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Pion Production from π^--p Collisions in the Long-Range Interaction Model*

FREDA SALZMAN AND GEORGE SALZMAN†
CERN, Geneva, Switzerland and University of Colorado, Boulder, Colorado
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Long-range interactions of negative pions with protons resulting in single and double pion production are examined in terms of a meson field theory model in which it is assumed that there is a pion-pion interaction and that the single virtual pion exchange graphs are dominant in large-impact-parameter collisions. This pion exchange leads, in single pion production, to an "excited state," π^*, which "decays" into two pions, and, in double pion production, also to an "excited state," N^*, which "decays" into a pion and a nucleon. For sufficiently high relative energy of the incident π^- and proton, each of these processes can occur for small values of the invariant square of the virtual pion four-momentum, Δ^2, in which case the virtual pion carries very little transverse three-momentum. For small Δ^2 it is shown that it is reasonable to neglect final state interactions between the "decay products" of the π^* and those of the N^*. It is found that in the limit $\Delta^2 \rightarrow -\mu^2$, where μ is the pion rest mass, the virtual pion behaves kinematically, in the π^* barycentric system, as an incoming pion which scatters elastically with the incident π^-, and also behaves, in the N^* barycentric system, as an incoming pion which scatters elastically with the incident proton. Thus, for small Δ^2 in the physical region the π^* and N^* vertices are defined as the corresponding off-the-mass-shell scattering amplitudes. A ratio of appropriately defined double-to-single pion production cross sections is obtained which is independent of the details of the assumed pion-pion interaction, and depends only on the relative strengths of N^* and N formation. This ratio is estimated by means of the p-wave static nucleon model, applied in appropriate coordinate systems. For incident 5-Bev/c pions this model leads to double pion production which is important compared to single pion production, both because of the $\frac{3}{2}$-$\frac{3}{2}$ pion-nucleon resonance and because of important phase-space factors. Kinematical considerations similar to those described above suggest that this model may also be the theoretical genesis of the "two fireballs" model proposed for ultrarelativistic nucleon-nucleon collisions.

1. INTRODUCTION

THERE is increasing evidence for the importance of "peripheral" interactions, i.e., long-range interactions, in collisions of negative pions with protons in the Bev range.[1–5]

Furthermore, there is evidence which indicates that this long-range interaction may be due to the exchange of a single virtual pion between the two colliding particles. Elastic π^--proton diffraction scattering above 1.3 Bev indicates that inelastic interactions may occur at a large distance ($\sim 1\times10^{-13}$ cm) from the nucleon.[1,2] Attempts to fit the diffraction pattern at incident pion energies of ~ 1.5 Bev suggest that the nucleon consists of an absorptive core, of radius $\sim 0.5\times10^{-13}$ cm, in which the interaction is strong, and a highly transparent fringe, of radius as large as $\sim 1.4\times10^{-13}$ cm, in which the interaction is relatively weak.[1] The exchange of a single virtual pion between the incident pion and the incident nucleon is expected to give an interaction with the characteristic range $\mu^{-1}\approx 1.4\times10^{-13}$ cm, where μ is the pion rest mass in units with $\hbar=c=1$, which are used throughout. This is the same range that occurs in nucleon-nucleon scattering. The large "nucleon radius"[6] obtained from the $\pi^- - p$ diffraction scattering analyses is evidence for the presence of the pion exchange interaction and thus supports the conjecture of a $\pi - \pi$ interaction. The analyses indicate that despite the weakness of the long-range interaction, it makes an important contribution to inelastic pion production. In this case the Fermi statistical model would have to be modified.[5]

Single virtual pion exchange is possible only if a pion "excited state," π^*, is formed which decays into at least two pions.[7] This exchange process, which favors low values of the square of the virtual pion four-momentum, Δ^2, leads to backward peaking of the nucleon, N, and to forward peaking of the π^* in the over-all barycentric system, that is, to low kinetic energy peaking of the nucleon in the laboratory.[3] Analysis of the angular distribution in single pion production from 4.5-Bev π^--proton interactions suggests that this long-range interaction makes a dominant contribution to the cross section for small values of Δ^2.[2]

Exchange of a virtual pion may also produce a nucleon excited state, N^*, which subsequently decays into a pion and a nucleon. One expects N^* production to be important because of the $\frac{3}{2}$-$\frac{3}{2}$ pion-nucleon resonance. At incident pion energies of the order of 5 Bev, N^* production can occur for small values of Δ^2, leading to backward peaking of the N^* and to forward peaking of the π^* in the over-all barycentric system. The decay of

* Supported in part by a grant from the National Science Foundation.
† Present address, University of Colorado, Boulder, Colorado.
[1] L. M. Eisberg, W. B. Fowler, R. M. Lea, W. D. Shephard, R. P. Shutt, A. M. Thorndike, and W. L. Whittemore, Phys. Rev. **97**, 797 (1955); and W. D. Walker and J. Crussard, Phys. Rev. **98**, 1416 (1955).
[2] W. D. Walker, Phys. Rev. **108**, 872 (1957); and G. Maenchen, W. B. Fowler, W. M. Powell, and R. W. Wright, Phys. Rev. **108**, 850 (1957).
[3] C. Goebel, Phys. Rev. Letters **1**, 337 (1958).
[4] F. Bonsignori and F. Selleri, Nuovo cimento **15**, 465 (1960); and I. Derado, Nuovo cimento **15**, 853 (1960).
[5] V. S. Barashenkov, Nuovo cimento **14**, 656 (1959), Nuclear Phys. **15**, 486 (1960), V. S. Barashenkov and V. M. Maltsev, Joint Institute for Nuclear Research Report P-433, Dubna, 1959 (unpublished); and F. Cerulus and J. von Behr, CERN preprint (1960). These papers report that improved results in fitting the data for $N-N$ and $\pi-N$ collisions in the Bev range are obtained by modifying the statistical model to take into account peripheral collisions. We are indebted to the above authors for communicating their results to us.
[6] The nucleon radius obtained from the Stanford electron-nucleon scattering experiments is $\sim(2\mu)^{-1}\approx 0.7\times10^{-13}$ cm, because the nucleon must reabsorb the virtually emitted pion.
[7] T. D. Lee and C. N. Yang, Nuovo cimento **3**, 749 (1956).

the N^* then favors backward peaking of the nucleon, which is strongly indicated by the experimental data at this energy.[2] It therefore appears that in double pion production, as in single pion production, the part of the cross section in which the nucleon undergoes small deflections may be explained by the long-range interaction.

In several recent papers[5] statistical-model analyses of pion-nucleon and nucleon-nucleon collisions have been carried out in which a peripheral interaction is assumed. In this paper the reactions $\pi^-+p\rightarrow N+2\pi$ and $\pi^-+p\rightarrow N+3\pi$ are analyzed in terms of a model based on meson field theory, in which it is assumed that for incident pions with laboratory momentum of 5 Bev/c, the long-range interaction dominates both reactions for small Δ^2, and arises from the exchange of a virtual pion. The model is formulated in Sec. 2 for a restricted region of phase space in which it is reasonable to neglect final-state interactions. The kinematics and the phase space are discussed in Sec. 3.

In Sec. 4 the differential cross section for double pion production is obtained. It is shown that, for small values of Δ^2, the virtual pion possesses a kinematical symmetry with respect to each of the incident particles in that it behaves as an incoming "almost real" pion in both the π^* and the N^* barycentric systems.[8] The formation of the two excited states, π^* and N^*, is then expressed in terms of $\pi-\pi$ and $\pi-N$ "scattering."

An advantage of considering both the single and double pion production processes in this model is that suitably defined ratios of their differential cross sections are independent of the details of the as yet unknown $\pi-\pi$ interaction and depend only on the relative strengths of N^* and N formation. These ratios are defined in Sec. 4 and are evaluated by means of the static nucleon approximation in Sec. 5, where it is shown that the π^*+N^* production exceeds that of π^*+N for a certain range of values of the phase space variables introduced. The qualitative predictions of this model, for the case in which the $\frac{3}{2}$–$\frac{3}{2}$ pion-nucleon resonant state dominates, are given in Sec. 6. It is found that the $\frac{3}{2}$–$\frac{3}{2}$ pion-nucleon resonant state is an important final-state configuration of the N^*, but, because of large phase-space factors, other final-state configurations may also be important. Conclusions based on present data should be regarded as tentative because of large experimental uncertainties.

2. FORMULATION OF THE MODEL

The following reactions are considered:

$$\pi^-+p\rightarrow N+2\pi,$$

$$\pi^-+p\rightarrow N+3\pi.$$

[8] In order to emphasize this kinematical symmetry, we use the term "long-range interaction" in preference to "peripheral interaction" because the latter term suggests that the virtual pion is in the peripheral part of the cloud surrounding one (rather than the other) of the two incident particles.

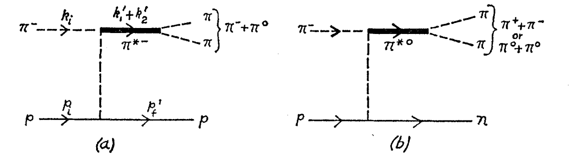

FIG. 1. Graphs for single pion production. π^{*-} and π^{*0} are negative and neutral excited pion states, respectively.

Since it will be necessary to consider several coordinate systems, the following notation is convenient. $q=(\mathbf{q},E)$ stands for the four-momentum of a particle and satisfies the invariant relation $q^2=(\mathbf{q})^2-E^2=-m^2$, where m is the rest mass of the particle. The components of q in a particular coordinate system, e.g., the laboratory system (L), are denoted with upper case subscripts $(\mathbf{q}_L,E_L)$, and q_L stands for $|\mathbf{q}_L|$. The relative energy of two particles with four-momenta p and q, i.e., their total energy in their own barycentric system, E_{pq}, is given by

$$E_{pq}{}^2=-(p+q)^2.$$

The initial state consists of the incident pion, with four-momentum $k_i=(\mathbf{k}_i,\omega_i)$, and the incident nucleon, with four-momentum $p_i=(\mathbf{p}_i,E_i)$, where $k_i^2=-\mu^2$, $p_i^2=-M^2$, and μ and M are the pion and nucleon rest masses, respectively. The two-pion final state $(N+2\pi)$ contains pions with $k_1'=(\mathbf{k}_1',\omega_1')$ and k_2' and a nucleon with $p_f'=(\mathbf{p}_f',E_f')$. The three-pion final state $(N+3\pi)$ contains pions $k_1=(\mathbf{k}_1,\omega_1)$, k_2, k_3 and a nucleon $p_f=(\mathbf{p}_f,E_f)$. For small values of the square of the virtual pion four-momentum, Δ^2, the diagrams considered for single and double pion production are shown in Figs. 1 and 2, respectively. The pion propagator is taken to be $(\Delta^2+\mu^2)^{-1}$.

In single pion production the value of Δ^2 is uniquely determined by the final state,

$$\Delta^2=(p_f'-p_i)^2,$$

because there is only one diagram (see Fig. 1) for each final-state configuration. This is not the case in double pion production because, in general, a given final-state pion may be produced either at the $\pi-\pi$ vertex or at the $\pi-N$ vertex, and each possibility corresponds to a different value of the square of the virtual pion four-momentum. Thus, if the final-state pion k is produced at the nucleon, the square of the virtual pion four-momentum is

$$\Delta_k{}^2=(p_f+k-p_i)^2.$$

One may, however, restrict the $(N+3\pi)$ final-state phase space, for $k_{iL}=5$ Bev/c, so that only one of the graphs, say that one in which k_3 is emitted at the $\pi-N$ vertex [see, e.g., Fig. 2(a)], is important. The simplest way to accomplish this is by (1) restricting the relative energy $E_{k_3p_f}$ of pion k_3 and the nucleon p_f, to be close to the energy of the $\frac{3}{2}$–$\frac{3}{2}$ pion-nucleon resonance, $W_{\text{res}}\approx 8.8\,\mu$; (2) restricting the relative energy $E_{k_1k_2}$ of pions k_1 and k_2, to be small, i.e., $\lesssim 3\,\mu$ or $4\,\mu$; and (3) re-

stricting $\Delta_{k_3}^2$ to as small values ($\sim 1\,\mu^2$) as are consistent with restrictions (1) and (2). These restrictions lead to the following results:

(1) The relative energies, $E_{k_1 p_f}$ of pion k_1 and the nucleon, and $E_{k_2 p_f}$ of pion k_2 and the nucleon, are much greater than W_{res}. In particular, if $E_{k_1 k_2} \approx 2\,\mu$, then $E_{k_1 p_f}$ and $E_{k_2 p_f}$ lie in the range from $13.6\,\mu$ to $16.4\,\mu$, the specific value depending on the nucleon momentum.

(2) The minimum value of $\Delta_{k_3}^2$ is much less than that of $\Delta_{k_1}^2$ and $\Delta_{k_2}^2$. Again, if $E_{k_1 k_2} \approx 2\,\mu$, the minimum value of $\Delta_{k_3}^2$ is $\approx 0.2\,\mu^2$ while that of $\Delta_{k_1}^2$ and $\Delta_{k_2}^2$ is $\approx 10\,\mu^2$.

With these restrictions one may thus neglect the graphs for k_1 or k_2 production at the nucleon compared to that for k_3 production at the nucleon, both because of their smaller virtual pion propagators and also because of their weaker pion-nucleon vertices. The natural grouping for such final-state configurations [see, e.g., Fig. 2(a)] is $\pi^*(k_1+k_2)$ and $N^*(k_3+p_f)$. This restricted final state then determines $\Delta_{k_3}^2$, which can thus be denoted simply by Δ^2.

For larger values of $E_{k_1 k_2}$ one of the pions k_1 or k_2, say k_1, may have its four-momentum close to that of k_3, in which case $E_{k_1 p_f} \approx E_{k_3 p_f}$ and $\Delta_{k_1}^2 \approx \Delta_{k_3}^2$. This configuration occurs for events in which, in the over-all barycentric system, the pion k_3 from the N^* is emitted forward with low energy and the π^* decays into two forward pions, one with very high and the other with very low energy. This small part of the phase space may be avoided by examining only events in which one pion is emitted backward in the over-all barycentric system. For the purposes of this paper the restriction to smaller values of $E_{k_1 k_2}$, mentioned above, is used.

Only those $(N+2\pi)$ final state configurations will be considered for which the values of Δ^2 and $E_{k_1' k_2'}$ are the same as the values of Δ^2 and $E_{k_1 k_2}$ of the $(N+3\pi)$ final state as just restricted.

In these regions of the $(N+2\pi)$ and $(N+3\pi)$ final-state phase spaces it is reasonable to assume that the π^* decays without further interaction with the nucleon of the N or N^*. First, each of the pions from the π^* has relative energy with respect to the nucleon much greater than W_{res}, and one expects pion-nucleon rescattering to be unimportant far from the resonance. Second, because the π^* is created at large distances from the N or N^*, i.e., small Δ^2, where the interaction is weak, and the wave function of each of its pions tends to be small at the nucleon, where the interaction is strong, rescattering corrections are expected not to be important even for those energies at which an incident free pion would interact strongly with the nucleon.[9]

It is not possible at this time to estimate the importance of final-state interactions between the pions

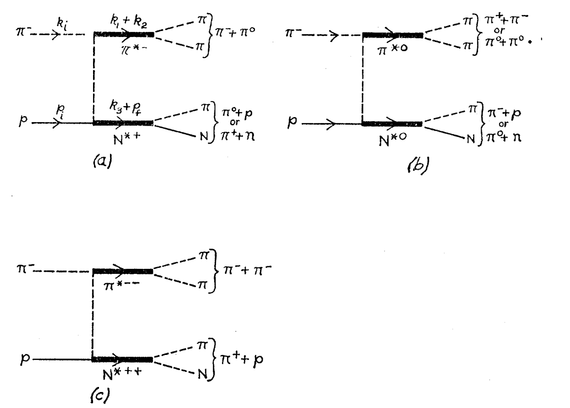

FIG. 2. Graphs for double pion production. N^{*+}, N^{*0}, and N^{*++} are positive, neutral, and doubly positive excited nucleon states, respectively.

from the π^* and that of the N^*. It seems likely that the long-range $\pi-\pi$ interaction will be weak, in which case it is also reasonable to neglect these final-state interactions.

The pion-nucleon vertex for the $(N+2\pi)$ final state is denoted by $\langle p_f' | \Gamma_5 | p_i \rangle$ and for the $(N+3\pi)$ final state by $\langle k_3, p_f | \Sigma_{\pi N} | p_i \rangle$. The $\pi-\pi$ vertex is denoted by $\langle k_1, k_2 | \Lambda_{\pi\pi} | k_i \rangle$. The index k_j stands for the charge state of the jth pion as well as its four-momentum. The model consists in assuming that for small values of Δ^2 the differential cross section for single pion production is

$$d\sigma_{N+2\pi} = \frac{M}{8(2\pi)^5 k_{iL}} |\langle k_1', k_2' | \Lambda_{\pi\pi} | k_i \rangle|^2 \times \frac{1}{(\Delta^2+\mu^2)^2} [|\langle p_f' | \Gamma_5 | p_i \rangle|^2]_{\text{av}} \times \delta^4(k_i + p_i - k_1' - k_2' - p_f') \times \frac{d^3k_1'}{\omega_1'} \frac{d^3k_2'}{\omega_2'} \frac{d^3p_f'}{E_f'}, \quad (2.1)$$

and for double pion production is

$$d\sigma_{N+3\pi} = \frac{M}{16(2\pi)^8 k_{iL}} |\langle k_1, k_2 | \Lambda_{\pi\pi} | k_i \rangle|^2 \times \frac{1}{(\Delta^2+\mu^2)^2} [|\langle k_3, p_f | \Sigma_{\pi N} | p_i \rangle|^2]_{\text{av}} \times \delta^4(k_i + p_i - k_1 - k_2 - k_3 - p_f) \times \frac{d^3k_1}{\omega_1} \frac{d^3k_2}{\omega_2} \frac{d^3k_3}{\omega_3} \frac{d^3p_f}{E_f}, \quad (2.2)$$

where the invariant flux[10] has been evaluated in the

[9] At lower incident pion energies, the rescattering of a pion produced far from the nucleon by a $\pi-\pi$ interaction is not found to be important, even for those pion energies close to the $\frac{3}{2}$–$\frac{3}{2}$ resonance. C. Goebel and H. J. Schnitzer (private communication). Similar results were obtained in photopion production by G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[10] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955), Sec. 8–6.

laboratory, and the subscript "av" indicates the sum and average over the final and initial nucleon spin states, respectively. If Eqs. (2.1) and (2.2) are evaluated for the same values of Δ^2, then k_1' and k_2' are different than k_1 and k_2.

3. KINEMATICS AND PHASE SPACE

The following coordinate systems and variables are introduced and defined for the $(N+3\pi)$ final state:

(L), the laboratory system, in which $\mathbf{p}_{iL}=\mathbf{0}$;

(U), the over-all barycentric system, in which $\mathbf{k}_{iU}+\mathbf{p}_{iU}=\mathbf{0}$, and in which the total energy of the complete system is designated by U, and given by

$$-U^2=(k_i+p_i)^2;$$

(V), the π^* barycentric system, in which $\mathbf{k}_{1V}+\mathbf{k}_{2V}=\mathbf{0}$, and in which the total energy of the π^* system is designated by V, and given by

$$-V^2=(k_1+k_2)^2;$$

(W), the N^* barycentric system, in which $\mathbf{k}_{3W}+\mathbf{p}_{fW}=\mathbf{0}$, and in which the total energy of the N^* system is designated by W, and given by

$$-W^2=(k_3+p_f)^2.$$

The natural phase space variables for describing double pion production are Δ^2 and V, as in single pion production,[11] and the additional variable W.

The total four-momentum of the π^* system is

$$K=k_1+k_2,$$

and that of the N^* system is

$$P=k_3+p_f.$$

From these definitions it follows that

$$\Delta^2=(K-k_i)^2=(P-p_i)^2. \tag{3.1}$$

The components of K and P in a coordinate system (X) are denoted $(\mathbf{K}_X,V_X)$ and $(\mathbf{P}_X,W_X)$. We then have, remembering that V is the "rest mass" of the π^* system and W that of the N^* system,

$$V_X=(K_X^2+V^2)^{\frac{1}{2}},$$

and

$$W_X=(P_X^2+W^2)^{\frac{1}{2}}. \tag{3.2}$$

One then obtains the following relations:

In (L),

$$\Delta^2=-W^2-M^2+2MW_L, \tag{3.3}$$

$$-V^2=-U^2-W^2-2\mathbf{k}_{iL}\cdot\mathbf{P}_L+2(\omega_{iL}+M)W_L; \tag{3.4}$$

in (U),

$$E_{iU}=(U^2+M^2-\mu^2)/2U,$$
$$\omega_{iU}=(U^2+\mu^2-M^2)/2U, \tag{3.5}$$
$$U^2=2\omega_{iL}M+M^2+\mu^2,$$

$$V_U=(U^2+V^2-W^2)/2U,$$
$$W_U=(U^2+W^2-V^2)/2U, \tag{3.6}$$

$$\Delta^2=-V^2-\mu^2-2\mathbf{K}_U\cdot\mathbf{k}_{iU}+2V_U\omega_{iU}; \tag{3.7}$$

in (V),

$$\omega_{iV}=(V^2+\mu^2+\Delta^2)/2V, \tag{3.8}$$

$$\omega_{1V}=\omega_{2V}=V/2; \tag{3.9}$$

in (W),

$$E_{iW}=(W^2+M^2+\Delta^2)/2W, \tag{3.10}$$

$$E_{fW}=(W^2+M^2-\mu^2)/2W,$$
$$\omega_{3W}=(W^2-M^2+\mu^2)/2W. \tag{3.11}$$

The limits of the phase space, for a given value of the total energy U, are determined as follows. From Eq. (3.7) one finds the maximum and minimum values of Δ^2, for given V and W, to be

$$[\Delta^2(V,W)]_{\min}^{\max}=-V^2-\mu^2\pm 2K_Uk_{iU}+2V_U\omega_{iU}, \tag{3.12}$$

where V_U is given by Eq. (3.6), K_U is then given by Eq. (3.2), ω_{iU} by Eq. (3.5), and $k_{iU}=(\omega_{iU}^2-\mu^2)^{\frac{1}{2}}$. The minimum value of Δ^2, for given V and W, occurs for the configuration, in the system (U), in which the center of

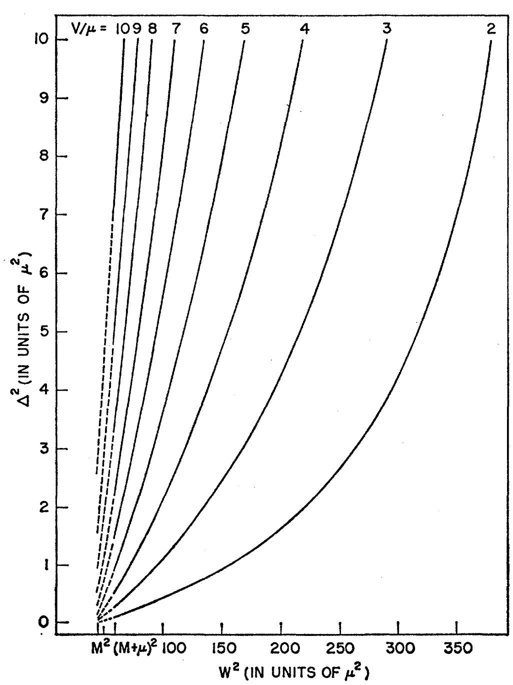

FIG. 3. Phase space for incident pion laboratory momentum, k_{iL}, equal to 5 Bev/c. The solid curves represent the phase space for N^* production and give, for given values of V, the minimum value of Δ^2 as a function of W^2, and correspondingly, the maximum value of W^2 as a function of Δ^2. The values of Δ^2 to which these curves extrapolate at $W^2=M^2$ are the corresponding minimum values of Δ^2 for N production.

[11] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

mass of the π^* moves in the same direction as the incident pion and the center of mass of the N^* moves in the same direction as the incident nucleon. The minimum value of V is 2μ and that of W is $\mu+M$. The maximum value of W for a given value of V is $U-V$. The maximum value of V is $U-(\mu+M)$.

Part of the phase space for these three variables is shown in Fig. 3 for $k_{iL}=5$ Bev/c. The curves give the minimum Δ^2 for a given V as a function of W^2. One can also obtain from a given curve the maximum value of W for given values of V and Δ^2. The lowest value of Δ^2, $0.1\,\mu^2$, occurs for $V=2\,\mu$ and $W=\mu+M$, and increases with V and with W. From Eqs. (3.2), (3.5), (3.6), and (3.7), one finds that for low values of V and W, $K_U\approx k_{iU}\approx 10\,\mu$, and that Δ^2 increases rapidly with the angle between $\mathbf{K}_U$ and $\mathbf{k}_{iU}$. Because of the strong angular dependence, one may assume, for small Δ^2, that the π^* is produced in the forward direction in the system (U).

Similar coordinate systems and variables are introduced for the $(N+2\pi)$ final state, the only differences being that (V), the π^* barycentric system, is defined by the condition $\mathbf{k}_{1V}'+\mathbf{k}_{2V}'=\mathbf{0}$; $-V^2=(k_1'+k_2')^2$; (W) is simply the rest system of the N, $\mathbf{p}_{fW}'=\mathbf{0}$; and $W=M$. The minimum value of Δ^2 for $k_{iL}=5$ Bev/c is obtained from Eq. (3.12) if W is replaced by M, and is shown in Fig. 3 at $W^2=M^2$. The lowest value of Δ^2, $0.002\,\mu^2$, occurs for $V=2\,\mu$, and increases with V.

Analysis of the single-pion production cross section shows that important contributions come from values of Δ^2 greater than μ^2.[3,4,11] From Fig. 3 it can be seen that, even for values of Δ^2 as low as $1.5\,\mu^2$, the $(N+3\pi)$ final-state phase space can include production of $\pi^*(V_{\mathrm{res}})+N^*(W_{\mathrm{res}})$, where V_{res}, the conjectured $\pi-\pi$ resonance energy, is of the order of $3\,\mu$ to $4\,\mu$.[12] In any case, there is a considerable range of values of V, including V_{res}, for which, even for small values of Δ^2, the phase space includes the $\frac{3}{2}$–$\frac{3}{2}$ pion-nucleon resonant state, $N^*(W_{\mathrm{res}})$. We therefore expect to obtain a range of values of Δ^2 for which the N^* production may be comparable to the N production.

4. REDUCTION OF THE DOUBLE-PION PRODUCTION CROSS SECTION

In this section the double-pion production cross section, Eq. (2.2), is expressed in terms of the variables of interest, Δ^2, V, and W, and then compared to the single-pion production cross section, expressed in terms of Δ^2 and V. The total and relative three-momenta for the π^* and N^* systems are

$$\begin{aligned}\mathbf{K}&=\mathbf{k}_1+\mathbf{k}_2, & \mathbf{P}&=\mathbf{k}_3+\mathbf{p}_f,\\ \mathbf{k}&=(\mathbf{k}_1-\mathbf{k}_2)/2, & \mathbf{p}&=(\mathbf{k}_3-\mathbf{p}_f)/2.\end{aligned}\tag{4.1}$$

By writing Eq. (2.2) in terms of these variables and performing the d^3K integration, one obtains

$$\begin{aligned}d\sigma_{N+3\pi}=&\frac{M}{16(2\pi)^8k_{iL}}|\langle k_1,k_2|\Lambda_{\pi\pi}|k_i\rangle|^2\\ &\times\frac{1}{(\Delta^2+\mu^2)^2}[|\langle k_3,p_f|\Sigma_{\pi N}|p_i\rangle|^2]_{\mathrm{av}}\\ &\times\frac{\delta(E_i+\omega_i-\omega_1-\omega_2-\omega_3-E_f)d^3k}{\omega_1\omega_2}\\ &\times\frac{d^3Pd^3p}{\omega_3E_f}.\end{aligned}\tag{4.2}$$

Each of the two phase space factors, as grouped in the last equation, is Lorentz invariant. The first factor is evaluated in the system (V). Integration over dk_V then yields

$$\int\frac{d^3k_V}{\omega k_V{}^2}\delta(E_{iV}+\omega_{iV}-\omega_{1V}-\omega_{2V}-\omega_{3V}-E_{fV})=\frac{k_V}{V}d\Omega_{kV}.\tag{4.3}$$

The second factor is evaluated in the system (L), but expressed in terms of $\mathbf{P}_L$ and the relative three-momentum in W, which, from Eq. (4.1), is equal to $\mathbf{k}_{3W}$. Calculation of the Jacobian then yields for the transformation,

$$d^3P_Ld^3p_L=d^3P_Ld^3k_{3W}\frac{1}{W_LW}\left\{W_L{}^2-\frac{(\mathbf{P}_L\cdot\mathbf{k}_{3W})^2}{\omega_{3W}E_{fW}}-\frac{\mathbf{P}_L\cdot\mathbf{k}_{3W}W_L(\omega_{3W}-E_{fW})}{\omega_{3W}E_{fW}}\right\},\tag{4.4}$$

where W_L is defined by Eq. (3.2). The expression for $\omega_{3L}E_{fL}$ in terms of the variables $\mathbf{P}_L$ and $\mathbf{k}_{3W}$ is obtained from the Lorentz transformation and is given by

$$\omega_{3L}E_{fL}=\frac{\omega_{3W}E_{fW}}{W^2}\left\{W_L{}^2-\frac{(\mathbf{P}_L\cdot\mathbf{k}_{3W})^2}{\omega_{3W}E_{fW}}-\frac{\mathbf{P}_L\cdot\mathbf{k}_{3W}W_L(\omega_{3W}-E_{fW})}{\omega_{3W}E_{fW}}\right\}.\tag{4.5}$$

Equations (4.4) and (4.5) then give for the second phase space factor

$$\frac{d^3P_Ld^3p_L}{\omega_{3L}E_{fL}}=\frac{d^3P_L}{W_L}\frac{d^3k_{3W}W}{\omega_{3W}E_{fW}}.\tag{4.6}$$

The variables P_L and k_{3W} are linked by $W_L=(P_L{}^2+W^2)^{\frac{1}{2}}$, where

$$W=(k_{3W}{}^2+\mu^2)^{\frac{1}{2}}+(k_{3W}{}^2+M^2)^{\frac{1}{2}}.\tag{4.7}$$

It is now convenient to reduce Eq. (4.6) to the vari-

[12] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959).

ables of interest. By noting that

$$\frac{d^3P_L}{W_L}=\frac{P_L^2}{W_L}dP_Ld(\cos\theta_{PL})d\Phi_{PL},$$

and by using Eqs. (3.2) and (3.3) to express dP_L in terms of $d(\Delta^2)$, and Eq. (3.4) to express $d(\cos\theta_{PL})$ in terms of $d(V^2)$, one obtains

$$\frac{d^3P_L}{W_L}=\frac{1}{4Mk_{iL}}d(\Delta^2)d(V^2)d\Phi_{PL}. \tag{4.8}$$

From Eq. (4.7) one obtains

$$d^3k_{3W}=k_{3W}^2dk_{3W}d\Omega_{k3W}=\frac{k_{3W}\omega_{3W}E_{fW}}{2W^2}d(W^2)d\Omega_{k3W}. \tag{4.9}$$

Equation (4.2) is now expressed in terms of the variables of interest, by making use of Eqs. (4.3), (4.6), (4.8), and (4.9), as follows:

$$d\sigma_{N+3\pi}=\frac{1}{128(2\pi)^8k_{iL}^2}|\langle k_1k_2|\Lambda_{\pi\pi}|k_i\rangle|^2$$
$$\times\frac{1}{(\Delta^2+\mu^2)^2}[|\langle k_3,p_f|\Sigma_{\pi N}|p_i\rangle|^2]_{\mathrm{av}}$$
$$\times\frac{k_V}{V}\frac{k_{3W}}{W}d\Omega_{kV}d\Omega_{k3W}d(W^2)$$
$$\times d(V^2)d(\Delta^2)d\Phi_{PL}. \tag{4.10}$$

These variables are no longer linked.

The most general dependence of the invariant vertex functions on the variables of the above equation is as follows: $[|\langle k_3,p_f|\Sigma_{\pi N}|p_i\rangle|^2]_{\mathrm{av}}=\Sigma_{\pi N}(\Delta^2,W,\cos\theta_W)$, where $\cos\theta_W=[\mathbf{k}_{3W}\cdot(-\mathbf{p}_{iW})]/k_{3W}p_{iW}$, and $|\langle k_1,k_2|\Lambda_{\pi\pi}|k_i\rangle|^2=\Lambda_{\pi\pi}(\Delta^2,V,\cos\theta_V)$, where $\cos\theta_V=(\mathbf{k}_V\cdot\mathbf{k}_{iV})/k_Vk_{iV}$. In the case of the pion-nucleon vertex, this may be seen by constructing the three possible invariants from the three independent four-vectors p_i,p_f, and k_3 and by evaluating them in the system (W), with the help of Eqs. (3.10) and (3.11). The pion-pion vertex may be treated similarly.

For small values of Δ^2, one expects the virtual pion to behave as an "almost real" particle. Equations (3.10) and (3.11) show that as $\Delta^2\to-\mu^2$, $E_{iW}\to E_{fW}$, that is, as the virtual pion approaches the mass shell the kinematics of the pion-nucleon vertex approaches that for a real, energy-elastic scattering process in the system (W), in which the "virtual" pion carries energy equal to ω_{3W} into the vertex. It should be emphasized that, although from the viewpoint of invariant field theory the virtual pion cannot be assigned a unique sense of propagation, nevertheless for small values of Δ^2 it behaves, in the system (W), as an incoming "almost real" pion which scatters on the nucleon to produce the N^*. In the limit $\Delta^2\to-\mu^2$, if one assumes that such a limiting procedure is possible, the pion-nucleon vertex function will equal that for real pion-nucleon scattering at energy W and scattering angle θ_W, where θ_W is the angle of the outgoing pion with respect to the "incoming" one.

Surprising though it may at first appear, the same limiting procedure, $\Delta^2\to-\mu^2$, which shows that for small Δ^2 the virtual pion behaves as an incoming "almost real" pion *with respect to the incident nucleon in the system* (W), also shows that it behaves as an incoming "almost real" pion *with respect to the incident pion in the system* (V). To see this, note that Eqs. (3.7) and (3.8) give, as $\Delta^2\to-\mu^2$, $\omega_{iV}\to V/2=\omega_{1V}=\omega_{2V}$, that is, the kinematics of the pion-pion vertex approaches that for a real, energy-elastic scattering process in the system (V), in which the "virtual" pion carries energy equal to $V/2$ into the vertex. Again, in the limit $\Delta^2\to-\mu^2$, if one assumes that such a limiting procedure is possible, the pion-pion vertex function will equal that for real pion-pion scattering at energy V and scattering angle θ_V.

For sufficiently small values of Δ^2 in the physical region, the virtual pion continues to behave kinematically as an incoming "almost real" particle at each vertex in the appropriate coordinate system. To emphasize this property we introduce the following suggestive notation:

$$\int|\langle k_1,k_2|\Lambda_{\pi\pi}|k_i\rangle|^2d\Omega_{kV}=16(2\pi)^2V^2\sigma_{\pi\pi}(\Delta^2,V),$$
$$\int[|\langle k_3,p_f|\Sigma_{\pi N}|p_i\rangle|^2]_{\mathrm{av}}d\Omega_{k3W}=4(2\pi)^2(W/M)^2\sigma_{\pi N}(\Delta^2,W), \tag{4.11}$$

where the charge states are no longer indicated on the right-hand sides of the equations. They may be reintroduced later without difficulty. With these definitions, the differential cross section for double pion production becomes, after performing the $d\Omega_{kV}$, $d\Omega_{k3W}$, and $d\Phi_{PL}$ integrations,

$$\frac{\partial^3\sigma_{N+3\pi}}{\partial(\Delta^2)\partial(V^2)\partial(W^2)}=\frac{1}{2(2\pi)^3k_{iL}^2M^2}\sigma_{\pi\pi}(\Delta^2,V)k_VV$$
$$\times\frac{1}{(\Delta^2+\mu^2)^2}\sigma_{\pi N}(\Delta^2,W)k_{3W}W. \tag{4.12}$$

The corresponding differential cross section for single pion production is

$$\frac{\partial^2\sigma_{N+2\pi}}{\partial(\Delta^2)\partial(V^2)}=\frac{1}{2(2\pi)^2k_{iL}^2}\sigma_{\pi\pi}(\Delta^2,V)k_VV$$
$$\times\frac{1}{(\Delta^2+\mu^2)^2}[|\langle p_f'|\Gamma_5|p_i\rangle|^2]_{\mathrm{av}}, \tag{4.13}$$

where the factor $[|\langle p_f'|\Gamma_5|p_i\rangle|^2]_{av}$ is a function of Δ^2 only. Note that the π^* production depends only on the variables Δ^2 and V, and that the corresponding factors enter each of the last two equations in exactly the same way. Likewise the N^* production depends only on the variables Δ^2 and W. Of course, in the final state $\pi^*(\Delta^2,V)+N$ the π^* has more translational energy in the laboratory than does the π^* in the final state $\pi^*(\Delta^2,V)+N^*(\Delta^2,W)$, and accordingly the four vector Δ is different in the two processes despite the equality of Δ^2. In terms of the natural variables Δ^2, V, and W, Eq. (4.12) exhibits, as does Fig. 2, the kinematically symmetric way in which the $\pi-\pi$ and the $\pi-N$ interactions enter into the $N+3\pi$ differential cross section. This again reflects the exchange nature of the virtual pion.

From Eqs. (4.12) and (4.13) one obtains, for the same π^* charge configuration, Δ^2, and V, and for a given value of W, the ratio

$$\frac{\partial^3\sigma_{N+3\pi}}{\partial(\Delta^2)\partial(V^2)\partial(W^2)}\bigg/\frac{\partial^2\sigma_{N+2\pi}}{\partial(\Delta^2)\partial(V^2)}=\frac{(1/2\pi M^2)k_{3W}W\sigma_{\pi N}(\Delta^2,W)}{[|\langle p_f'|\Gamma_5|p_i\rangle|^2]_{av}}, \quad (4.14)$$

which is independent of the $\pi-\pi$ interaction. The advantage of comparing this ratio to experiment is that one may test the validity of the model without making any assumptions about the detailed nature of the $\pi-\pi$ interaction. To indicate a particular final charge configuration of the N^*, for example π^0+n, $\sigma_{\pi N}(\Delta^2,W)$ will be replaced by $\sigma_{\pi^-+p\to\pi^0+n}(\Delta^2,W)$. We define

$$\sigma_{\pi^-p}{}^{\text{tot}}(\Delta^2,W)=\sigma_{\pi^-+p\to\pi^-+p}(\Delta^2,W)+\sigma_{\pi^-+p\to\pi^0+n}(\Delta^2,W),$$

for later use.

The ratio of the total probability for producing the N^* to that for producing the N, with the same π^* charge configuration, Δ^2, and V, is

$$\mathrm{R}(N^*/N)=\frac{\partial^2\sigma_{N+3\pi}}{\partial(\Delta^2)\partial(V^2)}\bigg/\frac{\partial^2\sigma_{N+2\pi}}{\partial(\Delta^2)\partial(V^2)}=\frac{1}{\pi M^2}\int_{W=\mu+M}^{W=f(k_{iL},\Delta^2,V)}\sigma_{\pi N}{}^{\text{tot}}(\Delta^2,W)k_{3W}W^2dW\bigg/[|\langle p_f'|\Gamma_5|p_i\rangle|^2]_{av}, \quad (4.15)$$

where $f(k_{iL},\Delta^2,V)$ is an algebraically involved function that determines the upper limit of the integration variable, W.

Because of the experimental difficulties involved in analyzing events in which neutral particles are produced, the four-prong events of Fig. 2(b) and the corresponding two-prong events of Fig. 1(b) may be of particular interest. For these events the appropriate ratio, which will be denoted by $\mathrm{R}([N^{*0}\to\pi^-+p]/n)$, where N^{*0} stands for the excited nucleon state with zero total charge, is obtained from Eq. (4.15) by replacing $\sigma_{\pi N}{}^{\text{tot}}(\Delta^2,W)$ in the numerator by $\sigma_{\pi^-+p\to\pi^-+p}(\Delta^2,W)$, and by evaluating the denominator for the case that p_i is a proton and p_f' a neutron.

Estimates of the pion-nucleon vertex functions are given in the next section.

Even if the virtual pion exchange graph does not dominate the double-pion production cross section for small Δ^2, one may still look for the presence of the second-order pole term, as given by Eq. (4.12), by studying the experimental differential cross section as a function of Δ^2, in a similar manner to that proposed for the single-pion production cross section.[3,4,11] The analyses carried out to date[4] have studied $d\sigma_{N+2\pi}/d(\Delta^2)$, that is the integration with respect to V has been performed, because of the limited statistics. With improved statistics it will be possible to study $(\Delta^2+\mu^2)^{-2}\partial^2\sigma_{N+2\pi}/\partial(\Delta^2)\partial(V^2)$ as a function of Δ^2, for a fixed value of V, and to attempt to obtain the limit as $\Delta^2\to-\mu^2$, which is directly related to the $\pi-\pi$ interaction. If $(\Delta^2+\mu^2)^{-2}\partial^3\sigma_{N+3\pi}/\partial(\Delta^2)\partial(V^2)\partial(W^2)$ is also studied as a function of Δ^2, for the same fixed value of V, and the same charge configuration of the π^*, and the limit as $\Delta^2\to-\mu^2$ is determined, then the ratio of this limit to that for the $N+2\pi$ function will be independent of the $\pi-\pi$ interaction. If it is assumed that

$$\lim_{\Delta^2\to-\mu^2}\sigma_{\pi^-p}(\Delta^2,W)=\sigma_{\pi^-p}{}^{\text{tot}}(W)=\sigma_{\pi^-+p\to\pi^-+p}(W)+\sigma_{\pi^-+p\to\pi^0+n}(W),$$

then for the processes shown in Figs. 2(b) and 1(b), one finds the following ratio

$$\frac{\lim_{\Delta^2\to-\mu^2}(\Delta^2+\mu^2)^{-2}\partial^3\sigma_{N+3\pi}/\partial(\Delta^2)\partial(V^2)\partial(W^2)}{\lim_{\Delta^2\to-\mu^2}(\Delta^2+\mu^2)^{-2}\partial^2\sigma_{N+2\pi}/\partial(\Delta^2)\partial(V^2)}=\frac{(1/2\pi M^2)\sigma_{\pi^-p}{}^{\text{tot}}(W)k_{3W}W}{-8\pi f^2}, \quad (4.16)$$

where f^2, the renormalized, unrationalized pion-nucleon coupling constant, is ≈ 0.08. Evaluation of this ratio can thus serve as a test of the meaningfulness of the extrapolation procedure.

5. PION NUCLEON VERTEX FUNCTIONS

Estimates of the pion-nucleon vertex functions for N and N^* production will be based on the p-wave static nucleon model, applied in a coordinate system in which the nucleon kinetic energy, both before and after the collision, is small compared to its rest mass. For the $\pi-N$ vertex this model gives, in the laboratory system,

$$[|\langle p_f'|\Gamma_5|p_i\rangle|^2]_{av}=4\pi f^2\frac{p_{fL}'^2}{\mu^2}\times\begin{Bmatrix}1 \text{ for } \pi^0\\ 2 \text{ for } \pi^\pm\end{Bmatrix}, \quad (p_{fL}'^2\ll M^2), \quad (5.1)$$

where the factor 1 is for emission or absorption of a neutral pion and the factor 2 for that of a charged pion. For the case of interest, $\Delta^2 \ll M^2$, one obtains from Eq. (3.3), by setting $W=M$,

$$\Delta^2 = 2M[(p_{fL}'^2+M^2)^{\frac{1}{2}} - M] \approx p_{fL}'^2,$$

which allows Eq. (5.1) to be rewritten as

$$[|\langle p_f'|\Gamma_5|p_i\rangle|^2]_{\text{av}} = 4\pi f^2 \frac{\Delta^2}{\mu^2} \times \begin{Bmatrix} 1 \text{ for } \pi^0 \\ 2 \text{ for } \pi^\pm \end{Bmatrix}. \quad (5.2)$$

This is the same result that one obtains from invariance considerations, with the usual definition of the renormalized coupling constant at $\Delta^2 = -\mu^2$,[11] but keeping only the Δ^2 dependence arising from the spinor functions, that is, taking the pion-nucleon vertex form factor equal to 1.

For the $\pi - N^*$ vertex the static nucleon model is applied in the coordinate system (W). For $\Delta^2 \ll M^2$ and $W \approx M$, Eqs. (3.10) and (3.11) give

$$E_{iW} \approx M \quad \text{and} \quad E_{fW} \approx M,$$

so that the energy, in the system W, of the virtual "incoming" pion, $\omega_{\Delta W}$, is

$$\omega_{\Delta W} = W - E_{iW} \approx W - M \approx \omega_{3W}.$$

The virtual pion is of course not on the mass shell, the magnitude of its momentum, Δ_W, being given by

$$\Delta_W = (\omega_{\Delta W}^2 + \Delta^2)^{\frac{1}{2}} \approx [(W-M)^2 + \Delta^2]^{\frac{1}{2}},$$

while that for the real pion k_3 is

$$k_{3W} = (\omega_{3W}^2 - \mu^2)^{\frac{1}{2}} \approx [(W-M)^2 - \mu^2]^{\frac{1}{2}},$$

and thus although $\omega_{\Delta W} \approx \omega_{3W}$, $\Delta_W > k_{3W}$. Because the energy, $\omega_{\Delta W}$, is almost correct for an energy-elastic scattering in the coordinate system (W), it is natural to think of the virtual pion as being off the mass shell because the magnitude of its three-momentum, Δ_W, is too large compared to $\omega_{\Delta W}$. Therefore we conjecture that in the p-wave static nucleon model, one obtains the virtual pion-nucleon scattering amplitude by modifying only the momentum dependence, at the absorption vertex, of the real pion-nucleon scattering amplitude. This conjecture, which amounts simply to multiplication of the real scattering amplitude by the factor Δ_W/k_{3W}, is verified in lowest order pseudoscalar coupling (γ_5) theory, and gives

$$\sigma_{\pi N}(\Delta^2,W) = \left(\frac{\Delta_W}{k_{3W}}\right)^2 \sigma_{\pi N}(W) = \left[\frac{(W-M)^2+\Delta^2}{(W-M)^2-\mu^2}\right] \sigma_{\pi N}(W), \quad (5.3)$$

where $\sigma_{\pi N}(W)$ is the physical cross section. However, $\sigma_{\pi N}(\Delta^2,W)$ will be made to go smoothly to zero as W approaches $\mu+M$, that is, the factor k_{3W}^{-2}, which arises from the p-wave static nucleon model, is suppressed at low scattering energies where the cross section is predominantly s wave.

For N^{*0} production the experimental value of $\sigma_{\pi^- p}{}^{\text{tot}}(W)$ will be used directly. For the 2-prong decay mode of N^{*0}, i.e., $N^{*0} \to \pi^- + p$, dominance of the $\frac{3}{2}$–$\frac{3}{2}$ pion-nucleon resonant state is assumed (as well as charge independence), so that

$$\sigma_{\pi^- + p \to \pi^- + p}(W) \approx \tfrac{1}{3} \sigma_{\pi^- p}{}^{\text{tot}}(W).$$

For N^{*+} production the same assumptions give

$$\sigma_{\pi^0 p}{}^{\text{tot}}(W) \approx \tfrac{2}{3} \sigma_{\pi^+ p}(W).$$

In the last two cases the experimental values of $\sigma_{\pi^- p}{}^{\text{tot}}(W)$ and $\sigma_{\pi^+ p}(W)$ will be used.

Curve $\pi^- + p$ (a) of Fig. 4 is the spectrum $\sigma_{\pi^- p}{}^{\text{tot}}(\Delta^2,W) k_{3W} W^2$, for $\Delta^2 = \mu^2$, obtained from Eq. (5.3), as a function of W. For Δ^2 and μ^2 much less than $(W-M)^2$, Eq. (5.3) gives $\sigma_{\pi N}(\Delta^2,W) \approx \sigma_{\pi N}(W)$. For purposes of comparison, the spectrum $\sigma_{\pi^- p}{}^{\text{tot}}(W) k_{3W} W^2$ is shown as curve $\pi^- + p$ (b) of Fig. 4. One obtains a broad spectrum about the resonance, $W \approx 8.8\,\mu$, because of the large phase space factor, $k_{3W} W^2$. Curves $\pi^+ + p$ (a) and (b) are the corresponding spectra $\sigma_{\pi^+ p}(\Delta^2,W) k_{3W} W^2$ and $\sigma_{\pi^+ p}(W) k_{3W} W^2$, respectively, and show the effect of the resonance more strongly than the $\pi^- p$ spectra.

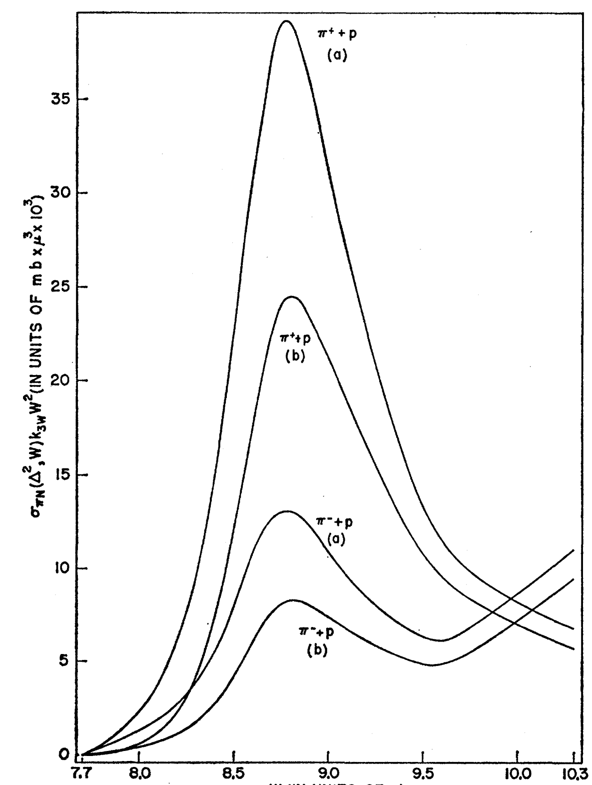

FIG. 4. N^* production spectra. Curves (a) are obtained with $\sigma_{\pi N}(\Delta^2,W)$ given by equation (5.3) for $\Delta^2=\mu^2$. Curves (b) are obtained by taking $\sigma_{\pi N}(\Delta^2,W)$ equal to the experimental cross section, $\sigma_{\pi N}(W)$.

Ratios for N^* to N production, obtained from Eq. (4.15) for several charge configurations, are now evaluated by use of Eq. (5.2) and the spectra of Fig. 4, for the values $k_{iL}=5$ Bev/c, $\Delta^2=\mu^2$, and $V=3\,\mu$. These are physically interesting values in the region of applicability of the model. The upper limit of the integration variable W, obtained from Fig. 3, is $9.8\,\mu$. For the ratio of the total neutral excited nucleon state production [all graphs of Fig. 2(b)] to neutron production [graphs of Fig. 1(b)] one finds

$$\mathrm{R}(N^{*0}/n) = \frac{1}{\pi M^2}\int_{W=7.7\,\mu}^{W=9.8\,\mu} \sigma_{\pi^- p}{}^{\mathrm{tot}}(\mu^2,W) k_{3W} W^2 dW \Big/ 8\pi f^2 = \begin{cases} \text{(a)} & 2.5, \\ \text{(b)} & 1.6, \end{cases} \tag{5.4}$$

where the values (a) and (b) correspond to curves π^-+p (a) and π^-+p (b) of Fig. 4. For the ratio of the total charged excited nucleon state production [all graphs of Fig. 2(a)] to proton "production" [graphs of Fig. 1(a)] one finds

$$\mathrm{R}(N^{*+}/p) = \frac{1}{\pi M^2}\int_{W=7.7\,\mu}^{W=9.8\,\mu} \sigma_{\pi^0 p}{}^{\mathrm{tot}}(\mu^2,W) k_{3W} W^2 dW \Big/ 4\pi f^2 = \begin{cases} \text{(a)} & 8.6, \\ \text{(b)} & 5.5, \end{cases} \tag{5.5}$$

where the values (a) and (b) correspond to ($\frac{2}{3}$ of) curves π^++p (a) and π^++p (b) of Fig. 4. For the ratio of the 4-prong events of Fig. 2(b) to the 2-prong events of Fig. 1(b) one finds

$$\mathrm{R}([N^{*0}\rightarrow \pi^-+p]/n) = \frac{1}{\pi M^2}\int_{W=7.7\,\mu}^{W=9.8\,\mu} \sigma_{\pi^-+p\rightarrow\pi^-+p}(\mu^2,W) k_{3W} W^2 dW \Big/ 8\pi f^2 = \begin{cases} \text{(a)} & 0.84, \\ \text{(b)} & 0.53, \end{cases} \tag{5.6}$$

where the values (a) and (b) correspond to ($\frac{1}{3}$ of) curves π^-+p (a) and π^-+p (b) of Fig. 4. There is of course no corresponding ratio for the doubly charged excited nucleon state [graph of Fig. 2(c)], N^{*++}, which decays into π^++p.

6. DISCUSSION

The large ratios of N^* to N production predicted above by the long-range interaction model for the particular values of the phase space variables considered is due both to the ability to excite the $\frac{3}{2}$–$\frac{3}{2}$ pion-nucleon resonant state, even for small values of Δ^2, and to the rapidly increasing phase-space factor, $k_{3W}W^2/(\pi M^2)$. For larger Δ^2 the N^* spectrum extends to higher values of W and the integration with respect to W will contain important contributions from the range above the $\frac{3}{2}$–$\frac{3}{2}$ pion-nucleon resonance because of the phase-space factor. Thus, although the denominator of each ratio increases with Δ^2, N^* production is expected to remain important compared to that of N. The calculations are being extended to include a larger region of phase space.

It should be noted that the ratios of N^* to N production based on the static nucleon model approximation for $\sigma_{\pi N}(\Delta^2,W)$ exceed by more than 50% those obtained by use of the approximation $\sigma_{\pi N}(\Delta^2,W)\approx\sigma_{\pi N}(W)$, for the region of phase space considered. While we believe that the "off-the-mass-shell" approximation is probably in the right direction, clearly a better approximation than that given by the static nucleon model should be sought.

Independent of the details of the N and N^* formation (e.g., the static nucleon approximation), certain other consequences follow from the long-range interaction model if one assumes that charge independence holds and that the N^* is produced entirely in the isotopic spin $T=\frac{3}{2}$ state. From these assumptions, it follows that:

(1) In the reaction $\pi^-+p\rightarrow\pi^*+N$, the relative strength of n production is twice that of p "production," and in the reaction $\pi^-+p\rightarrow\pi^*+N^*$ the relative strengths of N^{*+} and N^{*++} production are twice and three times that of N^{*0} production, respectively. The actual production ratios depend also upon the $\pi-\pi$ interaction. Despite the greater relative strength of the N^{*++} production, it should be noted that this mode requires that the π^* be produced in the isotopic spin $T=2$ state.

(2) There is no $N^{*-}(n+\pi^-)$ isobar production.

(3) Independent of the $\pi-\pi$ interaction, the pion emitted by the N^{*0}, as well as that emitted by the N^{*+}, is neutral in two-thirds of the cases.

(4) The ratios defined in Eqs. (5.4) and (5.5) are related as follows:

$$\mathrm{R}(N^{*+}/p)=4\mathrm{R}(N^{*0}/n).$$

Because of experimental uncertainties in presently available data, a detailed comparison of experiment with the above predictions, as well as those in the previous section, is not possible at this time.

The original motivation for trying to apply the single virtual pion exchange interaction to the reaction $\pi^-+p\rightarrow N+3\pi$ lay in the observation that the virtual pion could excite the $\frac{3}{2}$–$\frac{3}{2}$ pion-nucleon resonance. The specific numerical results presented are for values of the phase-space variables appropriate to this final state.

However, the model should not be thought of as being restricted solely to this part of the phase space. It is better visualized as a long-range collision in which a virtual, but "almost real" pion is exchanged, leading to the excitation of each of the two incident particles. These two spatially separated, excited particles (or

cores) proceed almost undeflected from their original trajectories without further interaction between them and subsequently decay independently into the final state particles. This model, with two cores, contrasts with the Fermi statistical model in which all final-state particles emerge from a single excited region.

In the single virtual pion exchange model the probability for the production of a particular final state depends upon (1) the magnitude of the virtual pion propagator, which favors small values of Δ^2; (2) the strengths of the interactions of the virtual pion with each of the incident particles for the formation of the particular "core" states; and (3) the phase space factors, which favor large values of the internal energy of each core. Even for small values of Δ^2 the internal energy of each core may be large compared to the rest mass of the corresponding incident particle. For larger values of Δ^2, larger values of the internal energies of the cores can occur with correspondingly lower translational energies of the cores in the over-all barycentric system. For such events the Lorentz transformations that connect the over-all barycentric system with each of the core systems will not compress the decay particles of each core into as narrow cones (in the over-all barycentric system) as is the case for smaller Δ^2.

There is also strong evidence for the importance of long-range interactions in ultrarelativistic nucleon-nucleon collisions ($\gtrsim 10^3$ Bev laboratory energy), in which the average transverse momentum of the secondary particles is independent of the primary energy, and in which the angular distribution of the secondary particles is consistent, in a majority of events, with their emission from two excited regions (or fireballs) which propagate in opposite directions, in the over-all barycentric system, along the collision axis.[13] In these events the primary nucleons proceed almost undeflected from their original trajectories, but lose the energy required to form their respective fire-balls.

These facts suggest a direct extension of the single virtual exchange pion model, which one may visualize as follows: The typical Feynman diagram for the process, neglecting charge configurations, is constructed by taking any two of the diagrams of Figs. 1 and 2 and replacing the two incident pions by a single virtual pion propagator which connects the two π^* vertices. This virtual pion acts as an incoming "almost real" pion with respect to each of the two incident nucleons, and exchanges a single virtual pion with each of them, leading to the formation of an excited π^* state associated with each incident nucleon. These two π^* states are the two fireballs. The kinematics and phase-space considerations required to justify this picture will be given in a subsequent paper. Of course, in these very high energy events the internal energies of the π^* fireballs will be much greater than those considered in detail in this paper, and they will in general decay into a large number of secondary particles. In this model one also expects N^* production to be important.

In conclusion it should be emphasized that the long-range interaction model is expected to apply only in restricted regions of the final-state phase spaces, and is thus proposed as an adjunct to, rather than as a replacement for, statistical theories.

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[13] P. Ciok, T. Coghen, J. Gierula, R. Holynski, A. Jurak, M. Miesowicz, T. Saniewska, O. Stanisz, and J. Pernegr, Nuovo cimento **8**, 166 (1958) and Nuovo cimento **10**, 741 (1958). G. Cocconi, Phys. Rev. **111**, 1699 (1958).