

# Vacuum Polarization Effects on Energy Levels in $\mu$ -Mesonic Atoms

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(Received June 13, 1960)

The shifts of the energy levels of  $\mu$ -mesonic atoms due to vacuum polarization are calculated for the states  $n=l+1$ . The first-order perturbation integrals are evaluated both in closed form and by means of series expansions. A more accurate calculation of the energy shifts which takes into account the alteration of the atomic wave functions by the vacuum polarization potential is described in the concluding section.

## 1. INTRODUCTION

ONE of the well-known predictions of quantum electrodynamics is that free space has certain properties analogous to those of a dielectric medium. The presence of an electrostatic field in free space induces a slight separation of virtual electrons and positrons, which is referred to as vacuum polarization. The vacuum polarization in turn leads to alteration of the electrostatic potential. This addition to the potential changes, for example, both the strength and the functional form of the interaction of two point charges at small distances from one another. It adds to the familiar Coulomb potential energy a term which, for small distances  $r$ , diverges as  $r^{-1} \ln r$ , and which decreases exponentially for distances larger than the electron Compton wavelength.

While the details of the vacuum polarization interaction are quite well established from a theoretical standpoint, the experimental evidence for the correctness of these predictions is fairly meager.<sup>1</sup> By far the most satisfactory verification, until recently, has been supplied by the occurrence of the Lamb shift in hydrogen.<sup>2</sup> The vacuum polarization contribution<sup>3-5</sup> to the calculated value of this shift is 27.1 Mc/sec. The measured and calculated values of the total shift are found to agree within about 0.2 Mc/sec. Although the Lamb shift thus furnishes an excellent check on the theory, it tells us in fact very little about the details of the vacuum polarization interaction. The latter interaction, as we have already noted, is negligibly small when the electron of a hydrogen atom is at a distance greater than one Compton wavelength from the proton, a distance 137 times smaller than the radius of the first Bohr orbit. The shifts of energy levels due to vacuum polarization are therefore only appreciable in the  $S$  states. For the  $S$  states the shift is proportional,

in first-order perturbation theory, to the volume integral of the vacuum polarization potential. The measurement of the Lamb shift furnishes an accurate check on the volume integral of the vacuum polarization potential, but tells us nothing further about its functional form.

The existence of hydrogen-like atoms in which the role of the electron is played by particles of considerably greater mass opens the possibility of probing the detailed structure of the vacuum polarization interaction. Recent experimental studies<sup>6,7</sup> of the spectra of  $\mu$ -mesonic atoms, in particular, have already been carried to a level of accuracy capable of furnishing information of this type. For such atoms the Bohr orbits are smaller in radius than their electronic counterparts by the ratio of the electron mass to the  $\mu$ -meson mass. In particular for nuclei of  $Z > 1$  the lowest Bohr orbits for  $\mu$ -mesons lie entirely within the vacuum polarization cloud surrounding the nucleus. The electrostatic potential in such states differs at all points from the Coulomb potential by an amount of order  $e^2/\hbar c = 1/137$ . Level shifts of the order of 1% of the total binding energy are thus to be expected for all of the low-lying states. The precise magnitudes of these shifts will vary from one level to another in accordance with the forms of their wave functions and the variation with distance of the vacuum polarization interaction. Thus if  $a_0$  is the  $\mu$ -mesonic Bohr radius, the level shift of a state of principal quantum number  $n$  is a measure of the vacuum polarization potential in the vicinity of  $r = na_0$ . Accurate prediction of a succession of level shifts should furnish a detailed check on the functional form of the vacuum polarization potential.

It may be noted that the small radii of the  $\mu$ -mesonic Bohr orbits render the effects of screening by electrons quite small and that indeed the correction due to vacuum polarization effects is the dominant one.<sup>8</sup> Because the calculation of the energy levels must preserve an accuracy considerably greater than 1%,

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<sup>1</sup> L. L. Foldy and E. Eriksen, Phys. Rev. **95**, 1048 (1954).

<sup>2</sup> C. M. Sommerfield, Phys. Rev. **107**, 328 (1957).

<sup>3</sup> E. A. Uehling, Phys. Rev. **48**, 55 (1938).

<sup>4</sup> J. Schwinger, Phys. Rev. **75**, 663 (1949).

<sup>5</sup> E. E. Salpeter, Phys. Rev. **89**, 92 (1953).

<sup>6</sup> S. Koslov, Nevis Cyclotron Report 19, 1956 (unpublished).

<sup>7</sup> A. J. Bearden, Phys. Rev. Letters **4**, 240 (1960).

<sup>8</sup> A. Petermann and Y. Yamaguchi, Phys. Rev. Letters **2**, 359 (1959).

it is important that use be made of an appropriately accurate value of the  $\mu$ -meson mass in calculating the energies of the hydrogenic states. At present we lack a direct measurement of the  $\mu$ -meson mass to such accuracy; indeed, recent  $\mu$ -meson mass determinations<sup>6-8</sup> have been based upon the measurement of the energy of the  $\mu$ -mesonic x-ray transition. It is, of course, necessary to correct for vacuum polarization effects in establishing such a mass. If the mass found in this way is employed, it requires the measurement of two or more x-ray energies to secure a check upon the theory of vacuum polarization. For the purposes of the present calculation a provisional value of the  $\mu$ -meson mass may be inferred from the recent, accurate measurement of its magnetic moment, by assuming the correctness of the gyromagnetic ratio predicted by quantum electrodynamics.<sup>9</sup>

If we write the Coulomb potential as  $V_c(r) = -Ze^2/r$ , the full electrostatic interaction, including the effect of vacuum polarization to first order in the fine structure constant  $\alpha = e^2/\hbar c$ , may be written in the form

$$V(r) = V_c(r)[1 + \beta v(r)]. \quad (1.1)$$

Here  $\beta = 2\alpha/3\pi$ , and  $v(r)$  is a function most conveniently stated in an integral representation,<sup>3,4</sup>

$$v(r) = \int_1^\infty d\xi \left(1 + \frac{1}{2\xi^2}\right) \frac{(\xi^2 - 1)^{\frac{1}{2}}}{\xi^2} e^{-2\xi\kappa r}, \quad (1.2)$$

where  $\kappa = mc/\hbar$  is the inverse Compton wavelength of the electron. These expressions will be of sufficient accuracy for the next two sections of this paper, which will be devoted to the calculation of the level shift by means of first-order perturbation theory. In the first of these sections a series expansion of the perturbation integral is developed, which is later used in more accurate calculations. In the second of these sections the first-order perturbation integral is evaluated exactly in closed terms. In the last section we describe a procedure of calculation which takes into account the alteration of the wave function by the vacuum polarization interaction. The error of the computed level shift is thereby reduced considerably.

We shall consider in particular the states of  $\mu$ -mesonic atoms for which  $n = l + 1$ . These are the states most easily attained experimentally, and furthermore the ones most useful in exploring the form of  $v(r)$ . The first-order vacuum polarization shift has previously been calculated for the  $2P$  and  $3D$  levels of phosphorus by means of numerical integration by Koslov<sup>6</sup> who also carried out the measurement of the line resulting from the transition between these levels. This line is particularly well adapted to precision measurement, and the necessary determination of the shape of the x-ray absorption edge in lead has recently been completed

by Bearden.<sup>7</sup> Petermann and Yamaguchi<sup>8</sup> have considered many of the corrections to the Bohr energy for this line but have not included the effect of vacuum polarization in distorting the wave functions of the  $\mu$ -mesonic atom.

## II. SERIES FOR FIRST ORDER ENERGY SHIFT

We first develop a series expression for the first-order perturbation value of the energy shift, a result that will later serve as the basis for a more exact calculation. As the first step, we present a series expansion<sup>10</sup> for  $v(r)$ . For this purpose we note that  $v(r)$  can be written

$$v(r) = E_1(2\kappa r) + \int_1^\infty \frac{d\xi}{\xi} \left\{ \left(1 - \frac{1}{\xi^2}\right)^{\frac{1}{2}} \times \left(1 + \frac{1}{2\xi^2}\right) - 1 \right\} e^{-2\kappa\xi r}, \quad (2.1)$$

where  $E_1(x)$  is the exponential integral,  $\int_1^\infty d\xi e^{-x\xi}/\xi$ . It may be noted that, with the  $E_1$  term extracted, the remaining integral is well behaved at  $r=0$  and hence is suitable for further expansion as follows. We write it as

$$S(r) = \int_1^\infty d\xi \chi(\xi) e^{-2\kappa\xi r}, \quad (2.2)$$

and perform an integration by parts to obtain

$$S(r) = e^{-2\kappa r} \int_1^\infty d\xi \chi(\xi) - 2\kappa r \int_1^\infty d\xi e^{-2\kappa\xi r} \int_\xi^\infty d\xi' \chi(\xi'). \quad (2.3)$$

The integral in the first term is elementary and gives the coefficient of  $e^{-2\kappa r}$  in the series being constructed. The integral in the remaining term may again be integrated by parts to give the coefficient of  $(2\kappa r)e^{-2\kappa r}$ . This process may be repeated to yield the required series. In general, however, at every other step the integrand obtained behaves like  $\xi^{-1}e^{-2\kappa\xi r}$  at  $\xi = \infty$ , so that the integral diverges at  $r=0$  like  $E_1(2\kappa r)$ . To continue the process, therefore, we subtract the necessary multiple of  $\xi^{-1}e^{-2\kappa\xi r}$  from the integrand, thus making it sufficiently well behaved at  $\xi = \infty$ , and compensate by adding the appropriate term in  $E_1(2\kappa r)$  to the series. In this fashion we obtain the following representation of  $v(r)$ :

$$v(r) = \sum_{j=0}^{\infty} [c_j(2\kappa r)^j e^{-2\kappa r} + d_j(2\kappa r)^{2j} E_1(2\kappa r)], \quad (2.4)$$

<sup>9</sup> R. L. Garwin, D. P. Hutchinson, S. Penman, and G. Shapiro, Phys. Rev. **118**, 271 (1960).

<sup>10</sup> This expansion was originally obtained in a different fashion by Dr. C. M. Sommerfield and one of us (W.R.).

where the  $c_j$ 's have the form

$$c_j = - \sum_{m=0}^j a_{j-m}/m! - b_j. \quad (2.5)$$

The first few constants entering (2.4) and (2.5) are

$$\begin{aligned} a_0 &= 5/6 - \ln 2; & a_1 &= -3\pi/8, & a_2 &= 3/8, & a_3 &= -\pi/24, \\ a_4 &= (7+3\ln 2)/192; & b_0 &= 0, & b_1 &= 1, & b_2 &= 3/4, \\ b_3 &= 11/36, & b_4 &= 25/288; & d_1 &= 1, & d_2 &= 0, & d_3 &= 1/64. \end{aligned}$$

It is now straightforward to obtain the corresponding series for the shift in the energy level due to vacuum polarization. We let  $E_l$  be the energy of the  $\mu$  meson in the state with orbital quantum number  $l$  and total quantum number  $n=l+1$ . The corresponding radial wave function (defined in  $r$  space so that  $\int |\Psi_l(r)|^2 dr = 1$ ) is

$$\Psi_l(r) = \left( \frac{(2\kappa\gamma_l)^{2l+3}}{(2l+2)!} \right)^{\frac{1}{2}} r^{l+1} e^{-\gamma_l r}, \quad (2.7)$$

with  $\gamma_l = Z\mu\alpha/m(l+1)$  where  $\mu$  is the reduced mass of the  $\mu$  meson and  $m$  is the mass of the electron. The first-order perturbation expression for the change in  $E_l$  due to vacuum polarization effects is then

$$\begin{aligned} \Delta^{(1)}E_l &= \beta \int dr V_e(r) v(r) |\Psi_l(r)|^2 \\ &= 2\beta\kappa Ze^2 \sum_{j=0}^{\infty} (c_j C_j + d_j D_j), \end{aligned} \quad (2.8)$$

with

$$C_j = \gamma_l^{3-j} \left( \frac{\gamma_l}{1+\gamma_l} \right)^{2l+j} \frac{(2l+j+1)!}{(2l+2)!}, \quad (2.9)$$

and

$$D_j = \gamma_l^{2l+3} \frac{(2l+2j+1)!}{(2l+2)!} I_{2l+2j+2}(\gamma_l). \quad (2.10)$$

The quantity  $I_s$  is defined by the recurrence relation

$$I_{s+1}(x) = -\frac{1}{s(1+x)^s} + \frac{I_s(x)}{x}, \quad s \geq 1, \quad (2.11)$$

and

$$I_1(x) = (1/x) \ln(1+x). \quad (2.12)$$

The application of these expressions is straightforward and will prove especially useful in the determination of the energy shift with the corrected wave function.

### III. EXACT EVALUATION OF FIRST-ORDER ENERGY SHIFT

An exact expression for the first-order level shift may be derived in terms of elementary functions without difficulty. The expectation value of the vacuum polarization potential in the state  $n=l+1$  is given by the integral noted earlier in Eq. (2.8). When the integral representation (1.2) is used to represent the vacuum polarization potential in this expression the integration

over the radial variable  $r$  may be carried out immediately. The expression for the level shift then becomes

$$\Delta^{(1)}E_l = -\frac{Ze^2\beta}{(l+1)\lambda} Q_{2l+2}(\lambda), \quad (3.1)$$

where the parameter  $\lambda$  is given by

$$\lambda = \frac{m}{Z\alpha\mu}(l+1) = \frac{1}{\gamma_l}, \quad (3.2)$$

and  $Q_{2l+2}(\lambda)$  is the integral

$$Q_{2l+2}(\lambda) = \int_1^\infty d\xi \frac{(\xi^2 + \frac{1}{2})(\xi^2 - 1)^{\frac{1}{2}}}{\xi^4(1+\lambda\xi)^{2l+2}}. \quad (3.3)$$

The integrals  $Q_k(\lambda)$  may be evaluated conveniently by changing the variable of integration to  $y=1/\xi$ , and separating them into two terms,

$$Q_k(\lambda) = J_k(\lambda) + \frac{1}{2}K_k(\lambda), \quad (3.4)$$

where

$$J_k(\lambda) = \int_0^1 \frac{y^{k-1}(1-y^2)^{\frac{1}{2}}}{(\lambda+y)^k} dy, \quad (3.5)$$

$$K_k(\lambda) = \int_0^1 \frac{y^{k+1}(1-y^2)^{\frac{1}{2}}}{(\lambda+y)^k} dy.$$

These integrals, which are to be evaluated for  $k=2l+2$ , may all be generated by simple operations of differentiation carried out upon the integrals  $J_1(\lambda)$  and  $K_1(\lambda)$ . For  $k \geq 1$  we may write

$$J_k(\lambda) = \frac{1}{(k-1)!} \frac{1}{\lambda^k} \left( \lambda^2 \frac{d}{d\lambda} \right)^{k-1} \lambda J_1(\lambda), \quad (3.6)$$

$$K_k(\lambda) = \frac{1}{(k-1)!} \frac{1}{\lambda^k} \left( \lambda^2 \frac{d}{d\lambda} \right)^{k-1} \lambda K_1(\lambda).$$

The integrals  $J_1(\lambda)$  and  $K_1(\lambda)$ , which are elementary, are given by

$$\begin{aligned} J_1(\lambda) &= (1-\lambda^2)^{\frac{1}{2}} \operatorname{sech}^{-1}\lambda - 1 + \frac{1}{2}\pi\lambda, \\ K_1(\lambda) &= \lambda^2(1-\lambda^2)^{\frac{1}{2}} \operatorname{sech}^{-1}\lambda + \frac{1}{3} - \frac{1}{4}\pi\lambda - \lambda^2 + \frac{1}{2}\pi\lambda^3. \end{aligned} \quad (3.7)$$

The combination of Eqs. (3.4), (3.6), and (3.7) yields similarly elementary expressions for the level shifts. In particular for the 1S state we find

$$\begin{aligned} Q_2 &= (1 - \frac{1}{2}\lambda^2 - 2\lambda^4)(1-\lambda^2)^{\frac{1}{2}} \operatorname{sech}^{-1}\lambda \\ &\quad - 11/6 + \frac{3}{4}\pi\lambda - 2\lambda^2 + \pi\lambda^3, \end{aligned} \quad (3.8)$$

and for the 2P state

$$\begin{aligned} Q_4 &= [1 - \frac{5}{2}\lambda^2 - (45/4)\lambda^4 + 22\lambda^6 - 10\lambda^8] \\ &\quad \times (1-\lambda^2)^{-\frac{1}{2}} \operatorname{sech}^{-1}\lambda + (5/4)(1-\lambda^2)^{-1} + \frac{3}{4}(1-\lambda^2)^{-2} \\ &\quad - (14/3) + \frac{3}{2}\pi\lambda - 10\lambda^2 + 5\pi\lambda^3. \end{aligned} \quad (3.9)$$

In practice the atomic states which interest us have values of  $l+1$  much smaller than  $Z$ , and consequently values of the parameter  $\lambda$  which are much smaller than unity. For such cases it is convenient to evaluate the integrals  $Q_{2l+2}$  for states of angular momentum  $l \geq 2$  by means of an expansion of  $J_1(\lambda)$  and  $K_1(\lambda)$  in terms of  $\ln(\lambda/2)$  and powers of  $\lambda$ . In this way we find, for the  $3D$  state,

$$Q_6 = - \left[ 1 - \frac{189}{4}\lambda^4 - \frac{231}{4}\lambda^6 + \dots \right] \ln(\lambda/2) - \frac{187}{60} + \frac{9\pi}{4} \\ - \frac{63}{4}\lambda^2 + 14\pi\lambda^3 + \frac{3747}{160}\lambda^4 + \frac{30671}{1440}\lambda^6 - \dots, \quad (3.10)$$

a result which is amply accurate for present applications.

The use of the above expressions for the level shift, or of the equivalent series development (Eq. 2.8) of Sec. II, leads to the prediction of increases of the x-ray frequencies for  $\mu$ -mesonic atoms. For the case of the  $3D-2P$  transition in phosphorus, with the  $\mu$ -meson mass taken as  $206.77m$ , we find a fractional frequency increase of 0.003758, which agrees with the result evaluated numerically by Koslov.<sup>6</sup>

#### IV. REFINED CALCULATION OF THE ENERGY SHIFT

The accuracy of the above results is necessarily limited by the fact that the energy shift was calculated to the first order. The correction to be attained by the use of the exact wave function is of the same order as that found by considering terms of order  $\alpha^2$  in the exact expression for the vacuum polarization potential.<sup>8</sup> An estimate of the maximum error introduced by this approximation can be obtained by fixing an upper bound for the second-order correction  $\Delta E_l^{(2)}$ . For this purpose one modifies the usual expression<sup>11</sup> by replacing all the energy denominators by the lowest one,  $E_l - E_{l+1}$ , so that they can be removed from the summation, and then applies closure to obtain

$$|\Delta^{(2)}E_l| \leq \frac{\int_0^\infty dr V_1^2(r) |\Psi_l(r)|^2}{E_{l+1} - E_l}, \quad (4.1)$$

where

$$V_1(r) = \beta V_c(r) v(r).$$

For instance, in the case of  $\mu$ -mesonic phosphorus we find, by numerical evaluation, for the  $l=1$  state,  $|\Delta^{(2)}E_1| \leq 0.058 \times 15\beta e^2$ , as compared with  $|\Delta^{(1)}E| \approx (4.64)15\beta e^2$ . (The unit  $Z\beta e^2$  is a measure of the magnitude of the energy of interaction between the meson and the nucleus due to vacuum polarization effects inside of the range of the vacuum polarization potential.) Our next task is to develop a simple pro-

cedure for diminishing this error. For this purpose we first remark that, for the range of  $r$  of importance in the cases contemplated here,  $V_1(r)$  may be very closely approximated by a function  $\phi(r)$  having the form

$$\phi(r) = Ar^{-2} + Br^{-1} + C. \quad (4.2)$$

If we take  $H$  to be the Hamiltonian in  $r$  space (i.e., after separation of the angular part for a fixed  $l$ ) and put

$$H = H_0 + V_1(r), \\ H_0' = H_0 + \phi(r), \quad (4.3)$$

$$V_1'(r) = V_1(r) - \phi(r),$$

we may write

$$H = H_0' + V_1'(r). \quad (4.4)$$

Our procedure is now based on the following circumstances. (1) The operator  $H_0'$  has the same form as  $H_0$  and hence has eigenfunction-eigenvalue solutions of the same form as the hydrogen atom. (2) To the extent that Eq. (4.2) gives a good approximation for  $V_1(r)$ ,  $V_1'(r)$  is a small perturbation and may therefore be treated by first-order perturbation theory. Specifically the energy of  $H_0'$  for orbital quantum number  $l$  is

$$E_l' = E_l(Z') + C \quad (4.5)$$

where  $E_l(Z')$  is the usual expression for the energy of a fictitious  $\mu$ -mesonic atom of atomic number  $Z'$  with a principal quantum number  $l'+1$ , i.e.,

$$E_l(Z') = -\mu Z'^2 e^4 / 2\hbar^2 (l'+1)^2, \quad (4.6)$$

with  $l'$  and  $Z'$  given by

$$l'(l'+1) = l(l+1) + 2\mu A / \hbar^2, \\ Z' = Z - B/e^2.$$

The corresponding wave function (in  $r$  space),  $\Psi_l'(r)$ , is the same as that given in Eq. (2.7) with  $l$  and  $Z$  replaced by  $l'$  and  $Z'$ , respectively. To obtain a better approximation of the eigenvalue of  $H$  we take  $V_1'(r)$  as a perturbation on  $H_0'$  and compute the corresponding shift in the energy  $\Delta^{(1)}E_l'$ :

$$\Delta^{(1)}E_l' = \int_0^\infty dr |\Psi_l'|^2 V_1'(r). \quad (4.8)$$

Again we can estimate an upper limit on the error as

$$|\Delta^{(2)}E_l'| \leq \frac{\int_0^\infty dr |\Psi_l'|^2 V_1'(r)}{E_{l+1} - E_l}. \quad (4.9)$$

Our object, of course, is to minimize this quantity and we undertake to select the coefficients  $A$ ,  $B$ , and  $C$  so as to achieve this goal. After approximating  $\Psi_l'$  by  $\Psi_l$  in the integral of Eq. (4.9), we demand that its partial derivatives with respect to  $A$ ,  $B$ , and  $C$  vanish. We thus obtain a set of three linear equations in  $A$ ,  $B$ , and  $C$  in which the coefficients depend on quantities of the

<sup>11</sup> L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., Chap. VII, p. 153.

form  $\int_0^\infty dr |\Psi_l|^2 V_1(r) r^{-n}$ . These can easily be evaluated by the techniques discussed in II.

We are now in a position to calculate  $\Delta^{(1)} E_l'$  and hence the energy shift which is  $E_l' + \Delta^{(1)} E_l' - E_l$ . The evaluation of the relevant integrals entering Eq. (4.8) is quite straightforward with the aid of the techniques given in Sec. II. It is necessary, however, to make some detailed changes in view of the fact that  $\Psi_l'$  is essentially a radial wave function corresponding to a nonintegral orbital quantum number. The first of these is the trivial one that any expression of the form  $(l'+k)!$  must be replaced by  $\Gamma(l'+k+1)$ . The other is that some special techniques are demanded in the calculation of  $I_s$ , the integral involving  $E_l(2kr)$ . The recurrence relation of Eq. (2.11) still holds. Therefore, the only alteration in the procedure required is that it is necessary to calculate  $I_{l+\Delta l}$  where  $\Delta l = l' - l$ . To do this we note that  $\Delta l$  is small and approximate  $I_{l+\Delta l}$  by

$$I_{l+\Delta l} = I_l + (dI_s/ds)_{s=l} \Delta l, \quad (4.10)$$

ignoring subsequent terms in the power series.

The quantity  $(dI_s/ds)_{s=l}$  can be written as

$$\left( \frac{dI_s(\gamma)}{ds} \right)_{s=l} = -(\gamma+1)^{-l} \{ \ln(1+\gamma) \times \int_0^\infty dx \frac{e^{-(l-1)x}}{e^x - \gamma/(\gamma+1)} + \int_0^\infty dx \frac{x e^{-(l-1)x}}{e^x - \gamma/(\gamma+1)} \}. \quad (4.11)$$

For  $l=1$  the first integral can be evaluated immediately in closed form while the second integral has the following series representation:

$$\int_0^\infty dx \frac{x}{e^x - \gamma/(\gamma+1)} = \frac{\pi^2}{6} + \frac{\epsilon}{1-\epsilon} \frac{\pi^2}{6} + \frac{1}{1-\epsilon} \left\{ \frac{1}{2} \ln^2(1-\epsilon) - \ln \epsilon \ln(1-\epsilon) - \left[ \frac{\epsilon}{1-\epsilon} - \frac{1}{4} \left( \frac{\epsilon}{1-\epsilon} \right)^2 + \dots \right] \right\}, \quad (4.12)$$

with  $\epsilon = 1/(\gamma+1)$  small for the cases of interest.

It is also possible to approximate in Eq. (4.11) directly for large  $l$  and thus to obtain convenient expressions for checking the iteration. As before, the first integral is evaluable in closed form while the second may be approximated by

$$\int_0^\infty dx \frac{x e^{-(l-1)x}}{e^x - \gamma/(\gamma+1)} = \sum_{m=1}^N \frac{1}{m} \times \left[ \sum_{\omega=0}^{m-1} (-\epsilon)^\omega B(m-\omega, l+\omega) + (-\epsilon)^m E(l+m) \right] + O[B(N-1, l+1)], \quad (4.13)$$

where  $N$  is an arbitrary integer to be selected to achieve the necessary accuracy,  $B(m-\omega, l+\omega)$  is the beta function of the indicated variables, and  $E(l+m)$  is defined by

$$E(l+m) = -(1-\epsilon)^{-(l+m)} \ln \epsilon - \sum_{t=1}^{l+m-1} \frac{(1-\epsilon)^{t-l-m}}{t}. \quad (4.14)$$

This procedure has been applied to the case considered earlier, i.e., the evaluation of the effect of the vacuum polarization potential on the frequency of the  $3D-2P$  line in  $\mu$ -mesonic P. This involved the determination of  $A$ ,  $B$ , and  $C$  as outlined above, and the subsequent evaluation of  $E_l'$  and  $\Delta^{(1)} E_l'$  for the  $P$  and  $D$  states. The frequency increases by 0.3767%. The maximum error as estimated from Eq. (4.9) in units of  $15\beta e^2 \kappa$  amounts to 0.00086 for the  $P$  state and 0.000030 for the  $D$  state, a decrease from the corresponding values of 0.058 and 0.0051 obtained without the corrected wave function.

The procedure of Secs. II and III lends itself to the rapid calculation of vacuum polarization energy level shifts while the technique of the present section permits a considerable improvement in accuracy with little extra labor. It seems likely that the general technique discussed in this section may have a wider range of applications for improving the accuracy of perturbation calculations.