Imaginary Part of the Delbrück Scattering Amplitude*

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A method for computing the imaginary part of the Delbrück scattering amplitude, based on an expression derived by Kessler, has been developed and applied for gamma-ray energies of 2.62 and 6.14 Mev. The significance of the calculation with regard to available experimental results is discussed and some possible further developments are outlined.

DELBRÜCK[1] was apparently the first to point out that quantum electrodynamics predicts that the interaction of electromagnetic radiation with the nuclear Coulomb field can contribute to the elastic scattering of gamma rays by nuclei. The process is describable, in lowest order, as the production of an electron-positron pair in the field of the nucleus followed by pair annihilation; it can be treated as a special case of the scattering of light by light,[2] with two of the photons being virtual. The effect is of fundamental interest since it is a direct demonstration of the polarization of the vacuum surrounding the nuclear charge.[3] Current evidence for the existence of the vacuum polarization consists largely of the 27-Mc/sec correction to the Lamb shift (Uehling effect), a small correction to the low-energy p-p scattering,[4] and the energy levels of mu-mesic atoms.[5] It is therefore somewhat indirect.

The calculation of the Delbrück scattering amplitude is extremely difficult and has not yet been carried out in all generality. Considerable simplifications occur for the strictly forward scattering which has been worked out by Rohrlich and Gluckstern[6] and by Toll.[7] An approximate treatment for momentum transfer small compared to mc and energies high compared to mc^2, where m is the electron mass, has also been given by Toll[7] and by Bethe and Rohrlich.[8] The corrections to be expected because of the use of the Born approximation and the neglect of electron screening in the calculation of forward scattering have been estimated by Rohrlich.[9] All these calculations are unfortunately of somewhat limited usefulness, since most experiments[10] have been done with gamma rays of a few Mev energy and with scattering angles of around 20°. The single exception appears to be the work of Moffat and Stringfellow[11] who observed the small-angle scattering of 87-Mev gamma rays and obtained results not inconsistent with the approximate theory of Bethe and Rohrlich.[8] An additional difficulty in the interpretation of the experimental results arises from the fact that coherent contributions to the elastic scattering also come from nuclear Thomson scattering, nuclear resonance scattering,[12] and Rayleigh scattering from the atomic electrons.[13] The last effect is the dominant one at energies below a few Mev.

The present calculations are based on the recent work of Kessler,[14] who was able to express the imaginary part of the scattering amplitude in the form of a five-dimensional integral. The imaginary part of the scattering amplitude arises, within the formalism of "old-fashioned" perturbation theory, from those transitions in which energy is conserved in the intermediate state. It is thus closely related to the cross section for pair production; in fact, for forward scattering the relation is trivial, being merely given by the "optical theorem." Kessler's work, therefore, may be regarded as a generalization of the optical theorem to finite scattering angles. The analytical evaluation of Kessler's integral does not appear to be feasible, but a few numerical results were obtained by Kessler by use of the Monte Carlo method.

The calculation of the real part of the scattering amplitude for finite scattering angles would be extremely difficult and has not so far been attempted. However, it is already known[8] that for energies large compared to mc^2 and momentum transfers of the order of mc or larger the imaginary part is large compared to the real part. For energies greater than 10 Mev the imaginary part is dominant even for zero momentum transfer.

The analytical work of Kessler has been verified and

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TABLE I. The imaginary part of the Delbrück scattering amplitude in units of $(\alpha Z)^2 r_0$ for gamma-ray energies of 2.62 Mev, as a function of the scattering angle θ.

θ	0	$\frac{1}{2}\theta_c$	θ_c	$\frac{3}{2}\theta_c$	$2\theta_c$	$3\theta_c$	30°	60°	90°	180°
a (para)	0.265	0.246	0.204	0.160	0.124	0.078	0.089	0.037	0.021	0.010
a (perp)	0.265	0.235	0.175	0.119	0.078	0.031	0.041	−0.002	−0.009	−0.010

TABLE II. The imagingry part of the Delbrück scattering amplitude in units of $(\alpha Z)^2 r_0$ for gamma-ray energies of 6.14 Mev, as a function of the scattering angle θ.

θ	0	$\frac{1}{2}\theta_c$	θ_c	$\frac{3}{2}\theta_c$	$2\theta_c$	$3\theta_c$	30°	60°	90°	180°
a (para)	2.81	2.14	1.43	1.02	0.77	0.47	0.17	0.05	0.03	0.01
a (perp)	2.87	1.76	0.92	0.49	0.21	0.14	0.10	−0.02	−0.02	−0.01

TABLE III. The Bethe-Rohrlich function $F_2(x)$.

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0
$F_2(x)$	2.920	2.227	1.826	1.542	1.325	1.149	1.003	0.8786	0.7712	0.6772	0.3447	0.1675

some further numerical results have been obtained. The method used was that of independent Gaussian quadrature. A program for this purpose was written for the IBM-704.

The scattering amplitudes were computed in units of $(Z\alpha)^2 r_0$, as Kessler did, where r_0 is the classical electron radius. The amplitudes for gamma-ray polarization parallel and perpendicular to the scattering plane are denoted by a (para) and a (perp), respectively. In order to facilitate comparison with the approximate theory of Bethe and Rohrlich, the amplitudes for small momentum transfer were computed for fixed momentum transfer rather than for fixed scattering angles. In this connection, it is convenient to define $\theta_c = mc^2/\hbar\omega$, where m is the mass of the electron and $\hbar\omega$ the gamma-ray energy.

The results of the computation for gamma-ray energies of 2.62 and 6.14 Mev are given in Tables I and II, respectively.

These results are believed to be correct to one or two percent. The results for 2.62 Mev at $\theta=0$ agree with the previous calculations of Rohrlich and Gluckstern.[6] The amplitudes for parallel and perpendicular polarization are equal in absolute magnitude, within the limits of computational error, at $\theta=0$ and $\theta=\pi$ as is required on the basis of elementary symmetry considerations. The results at 2.62 Mev agree reasonably well with the less accurate calculations of Kessler[14] with the exception of a (perp) at 60°. It should be noted that the angle θ used by Kessler is *half* the scattering angle but that it represents the full scattering angle in the present discussion.

The strong polarization dependence of the results is of particular interest, since it shows that Delbrück scattering will ordinarily produce appreciable polarization for unpolarized incident radiation. The results obtained so far suggest that for a given momentum transfer the polarization obtainable in this way increases with energy.

The approximate treatment of Bethe and Rohrlich[8] leads to a formula for the imaginary part of the scattering amplitude which may be written

$$a(\theta,\omega) = 0.2476 \frac{\hbar\omega}{mc^2} F_2(x)(Z\alpha)^2 r_0,$$

where

$$x = \theta(\hbar\omega/mc^2).$$

This formula is expected to apply when $\hbar\omega \gg mc^2$ and

$$(mc^2/\hbar\omega)^2 < \theta < mc^2/\hbar\omega.$$

The quantity $F_2(x)$ has been computed and is tabulated in Table III.

Comparison of the exact results with the approximate theory of Bethe and Rohrlich shows that the latter predicts results that are too large by a factor of about 8 at 2.62 Mev and by a factor of about 2 at 6.14 Mev. At 6.14 Mev, the present calculations of the ratio $a(\theta_c/2)/a(\theta_c)$ yield values of 1.5 and 1.9 for parallel and perpendicular polarization, respectively; the corresponding Bethe-Rohrlich result is 1.96 for both directions of polarization. This qualitative agreement with Bethe and Rohrlich's calculation of the imaginary part of the scattering amplitude gives one some grounds for believing that their predictions regarding the relative magnitude of the real and imaginary parts are also essentially correct at this energy. One may reasonably conclude, therefore, that for $\theta > \theta_c$, almost all of the Delbrück scattering arises from the imaginary part of the scattering amplitude.

The amplitudes given in Tables I and II must be combined with the amplitudes for the other coherent processes mentioned above before the cross section may be calculated. However, in order to give a feeling for the relative magnitude of the Delbrück effect, it is pointed out that for the scattering of 6.14-Mev gamma rays from lead at a scattering angle of 30°, the "absorptive

part of the differential cross section for Delbrück scattering," defined by

$$d\sigma/d\Omega = \tfrac{1}{2}[a^2(\text{para}) + a^2(\text{perp})](\alpha Z)^4 r_0^2,$$

is approximately equal to 0.29 mb/sr, which is comparable to the experimental value.[15] This agreement cannot be regarded as very significant, however, since the magnitudes and relative phases of the Rayleigh and nuclear-resonance scattering amplitudes are not known.

It remains to discuss the errors arising from the use of the Born approximation and the neglect of electron screening in the computation of the above results. Higher order corrections to the amplitudes might be expected to be of relative magnitude $(\alpha Z)^2$ so that they might become appreciable at the high values of Z that are required to make Delbrück scattering easily observable. However, Rohrlich[9] has shown that, for lead, inclusion of Coulomb corrections increases the imaginary part of the forward-scattering amplitude by about 25% at 2.62 Mev and has negligible effect at 6.14 Mev. The effects of electron screening have been estimated by Toll[7] and appear to be negligible for the energies considered in this paper. The above estimates are strictly valid only for forward scattering but may reasonably be expected to apply also for small angles.

The presently available computer program is not too suitable for calculations for higher gamma-ray energies, since the number of points at which the integrand must be evaluated in order to get a reasonably accurate result becomes rather large and the amount of computing time required becomes excessive.

[15] See S. G. Cohen, reference 10.

The reason for this may be qualitatively understood as follows. The five-dimensional integration is over the variables describing the intermediate state, i.e., the polar and azimuthal angles of the momentum vectors of the electron and positron and the energy of one of these particles. At high energies the integrand function becomes sharply "peaked" within small subspaces of the five-dimensional space so it is necessary to use a finer grid in the numerical quadrature. Some of this peaking arises from the emission of the pair into very small forward angles[16]; but this was dealt with by suitably dividing up the region of integration. The really troublesome peaks were apparently associated with the strong angular correlation of electrons with positrons at high energies. No simple way of dealing with this could be found.

It is felt that further progress along these lines might be made by employing more sophisticated methods of numerical quadrature.[17] If the imaginary part of the scattering amplitude could be computed at high energies also, it would probably be possible to compute the real part with the aid of the dispersion relations for fixed momentum transfer.[18]

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