Measurement of Quantum Mechanical Operators

HUZIHIRO ARAKI* AND MUTSUO M. YANASE†
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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The limitation on the measurement of an operator imposed by the presence of a conservation law is studied. It is shown that an operator which does not commute with a conserved (additive) quantity cannot be measured exactly (in the sense of von Neumann). It is also shown for a simple case that an approximate measurement of such an operator is possible to any desired accuracy.

1. INTRODUCTION

It was pointed out by Wigner[1] that the presence of a conservation law puts a limitation on the measurement of an operator which does not commute with the conserved quantity. The limitation is such that the measurement of such an operator is only approximately possible. An approximate measurement can be done by a measuring apparatus which is large enough in the sense that the apparatus should be a superposition of sufficiently many states with different quantum numbers of the conserved quantity. He has proved these statements for a simple case where the x component of the spin of a spin one-half particle is measured, the z component of the angular momentum being the conserved quantity. The aim of this paper is to present a proof of the above statement for the general case.

In Sec. 2, we will prove that an exact measurement of an operator M which does not commute with a conserved operator L_1 is impossible. In Sec. 3, we will prove that an *approximate* measurement of the operator M is possible if L_1 has discrete eigenvalues and is bounded in the Hilbert space of the measured object.

2. IMPOSSIBILITY OF AN EXACT MEASUREMENT OF AN OPERATOR WHICH DOES NOT COMMUTE WITH A CONSERVED QUANTITY

Suppose we measure a self-adjoint operator M for a system represented by a Hilbert space $\mathfrak{H}_1$. Assume that M has discrete eigenvalues μ and corresponding eigenvectors $\phi_{\mu\rho}$ which are orthonormal and complete in $\mathfrak{H}_1$,

$$M\phi_{\mu\rho}=\mu\phi_{\mu\rho}, \tag{2.1}$$

$$(\phi_{\mu\rho},\phi_{\mu'\rho'})=\delta_{\mu\mu'}\delta_{\rho\rho'}. \tag{2.2}$$

The measuring apparatus is represented by a Hilbert space $\mathfrak{H}_2$. Then a state of the combined system of the measured object and the measuring apparatus is represented by a unit vector in $\mathfrak{H}_1\otimes\mathfrak{H}_2$.

According to von Neumann,[2] the measurement of the operator M in a state ϕ is accomplished by choosing an apparatus in a state ξ (fixed normalized state independent of ϕ) in $\mathfrak{H}_2$ such that the combined system, if it is in the state $\phi_{\mu\rho}\otimes\xi$ before the measurement, goes over after a finite time t into

$$U(t)(\phi_{\mu\rho}\otimes\xi)=\textstyle\sum_{\rho'}\phi_{\mu\rho'}\otimes X_{\mu\rho\rho'}, \tag{2.3}$$

where $U(t)$ is a unitary operator describing the time-development of the combined system. In order to be able to distinguish the different measured values of the operator M in terms of states of measuring apparatus after the measurement, we require

$$(X_{\mu\rho\rho''},X_{\mu'\rho'\rho'''})=0,\quad\text{if}\quad \mu\neq\mu'. \tag{2.4}$$

We note that, because we are not measuring the degeneracy parameter ρ, we have to allow the possibility that the measuring object remains in any linear combination of $\phi_{\mu\rho'}$, with fixed μ but with arbitrary ρ.[3]

We now assume the existence of a universal conservation law for a self-adjoint operator L which is additive in the sense that

$$L=L_1\otimes 1+1\otimes L_2, \tag{2.5}$$

where L_1 and L_2 are self-adjoint operators in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Actually this additivity will be used only before and after the measurement, when the two systems are separated. By universal, we mean that, whatever measuring apparatus we take, $U(t)$ commutes with L,

$$[U(t),L]=0. \tag{2.6}$$

Our claim is that (2.3) is impossible unless

$$[L_1,M]=0. \tag{2.7}$$

For the proof, we first note that, because of the unitarity of $U(t)$ and the conservation law (2.6), we

* Present address: Department of Nuclear Engineering, Kyoto University, Kyoto, Japan.

† On leave of absence from Sophia University, Tokyo, Japan.

[1] E. P. Wigner, Z. Physik **131**, 101 (1952).

[2] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Verlag Julius Springer, Berlin, 1932; English ed.: Princeton University Press, Princeton, 1955).

[3] For any state of the measured object, we can write

$$U(t)(\phi\otimes\xi)=\Sigma_{\mu\rho}\,\phi_{\mu\rho}\otimes X_{\mu\rho}(\phi),$$

where $X_{\mu\rho}(\phi)$ depends on ϕ. The Eqs. (2.3) and (2.4) give the most general form of the above equation satisfying (1) the distinguishability of the measured result,

$$(X_{\mu\rho}(\phi),X_{\mu'\rho'}(\phi))=0,\quad\text{if}\quad\mu\neq\mu',$$

and (2) the requirement that the probability of an eigenvalue μ in the state ϕ, as measured by the state of the measuring apparatus after the measurement, should give the conventionally postulated value,

$$\Sigma_\rho\|X_{\mu\rho}(\phi)\|^2=\Sigma_\rho|(\phi_{\mu\rho},\phi)|^2.$$

have

$$
\begin{aligned}
&(\phi_{\mu'\rho'}\otimes\xi,\, L(\phi_{\mu\rho}\otimes\xi)) \\
&= (U(t)(\phi_{\mu'\rho'}\otimes\xi),\, U(t)L(\phi_{\mu\rho}\otimes\xi)) \\
&= (U(t)(\phi_{\mu'\rho'}\otimes\xi),\, LU(t)(\phi_{\mu\rho}\otimes\xi)) \\
&= (\textstyle\sum_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\, L\sum_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''}). \quad (2.8a)
\end{aligned}
$$

Hence, as the necessary condition for the conservation law for the operator L, we can write

$$
\begin{aligned}
&(\phi_{\mu'\rho'}\otimes\xi,\, L(\phi_{\mu\rho}\otimes\xi)) \\
&= (\textstyle\sum_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\, L\sum_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''}). \quad (2.8b)
\end{aligned}
$$

Using the additivity of L, (2.5), we obtain

$$
\begin{aligned}
&(\phi_{\mu'\rho'}\otimes\xi,\, L(\phi_{\mu\rho}\otimes\xi)) \\
&= (\phi_{\mu'\rho'},L_1\phi_{\mu\rho})(\xi,\xi)+(\phi_{\mu'\rho'},\phi_{\mu\rho})(\xi,L_2\xi) \\
&= \sum_{\rho''\rho'''}[(\phi_{\mu'\rho'''},L_1\phi_{\mu\rho''})(X_{\mu'\rho'\rho'''},X_{\mu\rho\rho''}) \\
&\qquad + (\phi_{\mu'\rho'''},\phi_{\mu\rho''})(X_{\mu'\rho'\rho'''},L_2X_{\mu\rho\rho''})]. \quad (2.8c)
\end{aligned}
$$

Because of the orthogonalities, (2.2) and (2.4), we finally obtain

$$(\phi_{\mu'\rho'},L_1\phi_{\mu\rho})=0, \quad \text{if} \quad \mu\neq\mu', \tag{2.9a}$$

or

$$(\phi_{\mu'\rho'},L_1\phi_{\mu\rho})=\delta_{\mu\mu'}(\phi_{\mu'\rho'},L_1\phi_{\mu\rho}). \tag{2.9b}$$

We are now ready to prove that L_1 commutes with M. For this purpose we decompose M into projection operators

$$M=\textstyle\sum_\mu \mu P_\mu; \quad P_\mu\phi_{\mu'\rho'}=\delta_{\mu\mu'}\phi_{\mu'\rho'}. \tag{2.10}$$

To prove the commutativity of L_1 and M, (2.7), it is sufficient to prove the commutativity of L_1 and P_μ,

$$P_\mu L_1-L_1P_\mu=0. \tag{2.11}$$

From the self-adjoint nature of P_μ, (2.9b) and (2.10), we see that

$$
\begin{aligned}
(\phi_{\mu'\rho'},P_\mu L_1\phi_{\mu''\rho''})&=\delta_{\mu\mu'}\delta_{\mu'\mu''}(\phi_{\mu'\rho'},L_1\phi_{\mu''\rho''}), \\
(\phi_{\mu'\rho'},L_1P_\mu\phi_{\mu''\rho''})&=\delta_{\mu\mu''}\delta_{\mu'\mu''}(\phi_{\mu'\rho'},L_1\phi_{\mu''\rho''}),
\end{aligned}
$$

which manifestly demonstrate (2.11). Thus we have succeeded in proving that (2.3)–(2.6) imply (2.7).[4]

Although we have assumed in the above proof that M has a discrete spectrum, the conclusion holds for any self-adjoint operator M. Namely, suppose

$$M=\int \mu dP(\mu)$$

is a spectral decomposition of M. If M can be measured exactly, $P(\mu)$ for each μ can obviously be measured exactly. Since the projection operator $P(\mu)$ has a discrete eigenvalue 1 or 0, the above proof tells us that $P(\mu)$ for each μ commutes with L_1, which in turn implies (2.7).

3. POSSIBILITY OF AN APPROXIMATE MEASUREMENT

In this section we will discuss the problem of whether the operator M, which does not commute with the conserved operator L_1 of the preceding section, can be measured approximately. We will prove that this is possible if L has a discrete spectrum and L_1 has only a finite number of eigenvalues.

We may assume that the eigenvalues of L_1 are[5] $0, \pm1, \pm2, \cdots \pm l$. We decompose L's into projection operators

$$L=\textstyle\sum_\lambda \lambda P(\lambda), \tag{3.1a}$$

$$L_i=\textstyle\sum_\lambda \lambda P_i(\lambda), \quad i=1,2. \tag{3.1b}$$

The additivity, (2.5), implies

$$P(\lambda)=\sum_{|\lambda'|\leq l} P_1(\lambda')P_2(\lambda-\lambda'). \tag{3.2}$$

As a first step of our proof, we state the following Lemma which will be proved at the end of this section.

Lemma. Given two sets of vectors $\Psi_\alpha{}^i$ and $\Psi_\alpha{}^f$ in a Hilbert space $\mathfrak{H}=\mathfrak{H}_1\otimes\mathfrak{H}_2$ satisfying

$$(\Psi_\alpha{}^i,P(\lambda)\Psi_\beta{}^i)=(\Psi_\alpha{}^f,P(\lambda)\Psi_\beta{}^f), \tag{3.3}$$

for every λ, then there exists a Hilbert space $\mathfrak{H}_2'$ con-

[4] If L_2 is unbounded, the above proof does not exclude the possibility that one can measure M, even if M does not commute with L_1, by a measuring apparatus (ξ or $X_{\mu\rho\rho'}$) in a state which is outside the domain of L, because (2.8) would then be meaningless.

However, even if L_2 is unbounded, as long as L_1 is bounded, we can modify the above argument in the following way. We introduce unitary operators

$$V(S)=\exp(iLS); \quad V_j(S)=\exp(iL_jS), \quad j=1,2. \tag{i}$$

Because of the additivity, (2.5),

$$V(S)=V_1(S)\otimes V_2(S). \tag{ii}$$

Then, by the conservation law (2.6), we have

$$
\begin{aligned}
&(\phi_{\mu'\rho'}\otimes\xi,\, V(S)(\phi_{\mu\rho}\otimes\xi)) \\
&= (\textstyle\sum_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\, V(S)(\sum_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''})). \quad (iii)
\end{aligned}
$$

By the orthogonality, (2.2),

$$
\begin{aligned}
&(\phi_{\mu'\rho'}\otimes\xi,\, V_2(S)(\phi_{\mu\rho}\otimes\xi)) \\
&= (\textstyle\sum_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\, V_2(S)(\sum_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''})) \quad (iv) \\
&= 0 \quad (\text{for } \mu\neq\mu')
\end{aligned}
$$

Combining the two equations above and using the additivity, (ii), we obtain for $\mu\neq\mu'$,

$$
\begin{aligned}
&(\phi_{\mu'\rho'}\otimes\xi,\, F(S)(\phi_{\mu\rho}\otimes\xi)) \\
&= (\textstyle\sum_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\, F(S)(\sum_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''})), \quad (v)
\end{aligned}
$$

where

$$F(S)=(1/iS)[V_1(S)-1]\otimes V_2(S). \tag{vi}$$

Since $F(S)\to L_1$, as $S\to 0$, we obtain

$$
\begin{aligned}
&(\phi_{\mu'\rho'}\otimes\xi,\, L_1(\phi_{\mu\rho}\otimes\xi)) \\
&= (\textstyle\sum_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\, L_1(\sum_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''})). \quad (vii)
\end{aligned}
$$

Because of the orthogonality, (2.4), we finally obtain (2.9a) from which we conclude (2.11) as before.

[5] The proof holds without this specification but notations become complicated, especially in dividing various regions of values of λ.

taining $\mathfrak{H}_2$ and a unitary operator U on $\mathfrak{H}'=\mathfrak{H}_1\otimes\mathfrak{H}_2'$ such that (1) a self-adjoint operator L_2' (representing the conserved quantity in $\mathfrak{H}_2'$) is defined on $\mathfrak{H}_2'$ coinciding with L_2 on $\mathfrak{H}_2$, (2) U commutes with the conserved quantity L' on $\mathfrak{H}'$, $L'=L_1\otimes 1+1\otimes L_2'$, and

$$\text{(3)}\quad \Psi_\alpha{}^f=U\Psi_\alpha{}^i. \tag{3.4}$$

If the set of the indices α is finite, $\mathfrak{H}_2'$ can be taken to be $\mathfrak{H}_2$.

This Lemma is used in the following way. We will construct states ξ, $X_{\mu\rho}$, ψ, and $\eta_{\mu\rho}$ satisfying

$$(X_{\mu\rho},X_{\mu'\rho'})=0,\quad \text{if}\quad \mu\neq\mu', \tag{3.5}$$

$$(X_{\mu\rho},\eta_{\mu'\rho'})=0,\quad \text{for any } \mu,\ \rho,\ \mu',\ \rho', \tag{3.6}$$

$$\|\psi\otimes\eta_{\mu\rho}\|^2<\epsilon,$$

$$(\eta_{\mu\rho},\eta_{\mu'\rho'})=0,\quad \text{for } (\mu,\rho)\neq(\mu',\rho') \tag{3.7}$$

in such a way that the two sets of vectors

$$\Psi_\alpha{}^i=\phi_{\mu\rho}\otimes\xi,\quad \Psi_\alpha{}^f=\phi_{\mu\rho}\otimes X_{\mu\rho}+\psi\otimes\eta_{\mu\rho},\quad \alpha=(\mu,\rho), \tag{3.8}$$

fulfill (3.3). If we succeed in constructing such states, then by the Lemma, there exists a unitary operator U in $\mathfrak{H}'$ which conserves L' and for which (3.4) holds. This implies, for any normalized state ϕ in $\mathfrak{H}_1$,

$$U(\phi\otimes\xi)=\sum_{\mu\rho}(\phi_{\mu\rho},\phi)(\phi_{\mu\rho}\otimes X_{\mu\rho})+\eta(\phi), \tag{3.9}$$

$$\eta(\phi)=\sum_{\mu\rho}(\phi_{\mu\rho},\phi)(\psi\otimes\eta_{\mu\rho}), \tag{3.10}$$

and, due to (3.7),

$$\|\eta(\phi)\|^2<\epsilon. \tag{3.11}$$

Thus if we choose the setup of a measurement in such a way that the Hilbert space of the measuring instrument is $\mathfrak{H}_2'$, the initial state of the instrument is ξ, and the time development of the combined system of the measured object and the measuring apparatus in a certain time interval t is described by $U(t)=U$, then we can measure the operator M in terms of the states $X_{\mu\rho}$ of the measuring apparatus after the measurement within the inaccuracy representing by $\eta(\phi)$. This inaccuracy can be made as small as one desires by making ϵ small enough. Because we are only concerned with the effect of the conservation law of L, we have assumed in the above argument that, if U is a unitary operator commuting with the conserved quantity, then there always exists an experimental setup whose time development in a certain time period is described by U. There may be many other conditions on U in addition to that it commutes with L. Hence, our argument does not assure that a system exists whose Hamiltonian leads to U.

We now give an explicit construction of states ξ, $X_{\mu\rho}$, ψ and $\eta_{\mu\rho}$. For this purpose we denote by $\mathfrak{H}_{2\lambda}$ the subspace of $\mathfrak{H}_2$ which is spanned by eigenvectors of L_2 with an eigenvalue λ. ψ is taken to be a normalized eigenstate of L_1 with the eigenvalue 0,

$$L_1\psi=0,\quad (\psi,\psi)=1. \tag{3.12}$$

ξ, $X_{\mu\rho}$, and $\eta_{\mu\rho}$ are given by

$$\xi=\textstyle\sum_\lambda \xi_\lambda,\quad X_{\mu\rho}=\sum_\lambda X_{\mu\rho\lambda},\quad \eta_{\mu\rho}=\sum_\lambda \eta_{\mu\rho\lambda}, \tag{3.13}$$

where ξ_λ, $X_{\mu\rho\lambda}$, and $\eta_{\mu\rho\lambda}$ are vectors in $\mathfrak{H}_{2\lambda}$ to be specified below.

ξ_λ is any state in $\mathfrak{H}_{2\lambda}$ with the norm given by

$$(\xi_\lambda,\xi_\lambda)=0,\qquad \text{for}\quad |\lambda|>N, \tag{3.14a}$$

$$=(2N+1)^{-1},\quad \text{for}\quad |\lambda|\leq N. \tag{3.14b}$$

N is any integer satisfying

$$N>2l/\epsilon-\tfrac{1}{2}. \tag{3.15}$$

The $X_{\mu\rho\lambda}$ are any states in $\mathfrak{H}_{2\lambda}$, orthogonal to each other and with the norm given by

$$(X_{\mu\rho\lambda},X_{\mu'\rho'\lambda})=0,\qquad \text{for}\quad |\lambda|>N-2l, \tag{3.16a}$$

$$=(2N+1)^{-1}\delta_{\mu\mu'}\delta_{\rho\rho'},\quad \text{for}\quad |\lambda|\leq N-2l. \tag{3.16b}$$

The orthogonal complement of the set $\{X_{\mu\rho\lambda}|\mu,\rho \text{ varying}\}$ in $\mathfrak{H}_{2\lambda}$ will be denoted by $\mathfrak{H}_{2\lambda}{}^\eta$.

$\eta_{\mu\rho\lambda}$ are taken from $\mathfrak{H}_{2\lambda}{}^\eta$ and defined in the following way

(I) For $|\lambda|>N+l$ or $|\lambda|\leq N-3l$,

$$\eta_{\mu\rho\lambda}=0. \tag{3.17a}$$

(II) For $N+l\geq|\lambda|>N-l$,[6]

$$(\eta_{\mu\rho\lambda},\eta_{\mu'\rho'\lambda})=(2N+1)^{-1}\sum_{\substack{|\lambda'|\leq l\\ |\lambda-\lambda'|\leq N}}(\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'}). \tag{3.17b}$$

(III) For $N-l\geq|\lambda|>N-3l$, $\eta_{\mu\rho\lambda}$ are any states in $\mathfrak{H}_{2\lambda}{}^\eta$ orthogonal to each other and with the norm given by

$$(\eta_{\mu\rho\lambda},\eta_{\mu'\rho'\lambda})=(2N+1)^{-1}(\phi_{\mu\rho},Q_\lambda\phi_{\mu\rho})\delta_{\mu\mu'}\delta_{\rho\rho'}, \tag{3.17c}$$

where Q_λ is a projection operator given by

$$Q_\lambda=\sum_{\substack{|\lambda'|\leq l\\ |\lambda-\lambda'|>N-2l}}P_1(\lambda'). \tag{3.18}$$

Note that $(\phi_{\mu\rho},Q_\lambda\phi_{\mu\rho})$ is non-negative (between 0 and 1).

We now show that ξ, $X_{\mu\rho}$, ψ, and $\eta_{\mu\rho}$ thus constructed have the desired properties. ξ is normalized due to (3.14). (3.5) and (3.6) are trivially satisfied by our

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[6] This means that $\eta_{\mu\rho\lambda}$ is defined by

$$\eta_{\mu\rho\lambda}=(2N+1)^{-\frac{1}{2}}\sum_{\substack{|\lambda'|\leq l\\ |\lambda-\lambda'|\leq N}}P_1{}^{(\lambda)}(\lambda')\phi_{\mu\rho}{}^{(\lambda)}$$

where we have made an isometric linear mapping of $\mathfrak{H}_1$ into $\mathfrak{H}_{2\lambda}{}^\eta$ and $\phi_{\mu\rho}$ and $P_1(\lambda')$ thus mapped are called $\phi_{\mu\rho}{}^{(\lambda)}$ and $P_1{}^{(\lambda)}(\lambda')$.

choice. To prove (3.3), we rewrite (3.3) using (3.2):

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})(\xi,\, P_2(\lambda-\lambda')\xi)$$

$$=\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})(\mathrm{X}_{\mu\rho},\, P_2(\lambda-\lambda')\mathrm{X}_{\mu'\rho'})$$

$$+(\eta_{\mu\rho},P_2(\lambda)\eta_{\mu'\rho'}), \quad (3.19)$$

where we have also used (3.12). By (3.13), this is equivalent to

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'}) \times[\|\xi_{\lambda-\lambda'}\|^2-(\mathrm{X}_{\mu\rho,\lambda-\lambda'},\mathrm{X}_{\mu'\rho',\lambda-\lambda'})]$$

$$=(\eta_{\mu\rho\lambda},\eta_{\mu'\rho'\lambda}). \quad (3.20)$$

We divide the range of λ into 4 parts and prove (3.20) separately for λ in each of these 4 regions.

(I) If $|\lambda|>N+l$, then $|\lambda-\lambda'|>N$ and (3.20) is trivially satisfied because all terms vanish.

(II) If $N+l\geq|\lambda|>N-l$, then $|\lambda-\lambda'|>N-2l$, and hence the term containing X still vanishes. Due to (3.14b), the left-hand side of (3.20) becomes

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})\|\xi_{\lambda-\lambda'}\|^2 =(2N+1)^{-1}\sum_{\substack{|\lambda'|\leq l\\ |\lambda-\lambda'|\leq N}} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'}),$$

which is equal to the right-hand side of (3.20) calculated by (3.17b).

(III) If $N-l\geq|\lambda|>N-3l$, then $|\lambda-\lambda'|\leq N$ and hence $\|\xi_{\lambda-\lambda'}\|^2$ is always $(2N+1)^{-1}$. By the orthogonality, (2.2), the definition (3.16b) and the equation

$$\sum_{|\lambda'|\leq l} P_1(\lambda)=1, \quad (3.21)$$

the left-hand side of (3.20) becomes

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})[(2N+1)^{-1}-(\mathrm{X}_{\mu\rho,\lambda-\lambda'},\mathrm{X}_{\mu'\rho',\lambda-\lambda'})]$$

$$=(2N+1)^{-1}\delta_{\mu\mu'}\delta_{\rho\rho'}\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu\rho})$$

$$\times[1-(2N+1)(\mathrm{X}_{\mu\rho,\lambda-\lambda'},\mathrm{X}_{\mu'\rho',\lambda-\lambda'})]. \quad (3.22)$$

Because of (3.16b), the inside of the square bracket of (3.22) vanishes for $|\lambda-\lambda'|\leq N-2l$ and is unity for $|\lambda-\lambda'|>N-2l$. Thus, due to (3.17c) and (3.18), (3.22) is equal to the right-hand side of (3.20).

(IV) If $N-3l\geq|\lambda|$, then $|\lambda-\lambda'|\leq N-2l$ and the left-hand side of (3.20) becomes

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda)\phi_{\mu'\rho'})(1-\delta_{\mu\mu'}\delta_{\rho\rho'}).$$

Because of (3.21) and the orthogonality, (2.2), this expression vanishes and hence is equal to the right-hand side of (3.20) which also vanishes due to (3.17a). This completes the proof of (3.4).

Finally, we will prove (3.7). Since ψ is normalized, $\|\psi\otimes\eta_{\mu\rho}\|$ is equal to $\|\eta_{\mu\rho}\|$. By (3.13), we have

$$(\eta_{\mu\rho},\eta_{\mu'\rho'})=\sum\nolimits_\lambda (\eta_{\mu\rho\lambda},\eta_{\mu'\rho'\lambda}).$$

By (3.20), we get

$$\sum_\lambda (\eta_{\mu\rho\lambda},\eta_{\mu\rho'\lambda'})$$

$$=\sum_{\substack{\lambda\lambda'\\ |\lambda'|\leq l}} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})[\|\xi_{\lambda-\lambda'}\|^2-\|\mathrm{X}_{\mu\rho,\lambda-\lambda'}\|^2\delta_{\mu\mu'}\delta_{\rho\rho'}]$$

$$=\sum_{\substack{\lambda\lambda'\\ |\lambda'|\leq l}} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})[\|\xi_\lambda\|^2-\|\mathrm{X}_{\mu\rho,\lambda}\|^2\delta_{\mu\mu'}\delta_{\rho\rho'}].$$

By (3.21) and (2.2),

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})=\delta_{\mu\mu'\rho\rho'}.$$

By (3.14) and (3.16)

$$\sum\nolimits_\lambda [\|\xi_\lambda\|^2-\|\mathrm{X}_{\mu\rho\lambda}\|^2]=4l(2N+1)^{-1}.$$

Combining these, and using (3.15), we obtain

$$\|\psi\otimes\eta_{\lambda\rho}\|^2=\frac{4l}{2N+1}<\epsilon.$$

$$(\eta_{\mu\rho},\eta_{\mu'\rho'})=0, \quad \text{for } (\mu,\rho)\neq(\mu',\rho')$$

In the above construction, $\mathfrak{H}_{2\lambda}$ for $|\lambda|\leq N-3l$ should have at least the dimension of $\mathfrak{H}_1$. We need higher dimension for $\mathfrak{H}_{2\lambda}$ with $N-3l<|\lambda|\leq N$.[7]

Finally we will give a proof of our Lemma. For this purpose, we denote the subspace of $\mathfrak{H}$ spanned by eigenvectors of L with eigenvalue λ by $\mathfrak{H}_\lambda$, the subspace spanned by $P(\lambda)\Psi_\alpha{}^i$ with varying α by $\mathfrak{H}_\lambda{}^i$, the subspace spanned by $P(\lambda)\Psi_\alpha{}^f$ with varying α by $\mathfrak{H}_\lambda{}^f$, the orthogonal complement of $\mathfrak{H}_\lambda{}^i$ in $\mathfrak{H}_\lambda$ by $\mathfrak{H}_\lambda{}^i{}_\perp$, and the orthogonal complement of $\mathfrak{H}_\lambda{}^f$ in $\mathfrak{H}_\lambda$ by $\mathfrak{H}_\lambda{}^f{}_\perp$. Obviously

$$\mathfrak{H}=\oplus_\lambda(\mathfrak{H}_\lambda{}^i\oplus\mathfrak{H}_\lambda{}^i{}_\perp)=\oplus_\lambda(\mathfrak{H}_\lambda{}^f\oplus\mathfrak{H}_\lambda{}^f{}_\perp). \quad (3.23)$$

We will first show that

$$U_\lambda(\sum\nolimits_\alpha C_\alpha P(\lambda)\Psi_\alpha{}^i)=\sum\nolimits_\alpha C_\alpha P(\lambda)\Psi_\alpha{}^f, \quad (3.24)$$

defines a unitary mapping U_λ of $\mathfrak{H}_\lambda{}^i$ onto $\mathfrak{H}_\lambda{}^f$, where $\{C_\alpha\}$ is a set of arbitrary complex numbers. To see this, we note that, due to (3.3), $\sum_\alpha C_\alpha P(\lambda)\Psi_\alpha{}^i$ and

[7] In the above construction, the measuring apparatus is a superposition of eigenstates of L_2 with different eigenvalues λ varying over the range of the order $1/\epsilon$. However, if one counts the number of equations to be satisfied, one finds a possibility of constructing a similar measuring apparatus which is a superposition of eigenstates of L_2 with eigenvalues, near a certain large value of the order $1/\epsilon$, but varying only over the range of the order of the dimension of $\mathfrak{H}_1$, provided that the latter is finite. Here we will not pursue the problem of such minimization, but we will only note that, if we do minimize the number of eigenvalues of L_2 to be used in the measuring apparatus, then $\mathrm{X}_{\mu\rho}$ will be nearly strictly determined and if that is the case, there is a fair chance that $\mathrm{X}_{\mu\rho}$ cannot be made macroscopically distinguishable any better than $\phi_{\mu\rho}$.

$\sum_\alpha C_\alpha P(\lambda)\Psi_\alpha^f$ converge, diverge, or vanish simultaneously. Hence, U_λ is a one-to-one mapping of $\mathfrak{H}_\lambda^i$ onto $\mathfrak{H}_\lambda^f$. Since this mapping is linear and, due to (3.3), isometric, U_λ is a unitary mapping of $\mathfrak{H}_\lambda^i$ onto $\mathfrak{H}_\lambda^f$ as was to be proved. This also proves that the dimensions of $\mathfrak{H}_\lambda^i$ and $\mathfrak{H}_\lambda^f$ are the same.

If this dimension is finite, the dimensions of $\mathfrak{H}_\lambda{}^i{}_\perp$ and $\mathfrak{H}_\lambda{}^f{}_\perp$ are the same. Then there always exists a unitary mapping $U_{\lambda\perp}$ of $\mathfrak{H}_\lambda{}^i{}_\perp$ onto $\mathfrak{H}_\lambda{}^f{}_\perp$. Now we define an operator U in $\mathfrak{H}$.

$$U=\oplus_\lambda(U_\lambda\oplus U_{\lambda\perp}). \tag{3.25}$$

Because of the unitarity of U_λ and $U_{\lambda\perp}$ and the decomposition, (3.23), U is obviously unitary. For any $\Psi\epsilon\mathfrak{H}$,

$$U\Psi=\sum_\lambda (U_\lambda\Psi_\lambda{}^i+U_{\lambda\perp}\Psi_\lambda{}^i{}_\perp), \tag{3.26}$$

where

$$\Psi=\sum_\lambda(\Psi_\lambda{}^i+\Psi_\lambda{}^i{}_\perp),\ \Psi_\lambda{}^i\epsilon\mathfrak{H}_\lambda{}^i,\ \Psi_\lambda{}^i{}_\perp\epsilon\mathfrak{H}_\lambda{}^i{}_\perp, \tag{3.27}$$

is a unique decomposition of Ψ according to the first equation of (3.23). Since the subspace $\mathfrak{H}_\lambda$ of $\mathfrak{H}$ spanned by eigenvectors of $L=L_1\otimes 1+1\otimes L_2$ with the eigenvalue λ is mapped onto itself by U, U commutes with L. This completes the proof for the case where the dimension of $\mathfrak{H}_\lambda^i$ and $\mathfrak{H}_\lambda^f$ is finite.

If this dimension is infinite, then the dimensions of $\mathfrak{H}_\lambda{}^i{}_\perp$ and $\mathfrak{H}_\lambda{}^f{}_\perp$ can be different. In such a case we introduce a new Hilbert space $\mathfrak{H}_2{}^r$ (on which the conserved quantity $L_2{}^r$ is defined) in such a way that the dimension of $\mathfrak{H}_\lambda{}^r$ is at least the number of indices α where $\mathfrak{H}_\lambda{}^r$ is the subspace of $\mathfrak{H}^r\equiv\mathfrak{H}_1\otimes\mathfrak{H}_2{}^r$ spanned by the eigenstates of $L^r=L_1\otimes 1+1\otimes L_2{}^r$ with eigenvalues λ. Then since the dimension of $\mathfrak{H}_\lambda^i$ and $\mathfrak{H}_\lambda^f$ does not exceed the cardinal number of the set of the indices α, $\mathfrak{H}_\lambda{}^{ir}\equiv\mathfrak{H}_\lambda{}^i{}_\perp\oplus\mathfrak{H}_\lambda{}^r$ and $\mathfrak{H}_\lambda{}^{fr}\equiv\mathfrak{H}_\lambda{}^f{}_\perp\oplus\mathfrak{H}_\lambda{}^r$ have the same dimension. Hence, there always exists a unitary mapping $U_{\lambda\perp}$ of $\mathfrak{H}_\lambda{}^{ir}$ onto $\mathfrak{H}_\lambda{}^{fr}$.

We are now in the position to construct the Hilbert space $\mathfrak{H}_2'$ and the unitary operator U for this case. $\mathfrak{H}_2'$ is taken to be $\mathfrak{H}_2\oplus\mathfrak{H}_2{}^r$. L_2' on $\mathfrak{H}_2'$ is taken to be $L_2\oplus L_2{}^r$. $\mathfrak{H}'$ can be decomposed as

$$\mathfrak{H}'=\oplus_\lambda(\mathfrak{H}_\lambda{}^i\oplus\mathfrak{H}_\lambda{}^{ir})=\oplus_\lambda(\mathfrak{H}_\lambda{}^f\oplus\mathfrak{H}_\lambda{}^{fr}). \tag{3.28}$$

U is defined as unitary mapping

$$U=\oplus_\lambda(U_\lambda\oplus U_{\lambda\perp}). \tag{3.29}$$

Instead of (3.26), (3.27), we have, for any $\Psi\epsilon\mathfrak{H}'$

$$U\Psi=\sum_\lambda (U_\lambda\Psi_\lambda{}^i+U_{\lambda\perp}\Psi_\lambda{}^{ir}), \tag{3.30}$$

$$\Psi=\sum_\lambda (\Psi_\lambda{}^i+\Psi_\lambda{}^{ir}),\ \Psi_\lambda{}^i\epsilon\mathfrak{H}_\lambda{}^i,\ \Psi_\lambda{}^{ir}\epsilon\mathfrak{H}_\lambda{}^{ir}. \tag{3.31}$$

Then by the same argument as in the previous case, we can show the unitarity of U, and commutativity with L', where L' is defined as $L'\equiv L_1\otimes 1+1\otimes L_2'$.

We note that in our application of the Lemma, the number of the indices α is the same as the dimension of $\mathfrak{H}_1$.

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