which, on the basis of a 0^+ assignment, is calculated[18] to be 7.5 ev. Ferrell[19] has calculated that the Γ_γ for the $7.66 \rightarrow 4.43$ ($0^+ \rightarrow 2^+$) transition is $\Gamma_\gamma = 0.0014$ ev (with an uncertainty of the order of a factor of two). Assuming this value of Γ_γ, one can calculate the percentage decay of the 7.66-Mev state by a γ cascade via the 4.43-Mev state:

$$\Gamma_\gamma/\Gamma = 0.0014/8 = 0.02\% \quad \text{(with an uncertainty of a factor of two).} \quad (4)$$

As has been mentioned earlier the best experimental upper limit[6] on this number is 0.1%. The 7.66-Mev state decays in $\sim 7 \times 10^{-4}\%$ of the cases [see (2)] by pair emission.

In summary, it is now clear from the width of the state and from its decay behavior that the 7.66-Mev state of C^{12} is a 0^+ state and that it can participate in the process of the buildup of the elements in red giant stars.

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[18] W. A. Fowler and T. Lauritsen (private communication).
[19] R. A. Ferrell, private communication, quoted in Cook et al., see reference 5.

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Shell Model and Pb^{208}†

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The lowest odd-parity excited energy levels of Pb^{208} have been calculated by a shell-model approach considering a single proton or a single neutron to be excited out of the Pb^{208} core. Both a singlet-even plus triplet-even force and a Rosenfeld force were used as the two-particle interaction. A zero-range force was also considered. There were no other arbitrary parameters. The results with the various forces indicate that it is impossible to get a 3− state low enough to be interpretated as the observed 2.615-Mev 3− level. The results, therefore, support the conclusion that the 3− level at 2.615 Mev in Pb^{208} is primarily the result of a collective octupole oscillation.

I. INTRODUCTION

RECENTLY there has been renewed interest in the low excited states of Pb^{208}. Several people have contributed to the position and spins of the experimentally determined levels.[1–3] Tauber[4] has tried to fit the excited states of Pb^{208} theoretically from a shell-model approach. He had difficulty in drawing any conclusions due to too many undetermined parameters. Tamura and Choudhury[5] have assumed shell-model configurations to explain some of the results of Cohen et al.[3] on the inelastic scattering of particles by heavy elements. They conclude that a collective octupole oscillation can affect their results for Pb^{208}. Lane and Pendlebury[6] have recently done some calculations which support the idea of C. Levinson that the first excited state of Pb^{208}, a 3− level at 2.615 Mev, is a surface vibration of the octupole type.

In the present paper, we calculate the energy spectrum of the lowest odd-parity energy levels of Pb^{208} according to the jj-coupling shell model with configuration mixing. The jj-coupling states included in our study are all those which can arise from promoting a $p_{1/2}$ or $f_{5/2}$ neutron into the $g_{9/2}$ or $i_{11/2}$ shell and from promoting an $s_{1/2}$ or $d_{3/2}$ proton into the $h_{9/2}$ or $f_{7/2}$ shell. The absolute positions of these jj configurations are obtained from empirical data; thus, the only arbitrary parameters in the calculation are those of the two-body potential between particles and of the nucleon radial wave functions. It is reasonable to take for these parameters values which have worked well in the past in theoretical calculations on nuclei in this mass region.[7–10]

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Section II of this paper will deal briefly with the evaluation of the matrix elements needed for the calculation. Section III discusses how the zero-order energies were obtained from experimental data. Finally, Secs. IV and V discuss the results and conclusions of this calculation.

II. THEORY OF CORE EXCITED STATES

In this paper we use a terminology which is more or less standard in nuclear shell theory. The ground state of Pb^{208} consists entirely of closed proton and neutron shells. We think of this state as the "core" about which the particles move. When higher single-particle states are occupied we use the terminology, "extra-core state." Vacancies in the core states are, of course, "holes." A state formed by promoting a core particle into an extra-core state is a "core-excited state."

We shall not go into details of the calculations of the matrix elements of core-excited states because the main aspects have been studied before.[11–13] We do wish, however, to point out some results which are unique to this type of core-excited calculation.

There are two approaches to problems involving holes in closed shells. Both approaches, of course, yield the same results. One method involves calculating with a wave function which explicitly involves the coordinates of all the particles present. If there are N single-particle states in the core and we wish to calculate matrix elements for states of $N-q$ particles, it is well known that these can be related to the matrix elements for q particles. This contraction is made possible by the antisymmetry of the $N-q$ particle wave functions and can be effected using either determinant wave functions[11] or fractional parentage techniques.[12] The reduction of the Hamiltonian matrix elements is brought about by the use of the relation

$$\sum_{i>j=1}^{n} V(r_{ij})=\frac{n(n-1)}{2}V(r_{12}), \tag{1}$$

which is valid between totally antisymmetric states. The main difficulty of this method is the necessity of counting the number of times that sets of quantum numbers, j_1, j_2, etc., have particle labels 1 and 2 associated with them. This counting process is especially complicated and tedious for cases involving off-diagonal matrix elements.

The other approach is that of second quantization. This method has been outlined recently by Brink and Satchler.[13] In particular, these authors derive the consequences of the rotational properties of the states. This method avoids most of the difficulties encountered in the other method.

[11] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1953), Chaps. XII and XIII.

[12] G. Racah, Phys. Rev. **62**, 438 (1942); **63**, 367 (1943).

[13] D. M. Brink and G. R. Satchler, Nuovo cimento **4**, 549 (1956).

The second-quantization approach employs the use of creation and annihilation operators which obey anticommutation relations as follows:

$$\begin{aligned}\{\eta^{\dagger}_{jm},\eta^{\dagger}_{j'm'}\}&=0,\\ \{\eta_{jm},\eta_{j'm'}\}&=0,\\ \{\eta^{\dagger}_{jm},\eta_{j'm'}\}&=\delta_{jj'}\delta_{mm'}.\end{aligned} \tag{2}$$

The operator $\eta^{\dagger}_{jm}$ creates a particle with angular momentum j and a z component m, while η_{jm} annihilates this particle.[14] One should refer to the paper of Brink and Satchler[13] for a detailed account of how to use these operators.

Brink and Satchler also introduce creation and annihilation operators for holes. That is, they consider a filled shell with $2j+1$ particles as a state with no holes. Then the operator[15] $(-1)^{j-m}\eta_{jm}$ will create a hole with angular momentum j and a z component $-m$. As these operators obey anticommutation relations, all particles are automatically antisymmetrized with respect to one another. It is this antisymmetrization and the introduction of the hole creation and annihilation operators, as well as the particle creation and annihilation operators, which enables one to bypass most of the complications which arise in the Racah approach mentioned above.

We will consider only the excited states in Pb^{208} where one neutron or one proton is excited out of the core. That is, we neglect all excitations of two neutrons, two protons, or a proton and a neutron, etc., out of the core. We shall describe a state which has a particle missing from the core with angular momentum j_1 and a particle outside the core with angular momentum j_2 as $|\tilde{j}_1 j_2 JM\rangle$, where all the particles are coupled to an angular momentum J with a z component M, and all equivalent particles are antisymmetrized with respect to one another.

We shall use the notation

$$\begin{aligned}{}_a\langle j_1 j_2 JM|V(r)|j_3 j_4 JM\rangle_a&=\langle j_1 j_2 JM|V(r)|j_3 j_4 JM\rangle\\ &-(-1)^{j_3+j_4-J}\langle j_1 j_2 JM|V(r)|j_4 j_3 JM\rangle,\end{aligned} \tag{3}$$

for a two-body matrix element calculated with antisymmetric states. Also, j_c will refer to a particle in the Pb^{208} core. Then one can show that for like particles

$$\begin{aligned}&\langle\tilde{j}_1 j_2 JM|\sum_{i>j}V(r_{ij})|\tilde{j}_3 j_4 JM\rangle\\ &=\sum_{j_c,j_c'}[\tfrac{1}{2}{}_a\langle j_c j_c' JM|V(r)|j_c j_c' JM\rangle_a]_{\text{diag}}\\ &\quad+\sum_{j_c}[{}_a\langle j_c j_2 JM|V(r)|j_c j_2 JM\rangle_a]_{\text{diag}}\\ &\quad-\sum_{j_c}[{}_a\langle j_c j_1 JM|V(r)|j_c j_1 JM\rangle_a]_{\text{diag}}+V_{\text{int}},\end{aligned} \tag{4}$$

where V_{int} represents the interaction of the holes with

[14] Note that our notation for creation and annihilation operators is the reverse of that of Brink and Satchler.

[15] See reference 13 for details about the phase of the hole-creation operator $(-1)^{j-m}\eta_{jm}$.

the extra-core particles. The first three terms in Eq. (4) only occur when $|\tilde{j}_1 j_2 JM\rangle = |\tilde{j}_3 j_4 JM\rangle$ as indicated by the "diag," i.e., are nonzero only for diagonal matrix elements. The first term in Eq. (4) is the total core energy. The second term in Eq. (4) is the interaction of the core with the extra-core particles, while the third term is the interaction of the core with the holes. The third term has a minus sign because we have included in the first term the interaction of the particles which are missing from the core with all the other particles in the core. We consider the first term in Eq. (4), which is the energy of the Pb^{208} core, as our zero point of energy. Although Eq. (4) is written for the potential energy of the system, the total energy, kinetic, Coulomb, and nuclear potential energy, can be broken down in this way. The extension of Eq. (4) to the kinetic energy is almost trivial, since it is a one-body scalar operator. The second and third terms of Eq. (4) will be evaluated from the experimental data in Sec. III. The Coulomb interaction was considered explicitly in the hole-particle interaction term, V_{int}, in Eq. (4), and this will be discussed in Sec. IV.

The last term in Eq. (4) is the interesting one as it represents the hole-particle interaction. This term in jj coupling is [13,16]

$$V_{\text{int}} = -\sum_{J_1} (-1)^{j_1+j_2-j_3-j_4}[J_1] W(j_1 j_2 j_4 j_3; J J_1) \times {}_a\langle j_3 j_2 J_1 M_1 | V(r) | j_1 j_4 J_1 M_1\rangle_a. \quad (5)$$

Note that the minus sign which occurs in Eq. (5) indicates that the hole-to-particle interaction is repulsive for an attractive force.

We shall assume that the two-body potential is of the form,

$$V(r) = v(r)[A + BP^\sigma + CP^r + DP^r P^\sigma], \quad (6)$$

where P^σ and P^r are the spin-exchange and space-exchange operators, respectively. For a Wigner force, $B=C=D=0$, we can write

$$V(r) = \sum_k f_k(r_1,r_2) P_k(\cos\theta_{12}) = \sum_k f_k(r_1,r_2) \mathbf{C}_1^{(k)} \cdot \mathbf{C}_2^{(k)}. \quad (7)$$

If we insert Eq. (7) in Eq. (5) and consider the case when $j_3 = j_1$ and $j_4 = j_2$, we get[17]

$$V_{\text{int}} = -[\langle j_1 j_2 JM | V(r) | j_1 j_2 JM\rangle + (-1)^{j_1+j_2}[J]^{-1} \times R^J(j_1 j_2 j_2 j_1)\langle j_1 \| \mathbf{C}^J \| j_2\rangle \langle j_2 \| \mathbf{C}^J \| j_1\rangle], \quad (8)$$

where the first term is the direct matrix element and the second term arises from the exchange matrix element. Note that the sum over J_1 and k has been carried out explicitly in order to get this result. Note also the over-all minus sign which indicates that the hole-to-particle interaction is repulsive for an attractive potential. $R^k(j_1 j_2 j_2 j_1)$ is a Slater integral. When one considers non-Wigner forces, that is, forces with P^σ or P^r in them, such as Eq. (6), the reduction of the sum over J_1 and k as was done above for Wigner forces is no longer trivial. It is much simpler if at the beginning we transform the states $|\tilde{j}_1 j_2 JM\rangle$ and $|\tilde{j}_3 j_4 JM\rangle$ to the LS representation.

In the Pb^{208} core, all shells are filled both as LS shells and jj shells except the $1i$ neutron shell and the $1h$ proton shell. The $1i_{13/2}$ neutron shell is full while the $1i_{11/2}$ shell is empty. Likewise, the $1h_{11/2}$ proton shell is full and the $1h_{9/2}$ shell is empty. We shall only consider the possibility of the $3p_{1/2}$ or $2f_{5/2}$ neutron or the $3s_{1/2}$ or $2d_{3/2}$ proton being excited out of the core. Then the only shells which get broken up in the core are those which are LS-closed as well as jj-closed shells. For this special type of excitation, one can transform the original many-particle state from the jj to LS coupling scheme as if one were considering just two-particle wave functions. That is,

$$\langle \tilde{j}_1 j_2 J | \sum_{i>j} V(r_{ij}) | \tilde{j}_3 j_4 J\rangle = \sum_{L,S} (j_1 j_2 J [\![l_1 l_2, s_1 s_2, LSJ)(l_3 l_4, s_3 s_4, LSJ]\!] j_3 j_4 J) \times \langle \tilde{l}_1 \tilde{s}_1, l_2 s_2, LSJ | \sum_{i>j} V(r_{ij}) | \tilde{l}_3 \tilde{s}_3, l_4 s_4, LSJ\rangle, \quad (9)$$

where we have suppressed the M dependence and the coefficients $(j_1 j_2 J [\![l_1 l_2, s_1 s_2, LSJ)$ are the jj-to-LS recoupling coefficients. We can now work in LS representation which we shall see is easier. The breakdown of the energy as in Eq. (4) is essentially unchanged.

We now wish to study a term in V_{int} for a given L and S, which we can call $V_{\text{int}}(L,S)$. Then

$$V_{\text{int}}(L,S) = \langle \tilde{l}_1 \tilde{s}_1, l_2 s_2, LS | \sum_{i>j} V(r_{ij}) | \tilde{l}_3 \tilde{s}_3, l_4 s_4, LS\rangle = -\sum_{L_1,S_1} (-1)^{l_1+l_2-l_3-l_4}[L_1][S_1] W(l_1 l_2 l_4 l_3; L L_1) \times W(s_1 s_2 s_4 s_3; S S_1) {}_a\langle l_3 s_3, l_2 s_2, L_1 S_1 | V(r) | l_1 s_1, l_4 s_4, L_1 S_1\rangle_a, \quad (10)$$

where the subscript a means an antisymmetric two-particle state as before. Consider now a Wigner force. This force can be expanded as in Eq. (7) and inserted in Eq. (10). The sum over L_1, S_1, and k can be performed and the result is

$$V_{\text{int}}(L,S) = -[(-1)^{l_2+l_4}\langle l_1 l_2 LS | V(r) | l_3 l_4 LS\rangle - 2\delta_{S0}(-1)^L [L]^{-1} R^L(l_3 l_2 l_4 l_1) \times \langle l_3 \| \mathbf{C}^L \| l_4\rangle \langle l_2 \| \mathbf{C}^L \| l_1\rangle], \quad (11)$$

which we see is similar in form to Eq. (8). Because we are in an LS representation, we can also in a straightforward manner repeat the above procedure for the general type of force which involved a P^σ and P^r, or both. The result for a potential as given by Eq. (6) is

$$V_{\text{int}}(L,S) = -[(A-D)+2(B-C)\delta_{S0}](-1)^{l_2+l_4} \times \langle l_1 l_2 LS | v(r) | l_3 l_4 LS\rangle + [2(A-D)\delta_{S0} + (B-C)](-1)^L [L]^{-1} R^L(l_3 l_2 l_4 l_1) \times \langle l_3 \| \mathbf{C}^L \| l_4\rangle \langle l_2 \| \mathbf{C}^L \| l_1\rangle, \quad (12)$$

where A, B, C, and D are defined in Eq. (6).

[16] We write $(2J+1)$ as $[J]$.

[17] $\mathbf{C}_1^{(k)}$ is a Racah tensor operator and $\langle j_1 \| \mathbf{C}^k \| j_2\rangle$ is a reduced matrix element as defined in reference 12.

TABLE I. Ground-state and first excited state configurations[a] of Pb^{207}, Pb^{209}, Tl^{207}, and Bi^{209}.

Nucleus	Ground-state configuration	First excited state: Configuration	First excited state: Energy above ground-state (Mev)
Pb^{207}	$3p_{1/2}$	$2f_{5/2}$	0.57
Pb^{209}	$2g_{9/2}$	$1i_{11/2}$ [b]	0.79
Tl^{207}	$3s_{1/2}$	$2d_{3/2}$	0.351
Bi^{209}	$1h_{9/2}$	$2f_{7/2}$	0.91

[a] The data in this table with the exception of the first excited state of Pb^{209} were taken from D. Strominger, J. M. Hollander, and G. T. Seaborg, Revs. Modern Phys. **30**, 585 (1958); *Nuclear Data Sheets* (National Research Council, Washington, D. C.); and D. Strominger and J. M. Hollander, University of California Radiation Laboratory Report UCRL-8289, June, 1958 (unpublished).
[b] This level was assumed from the experimental results of M. T. McEllistrem, H. J. Martin, D. W. Miller, and M. B. Sampson, Phys. Rev. **111**, 1636 (1958), and is further supported by the results of reference 2.

The only other type of matrix element which we must consider for our calculations in Pb^{208} is the off-diagonal matrix element between a state with a neutron excited and a state with a proton excited. More complicated cases are ruled out by our not considering the states where two neutrons, two protons, or a neutron and a proton are excited at the same time. In this case, we only have V_{int} in Eq. (4) and we can transform from jj coupling to LS coupling as before. We find that

$$\begin{aligned}V_{\text{int}}(L,S)&=\langle \bar{l}_1\bar{s}_1(n),l_2s_2(n),LS|\sum_{n,p}V(r_{np})|\bar{l}_3\bar{s}_3(p),l_4s_4(p),LS\rangle\\&=\sum_{L_1,S_1}(-1)^{l_2+l_3-L_1+2s_1-S_1}[L_1][S_1]W(l_1l_2l_4l_3;LL_1)\\&\quad\times W(s_1s_2s_4s_3;SS_1)\\&\quad\times\langle l_3s_3(p),l_2s_2(n),L_1S_1|V(r)|l_4s_4(p),l_1s_1(n),L_1S_1\rangle,\end{aligned}\tag{13}$$

where the l and s associated with a neutron or a proton has been so indicated.

Using Eqs. (6), (7), and (13), performing the sum over L_1, S_1, and k, we get

$$\begin{aligned}V_{\text{int}}(L,S)&=\langle \bar{l}_1\bar{s}_1(n),l_2s_2(n),LS|\sum_{n,p}V(r_{np})|\bar{l}_3\bar{s}_3(p),l_4s_4(p),LS\rangle\\&=(D+2C\delta_{S0})(-1)^{l_2+l_4}\langle l_1l_2LS|v(r)|l_3l_4LS\rangle\\&\quad+(B+2A\delta_{S0})(-1)^L[L]^{-1}R^L(l_3l_2l_4l_1)\\&\quad\times\langle l_3\|\mathbf{C}^L\|l_4\rangle\langle l_2\|\mathbf{C}^L\|l_1\rangle\end{aligned}\tag{14}$$

for the neutron-proton interaction.

We have now essentially reduced a many-particle matrix element to a single two-particle matrix element where we have split the two-particle matrix element up into a direct term and an exchange term. Equations (12) and (14) are now in a form where we can calculate matrix elements. For a singlet-even force, Eq. (6) will have the following form

$$V(r)=v(r)[\tfrac{1}{4}-\tfrac{1}{4}P^\sigma+\tfrac{1}{4}P^r-\tfrac{1}{4}P^rP^\sigma].\tag{15}$$

TABLE II. Experimental data used in fixing the absolute energies.

Reaction	Q value (Mev)	Reference
$Pb^{208}(\gamma,n)Pb^{207}$	$Q_1=-7.380$	a
$Pb^{207}(d,p)Pb^{208}$	$Q_2=5.14$	a
$Bi^{209}(\gamma,n)Bi^{208}$	$Q_3=-7.28$	a
$Pb^{208}(d,p)Pb^{209}$	$Q_4=1.71$	b, c
$Pb^{209}\rightarrow Bi^{209}+\beta^-$	$Q_5=0.63$	c
$Bi^{208}+\text{E.C.}\rightarrow Pb^{208}$	$Q_6=2.9$	c
$Tl^{207}\rightarrow Pb^{207}+\beta^-$	$Q_7=1.45$	c

[a] D. M. Van Patter and W. Whaling, Revs. Modern Phys. **26**, 402 (1954).
[b] M. T. McEllistrem, H. J. Martin, D. W. Miller, and M. B. Sampson, Phys. Rev. **111**, 1636 (1958).
[c] *Nuclear Data Sheets* (National Research Council, Washington, D.C.); D. Strominger and J. M. Hollander, University of California Radiation Laboratory Report UCRL-8289, June, 1958 (unpublished).

One might expect to get zero for Eqs. (12) and (14) with a singlet-even force when $S\neq 0$. This, however, is not the case as one can see by the inspection of Eqs. (15) and (12) or (14). Equations (12) and (14) represent interactions between an extra-core particle and the particles in the core, and it is expected that the extra-core particle and some particle in the core will have a relative S value of zero. So we do get a contribution to the matrix element for a singlet-even force even though the total S of the system is not zero. The same reasoning applies to a pure triplet force.

III. ABSOLUTE ENERGIES

As mentioned in Sec. II, we shall consider the Pb^{208} core as the zero point of energy, and refer all energies to this. It is also possible to predict the zero-order absolute energies (relative to the Pb^{208} core energy) of the core-excited states of Pb^{208} when one particle is excited out of the core. To do this, we note that the neutron separation energy is

$$S_n(A+1)=E(A)+n-E(A+1),\tag{16}$$

and the proton separation energy is

$$S_p(A+1)=E(A)+p-E(A+1),\tag{17}$$

where $E(A)$ is the energy of the ground state of the nucleus A. One can then easily show that the first three terms of Eq. (4) are just

$$E_n^*(\bar{p}_{1/2},g_{9/2})=S_n(Pb^{208})-S_n(Pb^{209}),\tag{18}$$

when we excite a $3p_{1/2}$ neutron out of the core into a $2g_{9/2}$ state, and that

$$E_p^*(\bar{s}_{1/2},h_{9/2})=S_p(Pb^{208})-S_p(Bi^{209}),\tag{19}$$

when we excite a $3s_{1/2}$ proton out of the core into a $1h_{9/2}$ state. From Table I, we see that it requires 0.57 Mev to excite a $3p_{1/2}$ neutron in Pb^{207} to the $2f_{5/2}$ state. We can then say that

$$E_n^*(\bar{f}_{5/2},g_{9/2})=E_n^*(\bar{p}_{1/2},g_{9/2})+0.57\text{ Mev},\tag{20}$$

and we can obviously extend this reasoning to other neutron (and proton) core-excited states.

Table II gives the experimental data used in deter-

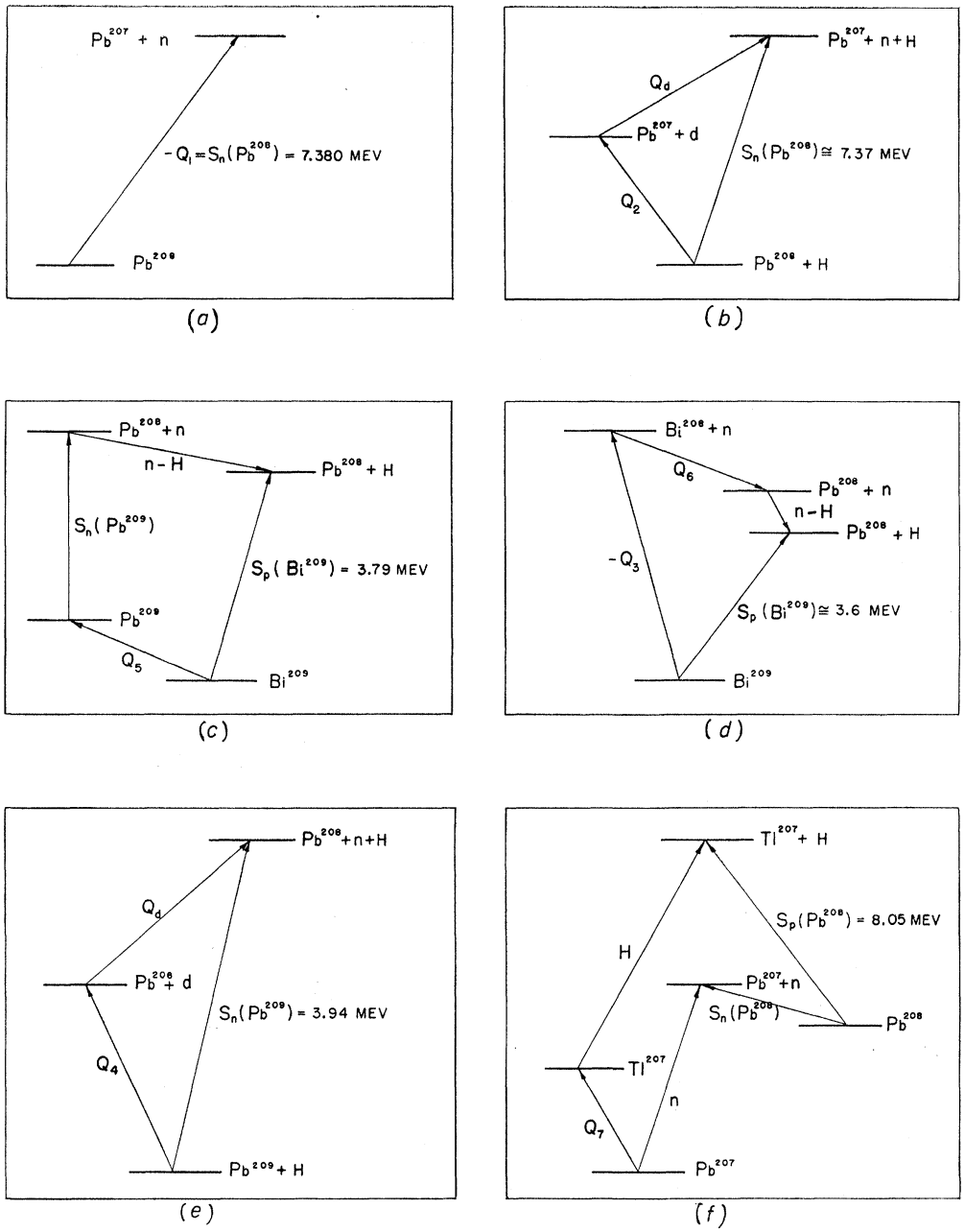

FIG. 1. Schematic mass diagrams used for determining neutron and proton separation energies. (a) shows how $S_n(Pb^{208})$ was determined with (b) as a check on this value; (c) determines $S_p(Bi^{209})$ with (d) as a check on this value; (e) and (f) determine $S_n(Pb^{209})$ and $S_p(Pb^{208})$, respectively. The various Q values are listed in Table II with the exception of Q_d, the deuteron binding energy, which was taken as 2.23 Mev. The neutron-hydrogen mass difference was taken as 0.78 Mev.

mining $E_n^*(\tilde{p}_{1/2},g_{9/2})$ and $E_p^*(\tilde{s}_{1/2},h_{9/2})$. Figure 1 shows schematically how the separation energies were determined. We shall take for these

$$S_n(Pb^{208}) = -Q_1 = 7.38 \text{ Mev}, \quad (21)$$

$$S_n(Pb^{209}) = Q_d + Q_4 = 3.94 \text{ Mev}, \quad (22)$$

$$S_p(Bi^{209}) = Q_5 + S_n(Pb^{209}) + H - n = 3.79 \text{ Mev}, \quad (23)$$

$$S_p(Pb^{208}) = Q_7 + S_n(Pb^{208}) + H - n = 8.05 \text{ Mev}. \quad (24)$$

Using Eqs. (18) to (24) and the values in Table I, the zero-order energies of the excited states of Pb^{208} were determined and are given in Table III. The energy values given in Table III are not expected to be in error by more than 0.1 Mev or 0.2 Mev at the most. We shall see that even if, for example, $E_p^*(\tilde{s}_{1/2},h_{9/2})$ were 4.00 Mev instead of 4.26 Mev, we would reach essentially the same conclusions about the excited states of Pb^{208}.

IV. RESULTS

A. Coulomb Energies

As was seen in Sec. III, the Coulomb energy contribution to the matrix elements was automatically taken into account with the exception of the last term in Eq. (4). To calculate the Coulomb energy of V_{int} between the proton hole and a proton outside the core, we used Eq. (9) and Eq. (12) with $V(r)$ in Eq. (6) as

$$V(r)=e^2/r. \tag{25}$$

The Coulomb energies pertinent to our calculations were calculated and are listed in Table IV. One may note from Table IV that the diagonal matrix elements are quite constant and are about -0.23 Mev. This value is very close to the Coulomb energies obtained by Newby and Konopinski[10] for two $1h_{9/2}$ protons outside of the lead 208 core. Our negative sign of the Coulomb energy is due to the fact that we are considering a hole-particle interaction. The off-diagonal matrix elements are also relatively constant and are about 0.01 Mev in magnitude.

B. Finite Range Forces

At this point, we have determined all parameters except those used in the two-body potential, Eq. (6), and the radial wave functions. Previous work[7] in the lead region indicates that the force between like nucleons is a pure singlet-even force. Pryce[8] suggests that for unlike particles and a zero-range force, the triplet-even force is 1.5 times as strong as the singlet-even force. We shall first assume that the two-body potential, Eq. (6), is a singlet-even plus triplet-even potential with the triplet-even part 1.5 times as strong as the singlet-even part. The parameters in Eq. (6) then become $A=0.625$, $B=0.125$, $C=0.625$, and $D=0.125$. For the $v(r)$ in Eq. (6), we shall choose a Gaussian well of the same range and depth as that used by True and Ford,[7] viz.,

$$v(r)=V_0\exp(-r^2/\beta^2), \tag{26}$$

where $V_0=-32.5$ Mev and $\beta=1.85$ fermis. We shall furthermore assume that the single-particle wave functions are harmonic oscillator wave functions with the same radial falloff as used in the study[7] of Pb^{206}. This is usually done to simplify the calculation of the two-particle matrix elements, and one expects that the results are not too sensitive to the exact nature of the radial part of the wave functions.

TABLE III. Zero-order energies of the excited states of Pb^{208}.

Configuration	Particle excited	Possible spins (odd parity)	Energy (Mev)
$\tilde{p}_{1/2}, g_{9/2}$	neutron	4, 5	3.44
$f_{5/2}, g_{9/2}$	neutron	2, 3, 4, 5, 6, 7	4.01
$\tilde{p}_{1/2}, i_{11/2}$	neutron	5, 6	4.23
$f_{5/2}, i_{11/2}$	neutron	3, 4, 5, 6, 7, 8	4.80
$\tilde{s}_{1/2}, h_{9/2}$	proton	4, 5	4.26
$d_{3/2}, h_{9/2}$	proton	3, 4, 5, 6	4.61
$\tilde{s}_{1/2}, f_{7/2}$	proton	3, 4	5.17
$d_{3/2}, f_{7/2}$	proton	2, 3, 4, 5	5.52

The matrix elements for this singlet-even plus triplet-even force were calculated using Eqs. (9), (12), and (14). Using the zero-order matrix elements from Table III, and the matrix elements for the Coulomb potential from Table IV, the matrices were diagonalized for each spin from 2 to 7. The results of this diagonalization are listed in Table V and compared with the experimental energies in Fig. 2. The experimentally determined levels of Pb^{208} are listed in Table VI.

We may note several things from Fig. 2. No calculated 3− level lies in the vicinity of the experimental 3− level at 2.615 Mev. The lowest experimental 4− and 6− levels are predicted in the correct energy region, but about 0.15 Mev too high. The lowest two 5− levels are also predicted, but are also too high. The splitting of the lowest two 5− levels is fairly good. The other experimental levels for which the spins are unknown are shown by dashed lines in Fig. 2. We have only indicated these levels when they can be associated with a calculated level, and the 6.09-Mev level is not shown. Lane and Pendlebury[6] have had a private communication from R. H. Helm, J. Oeser, and M. R. Yearian, who have found evidence from inelastic electron scattering for a level at 4.3 Mev in Pb^{208}. The angular distribution indicates that this new level is a 3− level. If so, this new level fits with the second 3− calculated

TABLE IV. Matrix elements in Mev for the Coulomb potential, $V_c=e^2/r$.

Matrix element	Spin and parity 2−	3−	4−	5−	6−
$\langle \tilde{s}_{1/2}h_{9/2}J \mid V_c \mid \tilde{s}_{1/2}h_{9/2}J\rangle$			−0.2297	−0.2257	
$\langle \tilde{s}_{1/2}h_{9/2}J \mid V_c \mid d_{3/2}h_{9/2}J\rangle$			0.0174	−0.0094	
$\langle \tilde{s}_{1/2}h_{9/2}J \mid V_c \mid \tilde{s}_{1/2}f_{7/2}J\rangle$			0		
$\langle \tilde{s}_{1/2}h_{9/2}J \mid V_c \mid d_{3/2}f_{7/2}J\rangle$			0.0023	−0.0045	
$\langle d_{3/2}h_{9/2}J \mid V_c \mid d_{3/2}h_{9/2}J\rangle$		−0.2286	−0.2238	−0.2131	−0.2402
$\langle d_{3/2}h_{9/2}J \mid V_c \mid \tilde{s}_{1/2}f_{7/2}J\rangle$		0.0105	0.0021		
$\langle d_{3/2}h_{9/2}J \mid V_c \mid d_{3/2}f_{7/2}J\rangle$		−0.0087	−0.0007	−0.0017	
$\langle \tilde{s}_{1/2}f_{7/2}J \mid V_c \mid \tilde{s}_{1/2}f_{7/2}J\rangle$		−0.2272	−0.2400		
$\langle \tilde{s}_{1/2}f_{7/2}J \mid V_c \mid d_{3/2}f_{7/2}J\rangle$		0.0089	−0.0111		
$\langle d_{3/2}f_{7/2}J \mid V_c \mid d_{3/2}f_{7/2}J\rangle$	−0.2572	−0.2296	−0.2249	−0.2412	

TABLE V. Calculated energy levels (in Mev) of Pb^{208} for a singlet-even plus triplet-even force.

Spin and parity					
2−	3−	4−	5−	6−	7−
6.04	5.56	5.53	5.43	4.98	4.82
4.80	5.34	5.24	4.93	4.67	4.04
	4.94	4.96	4.42	4.38	
	4.27	4.55	4.34	4.08	
	3.65	4.23	4.11		
		4.21	3.77		
		3.68	3.41		

TABLE VI. Experimental levels of Pb^{208} (taken from references 1, 6, and c of Table II).

Energy (Mev)	Spin and parity	Energy (Mev)	Spin and parity
0	0+	3.961	6−
2.615	3−	4.3	(3−)
3.198	5−	5.13	?
3.475	4−	5.44	?
3.709	5−	6.09	?
3.750	?		

level as shown in Fig. 2. The 3.75-Mev level is predicted to be 3−. The 5.13-Mev level is also predicted to be 4−, but the theory is not good enough to rule out an assignment of 3−, 5−, or 6−. Finally, the 5.44-Mev level can be either a 3−, 4−, or 5− level. One also expects other levels (e.g., due to exciting two particles from the core) to appear and to become important in the neighborhood of 5 Mev.

To see how sensitive the calculated energy levels are to the force used, we next used a Rosenfeld force which was adjusted to give the same singlet-even force as above. Kearsley[9] used a Rosenfeld force to predict the energy levels of Pb^{206}. However, she used a Yukawa well for $v(r)$ instead of a Gaussian well.[18] For a Rosenfeld force, we have $A=-0.2222$, $B=0.7778$, $C=1.5556$, and $D=-0.4444$, with $v(r)$ the same as in Eq. (26). The resulting energy levels are listed in Table VII and they are compared with the experimental levels in Fig. 3. Comparison of Fig. 2 and Fig. 3 (or of Table V and Table VII) shows that this Rosenfeld force causes all calculated levels to be lower in energy by about 0 to 0.2 Mev than the levels calculated for the singlet-even plus triplet-even force above. Figure 3 shows that the 6− level at 3.961 Mev is fitted better. The 3.750-Mev level now appears to be a 7− level instead of 3−. In general, the experimental energy levels are not fitted

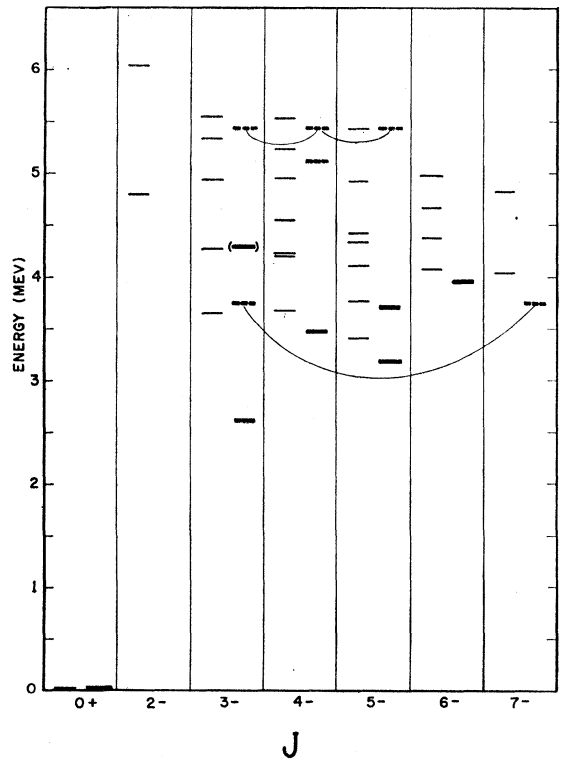

FIG. 2. Energy levels for Pb^{208}, I. For each spin, the first column gives the energy levels calculated for a singlet-even plus triplet-even force. The second column gives the experimental levels. The dashed experimental levels are levels for which the spins are not known.

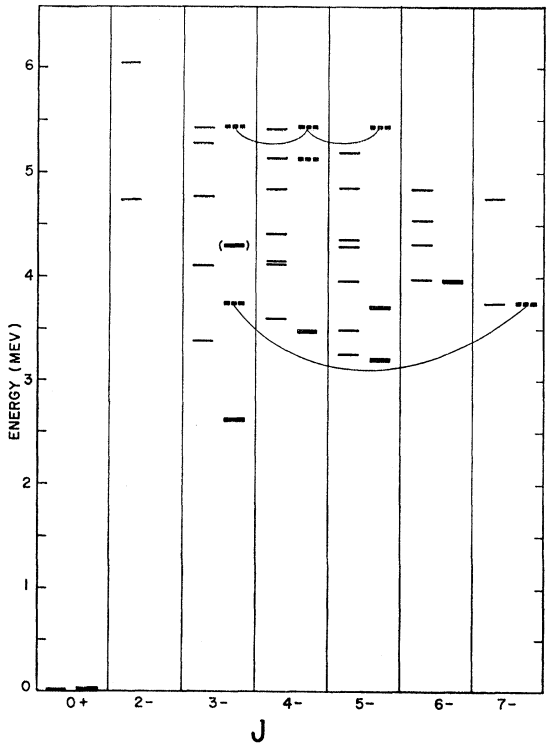

FIG. 3. Energy levels of Pb^{208}, II. For each spin, the first column gives the energy levels calculated for a Rosenfeld force. The second column gives the experimental levels. The dashed experimental levels are levels for which the spins are not known.

[18] See reference 7 for a comparison of the Yukawa well used by Kearsley with the Gaussian well used here.

as well as with the singlet-even plus triplet-even force above. We still do not have a calculated 3− level in the vicinity of the experimental 3− level at 2.615 Mev. The theoretical energy levels are not changed drastically by this rather abrupt change in the two-body force.

The left side of Table VIII shows a comparison of the diagonal and off-diagonal matrix elements for the $J=6$ states in Pb^{208} for the single-even plus triplet-even force and for the Rosenfeld force and this is a typical case. It is apparent from Table VIII why the Rosenfeld force gives different results. The diagonal matrix elements show that the Rosenfeld force is more repulsive (and so for our case of a hole-particle interaction, more attractive) than the singlet-even plus triplet-even force. The off-diagonal matrix elements are much the same for both forces and so about the same degree of configuration mixing takes place. The net result of the Rosenfeld force is to cause the resulting diagonal energy levels to be lower in energy.

In order to find out how sensitive these results are to changes in the zero-order energies, the energy

TABLE VII. Calculated energies (in Mev) for a Rosenfeld force.

Spin and parity					
2−	3−	4−	5−	6−	7−
6.04	5.42	5.41	5.19	4.84	4.75
4.73	5.28	5.14	4.85	4.55	3.75
	4.77	4.84	4.36	4.31	
	4.11	4.41	4.29	3.98	
	3.38	4.15	3.96		
		4.14	3.50		
		3.60	3.25		

$E_p^*(s_{1/2},h_{9/2})$ was arbitrarily changed from 4.26 Mev to 4.00 Mev. This also changed $E_p^*(d_{3/2},h_{9/2})$ to 4.35 Mev, $E_p^*(s_{1/2},f_{7/2})$ to 4.91 Mev, and $E_p^*(d_{3/2},f_{7/2})$ to 5.26 Mev. The singlet-even plus triplet-even force above was again used and the energy levels were calculated. The results of this change were compared to experiment and to the calculated results for no change. Some levels were barely affected while others were lowered by about 0.1 to 0.2 Mev. No real improvement was obtained in fitting the experimental energy levels. We conclude that even if the zero-order energies in Table III were in error by 0.1 Mev, the resulting energy spectrum would not be changed very much. (Note that in above calculation we have changed 4 zero-order energies by almost 0.3 Mev.)

C. Zero-Range Forces

Newby and Konopinski[10] have considered some of the lowest states in Bi^{210} and Po^{210} (two particles outside the Pb^{208} core). They found that better agreement with experiment was obtained in Bi^{210} with a zero-range force than with a finite-range central force. Newby and Konopinski also found that when they considered a tensor force in addition to the finite-range central force,

TABLE VIII. Comparison of the matrix elements for the $J=6-$ states of Pb^{208} for various forces. In the table, $|1\rangle=|p_{1/2}i_{11/2}J=6\rangle$, $|2\rangle=|f_{5/2},g_{9/2}J=6\rangle$, $|3\rangle=|f_{5/2}i_{11/2}J=6\rangle$, and $|4\rangle=|d_{3/2}h_{9/2}J=6\rangle$.

	Matrix elements in Mev		
	Finite-range forces		
	Singlet-even plus triplet-even force	Rosenfeld force	Zero-range limit of singlet-even plus triplet-even force
$\langle 1\|V(r)\|1\rangle$	0.175	0.095	0.309
$\langle 1\|V(r)\|2\rangle$	0.017	0.015	0.027
$\langle 1\|V(r)\|3\rangle$	0.124	0.068	0.282
$\langle 1\|V(r)\|4\rangle$	−0.038	0.035	−0.064
$\langle 2\|V(r)\|2\rangle$	0.075	−0.026	0.165
$\langle 2\|V(r)\|3\rangle$	0.006	0.003	0.016
$\langle 2\|V(r)\|4\rangle$	−0.015	−0.043	−0.004
$\langle 3\|V(r)\|3\rangle$	0.123	0.034	0.316
$\langle 3\|V(r)\|4\rangle$	−0.078	0.008	−0.095
$\langle 4\|V(r)\|4\rangle$	0.320	0.176	0.612

the resulting matrix elements changed in such a direction as to agree with the matrix elements of a zero-range force.

Due to these results of Newby and Konopinski,[10] it seemed desirable to repeat the above calculations with a zero-range force. In these zero-range calculations, we followed Newby and Konopinski and considered a potential which will conserve volume energy.[10] That is, Eq. (26) becomes

$$v(r)=V_0\beta^3\pi^{\frac{3}{2}}\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{27}$$

The above calculations were repeated with the force given by Eq. (27) and with the same parameters as were used in the singlet-even plus triplet-even force above. The parameters used by Newby and Konopinski[10] are not sufficiently different to drastically change the results. The energies resulting from this calculation with a zero-range force are listed in Table IX and are shown in Fig. 4. The matrix elements for this zero-range force for the $J=6$ states of Pb^{208} are also shown in Table VIII.

Figure 4, when compared with Figs. 2 and 3, show that the zero-range force gives the lowest 3− level of any of the forces used. This lowest 3− level is still 0.5 Mev above the experimentally observed 3− level at 2.615 Mev. In general, it is felt that the zero-range force does not give as good an over-all fit to the experi-

TABLE IX. Calculated energy levels (in Mev) of Pb^{208} for the zero-range limit of a singlet-even plus triplet-even force.

Spin and parity					
2−	3−	4−	5−	6−	7−
6.66	5.85	5.90	5.86	5.27	4.73
5.40	5.61	5.25	5.11	4.94	4.07
	5.06	5.10	4.42	4.42	
	4.12	4.85	4.30	4.17	
	3.13	4.52	3.95		
		4.22	3.58		
		3.75	3.08		

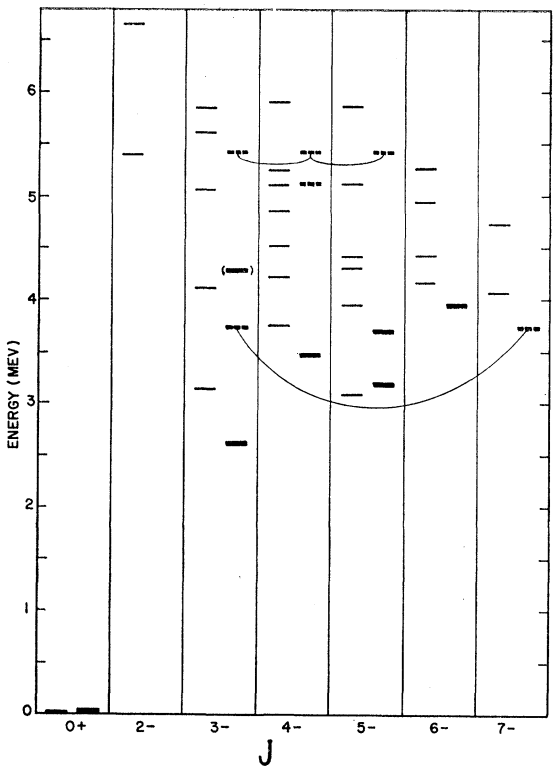

FIG. 4. Energy levels of Pb^{208}, III. For each spin, the first column gives the energy levels calculated for the zero-range limit of a singlet-even plus triplet-even force. The second column gives the experimental levels. The dashed experimental levels are levels for which the spins are not known.

mentally observed levels as the singlet-even plus triplet-even force (excepting the 3− level at 2.615 Mev).

V. CONCLUSIONS

In this paper, we have shown the results of theoretical calculations on the energy levels of Pb^{208} The only arbitrary parameters used were those to fix the nuclear force admixtures of Wigner, Majorana, Heisenberg, and Bartlett type forces. For the finite-range nuclear force, both a singlet-even plus triplet-even force and a Rosenfeld force were used. In addition, the energy levels were calculated for a zero-range nuclear force. Each of these three forces are quite different from one another. However, none of these three forces produced a 3− level near the experimentally observed 3− level at 2.615 Mev. This result supports the conclusion of Lane and Pendlebury,[6] that the 3− level at 2.615 Mev in Pb^{208} should be an octupole oscillation of the core instead of a core-excited state.

No transition rates were calculated betweeen the various levels in Pb^{208} because the conjectured core oscillations would make them invalid. One should really consider the effects of the collective oscillations on the shell-model configurations before calculating transition rates, as transition rates are very sensitive to configuration admixtures. A calculation of this sort has recently been done by Raz[19] for two $1f_{7/2}$ particles coupled to an oscillating core.

Since the singlet-even plus triplet-even force used in this paper is consistent with the forces used in other Pb isotopes, one might conjecture from Fig. 2 that the collective motion of the core will have a small effect on most of the shell-model energies.

This paper also points out that it is much simpler to use the second quantization approach with creation and annihilation operators to reduce a many-particle matrix element to a two-particle matrix element when one is considering core-excited states. In addition, when the core-excited particles are from LS- as well as jj-closed shells, it is much simpler to make the above reduction for non-Wigner type forces in the LS coupling scheme. One expects that this approach can be suitably modified to consider cases when the core-excited particles are from a jj-closed shell when the respective LS shell is not completely filled (e.g., promoting a $1i_{13/2}$ neutron out of the Pb^{208} core).

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[19] B. James Raz, Phys. Rev. **114**, 1116 (1959).