

Wingate, Gross, and Failla suggest that this energy dependence may be only an apparent one in air due to difficulties of ion collection. However, the electronegative behavior of a gas (responsible for large recombination losses) and the dependence of  $W_\alpha$  on energy were found to be uncorrelated effects. A good example is nitrogen, the major component of air, for which an

energy dependence was found by Jesse and Sadauskis.<sup>11</sup> On the other hand, nitrogen is nonelectronegative and permits an easy determination of saturation currents at relatively low-field strengths. Therefore, the observed energy dependence has to be considered real in nitrogen and can be accepted for air.

<sup>11</sup> W. P. Jesse and J. Sadauskis. *Phys. Rev.* **97**, 1668 (1955).

## Upper Bound on Total Electron Scattering Cross Sections in Hydrogen\*

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Dispersion relations for electron-hydrogen scattering are combined with existing scattering length calculations. The sign of the scattering length is shown to give an upper bound on  $\int \sigma(k) dk$  where  $\sigma(k)$  is the total cross section for scattering of electrons of incoming momentum  $\hbar k$ . Recent calculations of the scattering length are used to determine this limit. An experiment by Fite *et al.* satisfies this bound, agreeing with the recent calculations of the scattering length.

IN a recent paper<sup>1</sup> dispersion relations were applied to low-energy electron scattering by hydrogen atoms. The results of that work indicate a connection between angular distributions at a particular energy and total cross sections at all energies. In the present work knowledge of the scattering lengths is combined with the dispersion relations for rearrangement collisions to give a bound on the total cross section integrated over all energies. The arguments will be made for electron-hydrogen scattering; their extension to other systems will be obvious.

The appropriate dispersion relations for electron-hydrogen scattering at zero incoming electron energy are<sup>1</sup>

$$\text{Re}(f - \frac{1}{2}g) = (f - \frac{1}{2}g)_{\text{Born}} + \frac{1}{2\pi^2} \int_0^\infty \sigma_t(k) dk - \frac{1}{2}R, \quad (1)$$

where  $f$  and  $g$  are the zero-energy scattering amplitudes for direct and exchange electron scattering at angle  $\theta=0$ ,  $\sigma_t$  is the total cross section for scattering of electrons of incoming momentum  $\hbar k$ ,  $R$  is a positive number<sup>1</sup> which depends on the bound-state wave functions of the  $e$ -H system, and  $(f - \frac{1}{2}g)_{\text{Born}}$  is the first Born approximation to the forward amplitudes  $f$  and  $g$ .

At zero incoming electron energy we can express the singlet ( $f+g$ ) and triplet ( $f-g$ )  $s$ -wave phase shifts  $\delta_0^\pm$  in terms of a parameter  $a_\pm$ , the scattering length, by<sup>2,3</sup>

$$k \cot \delta_\pm = -1/a_\pm + \dots \quad (2)$$

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<sup>1</sup> E. Gerjuoy and N. A. Krall, *Phys. Rev.* **119**, 705 (1960).

<sup>2</sup> J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), pp. 56-65.

<sup>3</sup> T. Ohmura, Y. Hara, and T. Yamanouchi, *Progr. Theoret. Phys. (Kyoto)* **20**, 82 (1958).

Comparison of (2) with the partial wave expansion of the amplitudes<sup>4</sup> ( $f \pm g$ ) yields the elementary result (at  $k=0$ )

$$\text{Re}(f \pm g) = -a_\pm \quad \text{and} \quad -\text{Re}(f - \frac{1}{2}g) = \frac{3}{4}a_- + \frac{1}{4}a_+. \quad (3)$$

In the case of electron-hydrogen scattering  $(f - \frac{1}{2}g)_{\text{Born}}$  at zero energy equals  $-2a_0$ , where  $a_0$  is the Bohr radius.<sup>5</sup> Thus,

$$\begin{aligned} \text{Re}(f - \frac{1}{2}g) &= -\frac{3}{4}a_- - \frac{1}{4}a_+ = -(\frac{1}{2}R + 2a_0) \\ &+ \frac{1}{2\pi^2} \int_0^\infty \sigma_t dk. \end{aligned} \quad (4)$$

Now the scattering length is a zero-energy property of the system and has been calculated by a variety of researchers for  $e$ -H scattering.<sup>3,6,7,8</sup> While the numerical values of their results are not in complete agreement, they all predict that the singlet and triplet scattering lengths are both positive. It is well known that the scattering length is negative for a system with no bound states, and becomes positive for a system with a bound state near zero energy. The most recent calculations indicate  $a_- \sim 1.9a_0$  (triplet),<sup>6</sup>  $a_+ \sim 6.2a_0$  (singlet).<sup>6,8</sup> Some of these results<sup>6</sup> are based on elegant variational calculations, and are expected to be quite reliable. The singlet length might be expected to be positive because binding energy calculations and observations indicate the existence of one bound state near zero energy. A virtual  $s$  state within one ev of zero energy could change

<sup>4</sup> L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 105.

<sup>5</sup> E. Corinaldesi and L. Trainor, *Nuovo cimento* **9**, 940 (1952).

<sup>6</sup> L. Rosenberg, L. Spruch, and T. F. O'Malley, *Phys. Rev.* **119**, 164 (1960).

<sup>7</sup> S. Borowitz and H. Greenberg, *Phys. Rev.* **108**, 716 (1957).

<sup>8</sup> T. Ohmura and H. Ohmura, *Phys. Rev.* **118**, 154 (1960).

the sign of the scattering length. The triplet state is rather unusual. A state of negative energy does exist, but the exclusion principle rules out this state physically. In such a case the scattering theory accepts this state as legitimately bound, predicting a phase  $\delta_0^- \rightarrow \pi$  as  $k \rightarrow 0$  and predicting  $a_- > 0$ .

If both  $a_+$  and  $a_-$  are positive, we have an upper bound on the integrated total cross section, namely

$$\int_0^\infty \sigma_t dk \leq 2\pi^2 (\frac{1}{2}R + 2a_0). \quad (5)$$

Using the (approximate) value of  $R$  (at  $E=0$ ) computed in reference 1, the inequality (5) becomes

$$\frac{1}{2\pi^2} \int_0^\infty \sigma_t dk \leq 7.52a_0. \quad (6)$$

Using the most recent calculated values<sup>6</sup> of  $a_\pm$  and the same  $R$  as above, the equality (4) becomes

$$\frac{1}{2\pi^2} \int_0^\infty \sigma_t dk = 4.5a_0. \quad (7)$$

Two experiments, referred to as (B)<sup>9</sup> and (F),<sup>10</sup> exist

<sup>9</sup> B. Bederson, J. Hammer, and H. Malamud, New York University Technical Report No. 2, Electron Scattering Project (unpublished).

<sup>10</sup> R. T. Brackman, W. L. Fite, and R. H. Neynaber, Phys. Rev. **112**, 1157 (1958).

which measure  $\sigma_t$ , and an earlier paper<sup>1</sup> used these measurements to compute the integral in (5). These results were

$$\text{Experiment (B)} \quad \frac{1}{2\pi^2} \int \sigma_t dk = (10.3 \pm 0.5)a_0, \quad (8)$$

$$\text{Experiment (F)} \quad \frac{1}{2\pi^2} \int \sigma_t dk = (5.1 \pm 0.5)a_0.$$

It is apparent, on comparing (6) and (7) with (8), that experiment (B) violates the inequality, while experiment (F) easily satisfies the inequality, and comes reasonably close to agreeing with the equality based on existing calculations of the scattering length and our estimate of  $R$ . The indicated spread in (8) results from different choices of the cross sections outside the range of the experiments.

We conclude that knowledge of the sign of the scattering length can be used to place a limit on the integrated total cross sections, and accurate calculation of the scattering length and bound-state residue gives the value of  $\int_0^\infty \sigma_t dk$ . Thus there is a close relation between the scattering length, the integrated cross section, and the bound-state wave functions of the system (as reflected in the residue  $R$ ).

## Lower Bounds for Eigenvalues with Application to the Helium Atom\*

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A method is derived for finding lower bounds to the energy levels of the Schrödinger equation. This method is applied to the helium atom. The best lower bounds thus obtained are  $-3.063$ , and  $-2.165$ , atomic units for the energies  $E(1^1S)$  and  $E(2^1S)$ , respectively. If our lower bound for  $E(2^1S)$  is used together with the best published values of  $\langle H\psi, \psi \rangle$  and  $\langle H\psi, H\psi \rangle$  of the ground state, a rigorous lower bound  $-2.9037474$  atomic units is found for  $E(1^1S)$ .

### I. INTRODUCTION

THE basic ideas of our procedure for lower bounds go back to the work of A. Weinstein,<sup>1</sup> which introduces an explicitly solvable *base problem* with lower eigenvalues. Weinstein links the base problem to the given problem by a sequence of *intermediate problems* which can be solved in terms of the base problem and which improve the lower bounds. In his work on plates

these problems are obtained by changing the boundary conditions. By combining his lower bounds with the upper bounds given by the Rayleigh-Ritz method he obtained an accuracy of up to 0.18%.

Later Aronszajn<sup>2</sup> emphasized that a base problem can be obtained by a change of the operator and outlined the construction of the intermediate problems. In this paper we simplify Aronszajn's important extension of the method of intermediate problems by constructing these problems in such a way that in order to solve them in terms of the base problem one need only

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<sup>1</sup> A. Weinstein, *Memorial des Sci. Math.* No. 88 (1937).

<sup>2</sup> N. Aronszajn, *Proceedings of the Oklahoma Symposium on Spectral Theory and Differential Problems*, 179, 1951 (unpublished).