Measurement of the Curvature in a Two-Dimensional Universe

EUGENE P. WIGNER
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received June 1, 1960)

The usual definition of the curvature of space involves concepts, such as the measurement of the metric tensor or parallel displacement, which have no direct physical counterpart. The question of obtaining the curvature of a two-dimensional space by means of measurements which are possible, at least in principle, is reviewed and a formula given before is corrected and generalized.

BRUNO Bertotti has kindly pointed out to the present writer that the last formula (5.7) of the article, "Relativistic Invariance and Quantum Phenomena,"[1] contains an error. It gives the radius a of a two-dimensional de Sitter universe in terms of the times, t_1, t_2, and t_3, which three successive light quanta take to travel from one freely moving material body to another one, and back. The second light quantum is emitted when the first one arrives back, the third one is sent out when the second one returns. The left side of the equation contains the second difference $t_1-2t_2+t_3$, whereas it should contain the second difference of the logarithms of these time intervals. At any rate, the correct formula is ($c=1$)

$$\frac{t_1t_3-t_2^2}{t_2^4}=\frac{1}{a^2}=\tfrac{1}{2}R, \tag{1}$$

where R is Riemann's curvature tensor which has, in the case of two dimensions, only one (invariant) component. The other formulas are correct except that the incorrect (5.7) is quoted also as (2) on page 262.

Since it is, apparently, quite possible to make a mistake in the calculation, a few details of it will be reproduced here. There are four events in the course of the measurement: (1) the sending out of the first light quantum, (2) the return of this quantum and the simultaneous emission of the second quantum, (3) the return of the second quantum and the sending out of the third quantum, (4) the return of this last quantum. The corresponding proper times are given by the combination of (5.4) and (5.5)

$$\tau_i=a[\ln(1+\sin\varphi_i)-\ln\cos\varphi_i], \tag{2}$$

the successive φ_i differing, according to (5.3a), (5.3b), and (5.3c), by 2δ. We set

$$\varphi_1=\varphi-3\delta;\quad \varphi_2=\varphi-\delta;\quad \varphi_3=\varphi+\delta;\quad \varphi_4=\varphi+3\delta. \tag{3}$$

The τ_i have to be calculated accurately up to the third power of δ. In particular,

$$\frac{\tau_3}{a}=\ln\frac{1+\sin\varphi}{\cos\varphi}+\frac{\delta}{\cos\varphi}+\frac{\delta^2\sin\varphi}{2\cos^2\varphi}+\frac{\delta^3(1+\sin^2\varphi)}{6\cos^3\varphi}. \tag{4}$$

The quantities τ_1, τ_2, and τ_4 can be obtained by substituting -3δ, $-\delta$, and 3δ for δ in (4). Hence,

$$\frac{t_1}{a}=\frac{\tau_2-\tau_1}{a}=\frac{2\delta}{\cos\varphi}-\frac{4\delta^2\sin\varphi}{\cos^2\varphi}+\frac{13\delta^3(1+\sin^2\varphi)}{3\cos^3\varphi}, \tag{5a}$$

$$\frac{t_2}{a}=\frac{\tau_3-\tau_2}{a}=\frac{2\delta}{\cos\varphi}+\frac{\delta^3(1+\sin^2\varphi)}{3\cos^3\varphi}, \tag{5b}$$

$$\frac{t_3}{a}=\frac{\tau_4-\tau_3}{a}=\frac{2\delta}{\cos\varphi}+\frac{4\delta^2\sin\varphi}{\cos^2\varphi}+\frac{13\delta^3(1+\sin^2\varphi)}{3\cos^3\varphi}. \tag{5c}$$

The distance between the two material bodies is one-half of the time needed for the round trip of the light quantum

$$d\approx\tfrac{1}{2}t_2\approx a\delta/\cos\varphi, \tag{6}$$

their relative velocity is the change of the half return time, divided by the time in which this change took place

$$v\approx\frac{\tfrac{1}{2}(t_3-t_1)}{2t_2}\approx\frac{\delta\sin\varphi}{\cos\varphi}. \tag{7}$$

Both these expressions are accurate to the next power of δ.

Equation (1) is a direct consequence of Eq. (5). It should be noted, however, that the foregoing derivation applies only if the world lines of the two particles do not intersect. As a comparison of (7) and (6) shows, this means that their relative velocity, as defined above, is in terms of the velocity of light not larger than their distance in terms of the radius a of the universe. If this condition is not met, the world lines of the particles will intersect and the preceding derivation is not valid. It is possible, however, to derive a formula similar to (1) for the measurement of the radius of curvature by means of light signals between two particles with arbitrary velocities with respect to each other. If the scalar product of the velocities at the time of intersection is denoted by $\cosh\alpha$ (so that α is the hyperbolic angle between the paths at intersection), a calculation along the lines of the preceding calculation gives

$$\frac{t_1t_3-t_2^2}{t_2^4}=\frac{1}{a^2}[1+(7/3)\sinh^2\alpha+4/3\sinh^4\alpha]. \tag{8}$$

At the time of the measurement, the relative velocity of the particles, defined as in (7), is

$$v=\cosh\alpha\sinh\alpha. \tag{9}$$

[1] E. P. Wigner, Revs. Modern Phys. **29**, 255 (1957). There are several points of contact between the last section of this article and the more elaborate investigation of F. A. E. Pirani, Acta Phys. Polon. **6**, 389 (1956).