equation, θ must be chosen so that

$$\omega_\mu{}^\nu{}_{;\nu}+i\omega_\mu{}^\nu\theta_{,\nu}=0.$$

Multiplying by $\omega^\mu{}_\rho$ and using the identity

$$\omega_\mu{}^\nu\omega^\mu{}_\rho\equiv\tfrac{1}{2}\delta^\nu{}_\rho\omega_{\alpha\beta}\omega^{\alpha\beta}, \tag{A-2}$$

which is a consequence of the duality of $F_{\mu\nu}$ and $D_{\mu\nu}$, one obtains

$$\theta_{,\rho}=\frac{2i\omega_\mu{}^\nu{}_{;\nu}\omega^\mu{}_\rho}{\omega_{\alpha\beta}\omega^{\alpha\beta}}\equiv\beta_\rho. \tag{A-3}$$

The reality of the above expression follows from the Bianchi identity. It can now be shown that

$$\alpha_{[\rho,\sigma]}=\beta_{[\rho,\sigma]}. \tag{A-4}$$

The reality of β_ρ and the validity of (A-4) are the essential steps in the proof that is omitted here, the details having been given in a previous paper.[10] From (A-3) and (A-4), with the notation $\omega^2=\omega_{\alpha\beta}\omega^{\alpha\beta}$, one has

$$\alpha_{[\rho,\sigma]}\omega^{\rho\sigma}=\frac{-4i\omega_\mu{}^\nu{}_{;\nu}\omega^\mu{}_{\rho;\sigma}\omega^{\rho\sigma}}{\omega^2}-\frac{4i\omega_\mu{}^\nu{}_{;\nu}\omega^\mu{}_\rho\omega^{\rho\sigma}}{\omega^2}+\frac{4i\omega_\mu{}^\nu{}_{;\nu}\omega^\mu{}_\rho\omega^{\rho\sigma}(\omega^2)_{,\sigma}}{\omega^4}. \tag{A-5}$$

Using the identity (A-2) and the antisymmetry of $\omega^{\mu\rho}$, one obtains

$$\alpha_{[\rho,\sigma]}\omega^{\rho\sigma}=-4i\omega_\mu{}^\nu{}_{;\nu}\omega^\mu{}_{\rho;\sigma}\omega^{\rho\sigma}+\frac{2i\omega^{\sigma\nu}{}_{;\nu}(\omega^2)_{,\sigma}}{\omega^2}. \tag{A-6}$$

Differentiating (A-2) yields

$$\omega^\mu{}_{\rho;\sigma}\omega^{\rho\sigma}+\omega^\mu{}_\rho\omega^{\rho\sigma}{}_{;\sigma}=\tfrac{1}{2}(\omega^2)_{,\rho}. \tag{A-7}$$

Using this in (A-6) gives

$$\alpha_{[\rho,\sigma]}\omega^{\rho\sigma}=4i\omega_\mu{}^\nu{}_{;\nu}\omega^{\mu\rho}\omega_\rho{}^\sigma{}_{;\sigma}/\omega^2.$$

This vanishes because $\omega^{\mu\rho}$ is antisymmetric but $\omega_\mu{}^\nu{}_{;\nu}\omega_\rho{}^\sigma{}_{;\sigma}$ is symmetric in μ and ρ. So $\alpha_{[\rho,\sigma]}\omega^{\rho\sigma}=0$ identically; the real and imaginary parts correspond to Eqs. (35) and (36) which are identities.

[10] L. Witten, reference 6, p. 210.

PHYSICAL REVIEW VOLUME 120, NUMBER 2 OCTOBER 15, 1960

Invariance Under Antiunitary Operators*

G. Feinberg†
Department of Physics, Columbia University, New York, New York

(Received June 1, 1960)

It is shown that for transitions between "weakly interacting" states, the transition matrix $\mathcal{T}$ can be expressed in terms of a Hermitean operator $\mathcal{T}+\mathcal{T}^\dagger$, and so invariance of the Hamiltonian under antiunitary operators such as T or TCP implies invariance of transition rates under kinematic transformations, without changing the direction of time.

An application is made to π^0 decay into 2 photons, where it is shown that invariance under TCP alone implies equality in the number of left and right circularly polarized photons, to 1 part in 10^4.

THE invariance of a Hamiltonian under a unitary transformation leads to the invariance of the transition rates under certain "kinematic" transformations of the quantum numbers in the initial and final states. For example, invariance of H under space reflection implies invariance of the transition rates under the change of sign of all momenta in the initial and final states. On the other hand, invariance of H under an anitunitary operator such as T or TCP does not in general lead to such an invariance of the transition rate, but rather to a relation between the transition rate from an initial state to final state, and the transition rate from the "kinematically reversed" final state to the kinematically reversed initial state. This is a physically distinguishable process, unless the initial and final states contain the same particles. It is, however, known that under some circumstances, invariance under an antiunitary operator nevertheless does imply a relation between transition rates for the same process. This will be the case, for example, when the following two conditions are satisfied[1]:

1. The transition matrix $\mathcal{T}$ can be taken equal to a Hamiltonian, i.e., when first-order perturbation theory is used.
2. The initial and final states $|a\rangle$, $|b\rangle$ are weakly interacting states.

In this note we shall show that the second condition alone is sufficient. Specifically, we show that if the initial and final states are such that all products of the form $\langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^\dagger|b\rangle(|n\rangle\neq|a\rangle)$ can be neglected compared to $\langle a|\mathcal{T}|b\rangle$, then invariance under the antiunitary operator θ implies equality between the transition rates for $|a\rangle\to|b\rangle$ and for $|a_R\rangle\to|b_R\rangle$, where

* Work supported by the U. S. Atomic Energy Commission.
† Alfred P. Sloan Foundation Fellow.

[1] T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. 106, 340 (1957).

$|a_R\rangle$, $|b_R\rangle$ are the "kinematically reversed" states. This follows from the fact that under the assumption made, the $\mathcal{T}$ matrix element can be expressed in terms of the matrix element of the Hermitean operator $\mathcal{T}+\mathcal{T}^\dagger$, whose invariance under θ follows from the invariance of H.

To establish the theorem, let us consider the transition matrix element $\langle a|\mathcal{T}|b\rangle$, where

$$1+i\mathcal{T}=S,$$

the S matrix. The unitary condition on S then gives

$$i(\mathcal{T}-\mathcal{T}^\dagger)=\mathcal{T}\mathcal{T}^\dagger, \tag{1}$$

or

$$i\langle a|(\mathcal{T}-\mathcal{T}^\dagger)|b\rangle=\langle a|\mathcal{T}\mathcal{T}^\dagger|b\rangle. \tag{2}$$

Introducing a complete set of states $|n\rangle$ between $\mathcal{T}$ and $\mathcal{T}^\dagger$, one gets

$$i\langle a|(\mathcal{T}-\mathcal{T}^\dagger)|b\rangle=\textstyle\sum_n \langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^\dagger|b\rangle, \tag{3}$$

where by 4-momentum conservation, the states $|n\rangle$ must have the same 4-momentum as $|a\rangle$ and $|b\rangle$.

Let us separate out the state $|a\rangle$ itself from the sum over $|n\rangle$, so that

$$\sum_n \langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^\dagger|b\rangle=\langle a|\mathcal{T}|a\rangle\langle a|\mathcal{T}^\dagger|b\rangle + \sum_{n\neq a} \langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^\dagger|b\rangle. \tag{4}$$

The assumption that the final state $|b\rangle$ is weakly interacting will now be taken to mean that all terms in the remaining sum can be dropped. The second factor, $\langle n|\mathcal{T}|b\rangle$, in such terms represents the transition rate to the state $|b\rangle$ from the arbitrary state $|n\rangle$. Now if $|b\rangle$ contains particles which do not have strong interactions, such as photons or leptons, these factors will be proportional to some power either of the fine structure constant or the weak interaction constant, and so it is reasonable to neglect them. However, in some cases, this second factor may be of the same or lower order in small constants as the quantity $\langle a|\mathcal{T}|b\rangle$ which we eventually wish to compute. In this case, we must invoke the assumption that the state $|a\rangle$ is also weakly interacting, so that $\langle a|\mathcal{T}|n\rangle$ will also be proportional to some small constant, and then $\langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^\dagger|b\rangle$ will be doubly small, and so negligible. The validity of this procedure must necessarily be examined for each case where the theorem is to applied.

It remains to discuss the term $\langle a|\mathcal{T}|a\rangle\langle a|\mathcal{T}^\dagger|b\rangle$. There is good reason to drop this term as well. This is again because the assumption that $|a\rangle$ is weakly interacting implies that the matrix element $\langle a|\mathcal{T}|a\rangle$ is proportional to some small parameter, and thus $\langle a|\mathcal{T}|a\rangle\langle a|\mathcal{T}^\dagger|b\rangle$ will be small compared to $\langle a|\mathcal{T}|b\rangle$. For example, if $|a\rangle$ is a one-particle state, then $\langle a|\mathcal{T}|a\rangle$ is proportional to the decay rate of the particle.[2] We shall show in the Appendix that the final result still holds even without neglect of $\langle a|\mathcal{T}|a\rangle$.

The conclusion from the above analysis is that

$$\textstyle\sum_n \langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^\dagger|b\rangle$$

will be small compared to $\langle a|\mathcal{T}|b\rangle$, with the order of smallness varying from case to case. Thus we can set

$$i\langle a|(\mathcal{T}-\mathcal{T}^\dagger)|b\rangle=0,$$

or

$$\langle a|\mathcal{T}|b\rangle=\tfrac{1}{2}\langle a|(\mathcal{T}+\mathcal{T}^\dagger)|b\rangle, \tag{5}$$

and

$$\langle a|\mathcal{T}^\dagger|b\rangle=\tfrac{1}{2}\langle a|(\mathcal{T}+\mathcal{T}^\dagger)|b\rangle=\langle a|\mathcal{T}|b\rangle.$$

This is the necessary theorem for the application of invariance under θ. We now suppose that $[\theta,H]=0$. Then as usual

$$\theta\mathcal{T}^\dagger\theta^\dagger=\mathcal{T}. \tag{6}$$

Also, from (5)

$$\langle a|\mathcal{T}|b\rangle^*=\langle a|\mathcal{T}^\dagger|b\rangle^*. \tag{7}$$

But by antiunitarity of θ

$$\begin{aligned}\langle a|\mathcal{T}^\dagger|b\rangle^*&=\langle\theta a|\theta\mathcal{T}^\dagger|b\rangle\\&=\langle\theta a|\mathcal{T}|\theta b\rangle.\end{aligned} \tag{8}$$

We again invoke the assumption that $|a\rangle$, $|b\rangle$ are weakly interacting to conclude that $|\theta b\rangle$ and $|\theta a\rangle$ are just kinematically reversed states, (with particles changed to antiparticles if θ involves charge conjugation). Otherwise it would also be necessary to charge from "in" states to "out" states.

Denoting the reversed states by $|a_R\rangle$, $|b_R\rangle$ as before, we have

$$\langle a|\mathcal{T}|b\rangle^*=\langle a_R|\mathcal{T}|b_R\rangle, \tag{9}$$

or

$$|\langle a|\mathcal{T}|b\rangle|^2=|\langle a_R|\mathcal{T}|b_R\rangle|^2$$

which is a relation of the type implied by invariance under unitary transformations.

One place where the above theorem may be useful is when the first nonvanishing contribution to the $\mathcal{T}$ matrix for a particular process comes from some order in perturbation theory higher than the first. One such case is the decay of the π^0 into two photons. The present belief is that there is no term in the fundamental Hamiltonian of the form $\Phi\mathbf{E}\cdot\mathbf{H}$ which would give this decay in lowest order, but rather that the process happens through the combined effect of the π^0-baryon interaction and the electromagnetic interaction of charged baryons.

The initial π^0 state would be stable apart from electromagnetic interactions, so that

$$\langle\pi^0|\mathcal{T}|\pi^0\rangle=0(\alpha^2), \tag{10}$$

and thus

$$\langle\pi^0|\mathcal{T}|\pi^0\rangle\langle\pi^0|\mathcal{T}^\dagger|2\gamma\rangle=0(\alpha^3). \tag{11}$$

The final two-photon state is weakly interacting in the sense described above, and the states $|n\neq\pi^0\rangle$ for which $\langle n|\mathcal{T}^\dagger|2\gamma\rangle$ does not vanish can only be states

[2] This will be true after renormalization. For a stable particle, wave function renormalization implies that $\langle a|S|a\rangle=1$, or $\langle a|\mathcal{T}|a\rangle=0$.

with 2 or more photons, or with electron pairs. It is not hard to see that for these also

$$\langle\pi^0|\mathcal{T}|n\rangle\langle n|\mathcal{T}^\dagger|2\gamma\rangle=0(\alpha^3). \qquad (12)$$

Thus

$$\langle\pi^0|(\mathcal{T}-\mathcal{T}^\dagger)|2\gamma\rangle=0(\alpha^3). \qquad (13)$$

Since $\langle\pi^0|(\mathcal{T}+\mathcal{T}^\dagger)|2\gamma\rangle=0(\alpha)$, it follows that Eq. (9) will hold to order α^2 at least in this case.

Now let us assume invariance under TCP. Then $|\pi_R{}^0\rangle$ is the state obtained by acting on $|\pi^0\rangle$ with TCP. If the π^0 is taken at rest, this is just $|\pi^0\rangle$ (apart from an irrelevant phase), since π^0 is its own antiparticle.[3] For the final 2-photon state, if we analyze in terms of the photon helicites, it can be shown that[4]

$$TCP|ll\rangle=|rr\rangle,\quad TCP|rr\rangle=|ll\rangle, \qquad (14)$$

where $|ll\rangle$ means a state with 2 left circularly polarized photons and $|rr\rangle$ a state with 2 right circularly polarized photons. Thus in this case, Eq. (9) gives

$$|\langle\pi^0|\mathcal{T}|rr\rangle|^2=|\langle\pi^0|\mathcal{T}|ll\rangle|^2, \qquad (15)$$

and so if TCP invariance holds, the number of left and right circularly polarized photons in the π^0 decay must be equal (apart from possible corrections of order $\alpha^2=10^{-4}$). Thus the result of the experiments of Garwin *et al.*[5] demonstrating that the numbers of left and right circularly polarized photons are indeed equal, is implied by invariance of the strong and electromagnetic interactions under TCP alone.

One may write a $\mathcal{T}$-matrix element for the process $\pi^0\to2\gamma$ as in reference 4: $\langle\mathcal{T}\rangle=\bar{\alpha}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'+\bar{\beta}(\boldsymbol{\varepsilon}\times\boldsymbol{\varepsilon}')\cdot\mathbf{k}$, where $\bar{\alpha}$, $\bar{\beta}$ are constants, $\mathbf{k}$ is the photon momentum, and $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}'$ the photon linear polarizations. It then follows from Eq. (9), with $\theta=TCP$, that $\bar{\alpha}$ and $\bar{\beta}$ must be real, provided that one neglects terms of order e^4 compared to the leading ones.

ACKNOWLEDGMENTS

The author would like to thank Dr. J. Bernstein and Dr. L. Michel for a preprint and several communications which stimulated this investigation. He would like to thank Dr. N. Kroll for helpful discussions on the formulation and validity of these results.

[3] The antiparticle for a spinless particle at rest may in general be defined as its TCP transform. This definition will reduce to the usual ones if symmetries such as C hold as well. That the π^0 is its own antiparticle follows from the fact that no other particle of equal mass, spin, etc., is known.

[4] The transformation properties of 2-photon states under T, C, P, etc., is analyzed in detail by J. Bernstein and L. Michel [Phys. Rev. **118**, 871 (1960)].

[5] R. L. Garwin, G. Gidal, L. M. Lederman, and M. Weinrich, Phys. Rev. **108**, 1589 (1957).

APPENDIX

In this Appendix, we show that Eq. (9) is satisfied even when $\langle a|\mathcal{T}|a\rangle$ is not neglected.

From Eqs. (3), (4), we have, neglecting the remaining sum

$$\begin{aligned} i\langle a|\mathcal{T}-\mathcal{T}^\dagger|b\rangle &= \langle a|\mathcal{T}|a\rangle\langle a|\mathcal{T}^\dagger|b\rangle \\ &= \langle a|\mathcal{T}|a\rangle[\tfrac{1}{2}\langle a|(\mathcal{T}+\mathcal{T}^\dagger)|b\rangle+\tfrac{1}{2}\langle a|(\mathcal{T}-\mathcal{T}^\dagger)|b\rangle], \end{aligned} \qquad (A1)$$

or

$$\langle a|(\mathcal{T}-\mathcal{T}^\dagger)|b\rangle=\left\langle a\left|\frac{(\mathcal{T}+\mathcal{T}^\dagger)}{2}\right|b\right\rangle\left[\frac{\langle a|\mathcal{T}|a\rangle}{i+\frac{1}{2}\langle a|\mathcal{T}|a\rangle}\right], \qquad (A2)$$

so that

$$\langle a|\mathcal{T}|b\rangle=\left\langle a\left|\frac{(\mathcal{T}+\mathcal{T}^\dagger)}{2}\right|b\right\rangle\left[1+\frac{\langle a|\mathcal{T}|a\rangle}{2i+\langle a|\mathcal{T}|a\rangle}\right], \qquad (A3)$$

while

$$\langle a|\mathcal{T}^\dagger|b\rangle=\left\langle a\left|\frac{(\mathcal{T}+\mathcal{T}^\dagger)}{2}\right|b\right\rangle\left[1-\frac{\langle a|\mathcal{T}|a\rangle}{2i+\langle a|\mathcal{T}|a\rangle}\right].$$

Now

$$\begin{aligned} 1+\frac{\langle a|\mathcal{T}|a\rangle}{2i+\langle a|\mathcal{T}|a\rangle}&=\frac{2(i+\langle a|\mathcal{T}|a\rangle)}{2i+\langle a|\mathcal{T}|a\rangle}\equiv1+R, \\ 1-\frac{\langle a|\mathcal{T}|a\rangle}{2i+\langle a|\mathcal{T}|a\rangle}&=\frac{2i}{2i+\langle a|\mathcal{T}|a\rangle}\equiv1-R, \end{aligned} \qquad (A4)$$

$$\begin{aligned} |1+R|^2 &= \frac{4[1+i(\langle a|\mathcal{T}^\dagger|a\rangle-\langle a|\mathcal{T}|a\rangle)+\langle a|\mathcal{T}|a\rangle\langle a|\mathcal{T}^\dagger|a\rangle]}{4+2i(\langle a|\mathcal{T}^\dagger|a\rangle-\langle a|\mathcal{T}|a\rangle)+\langle a|\mathcal{T}|a\rangle\langle a|\mathcal{T}^\dagger|a\rangle}, \\ |1-R|^2 &= 4/[4+2i(\langle a|\mathcal{T}^\dagger|a\rangle-\langle a|\mathcal{T}|a\rangle) \\ &\qquad +\langle a|\mathcal{T}|a\rangle\langle a|\mathcal{T}^\dagger|a\rangle]. \end{aligned} \qquad (A5)$$

But

$$\begin{aligned} i\langle a|\mathcal{T}^\dagger-\mathcal{T}|a\rangle &= -\sum\langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^\dagger|a\rangle \\ &= -\sum\langle a|\mathcal{T}|a\rangle\langle a|\mathcal{T}^\dagger|a\rangle \\ &\qquad -\sum_{n\neq a}\langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^\dagger|a\rangle. \end{aligned} \qquad (A6)$$

The terms in the second sum are of the type we have neglected as being proportional to some higher power of the decay constant. (We do not assume this to be true about $\langle a|\mathcal{T}|a\rangle$ itself here.) Therefore

$$|1+R|^2=4/[4-\langle a|\mathcal{T}|a\rangle\langle a|\mathcal{T}^\dagger|a\rangle]=|1-R|^2, \qquad (A7)$$

and so

$$\begin{aligned} |\langle a|\mathcal{T}|b\rangle|^2 &= |\langle a|\mathcal{T}^\dagger|b\rangle|^2 \\ &= |\langle\theta a|\mathcal{T}|\theta b\rangle|^2 \\ &= |\langle a_R|\mathcal{T}|b_R\rangle|^2 \end{aligned} \qquad (A8)$$

as before.