation of the packet), leads to the self-consistent set of equations. It was essential to take into account (at least a part of) the attached field in the constituent of the normal mode in order to get rid of the appearance of a singular potential in the equation of motion.

Our results for the approximate excitation energy and the ground state energy represent the first (few) terms of these quantities exactly at low density. For higher densities, it is known that for the Fermi system when the force has long-tail (namely, a Coulomb-type interaction) we should take into account the pair (particle-hole) excitation [Sec. I(b)]. For the Bose system, the lifetime of the "phonon" excitation represented by d in Eq. (50) must be estimated by using (50) itself, and the lifetime should be long enough if the spectrum (45) is to represent elementary excitation accurately.

In this paper, we have not attempted to answer these questions but have only aimed at showing that we can investigate the interesting properties of many-body systems by the construction of (approximate) normal modes for particle-particle scattering (and particle-hole scattering for a charged Fermi system at high density) even when the interparticle interactions contain a hard-sphere singularity.

ACKNOWLEDGMENT

The author would like to express his sincere thanks to Professor Robert Oppenheimer for the hospitality extended to him at the Institute for Advanced Study.