Multiple Meson Production in Nucleon-Antinucleon Annihilations*†

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A two-parameter model is proposed for treating complicated production problems in a relatively simple way. It is assumed that the interaction may be characterized by a range of interaction and by a coupling strength. After the model is developed, it is applied to the problem of pion production in $N-\bar{N}$ annihilations. The two parameters are fixed by the experimental data for the multiplicity and energy spectra. It is found that all the data can be satisfied if one chooses the radius of interaction to be one pion Compton wavelength. Under certain restrictions the model reduces to the Fermi model.

I. INTRODUCTION

SINCE the discovery of the antiproton in 1955,[1] a large amount of experimental work has been devoted to its interaction with nucleons.[2–5] The results of these investigations present two conspicuous features: (a) cross sections that are large compared to similar $N-N$ interactions, and (b) multiplicities that appear large in light of calculations based on the Fermi model[6] if a radius of interaction is chosen to agree with current ideas of nuclear structure, i.e., $1/\mu$.[7]

Recently, several authors[8–10] have attempted to understand one or both of these features on the basis of phenomenological models that preserve our present understanding of the nucleon structure, and of these, two have been moderately successful, viz., those of Koba and Takeda[8] and of Ball and Chew.[9]

The model of Koba and Takeda attempts to describe both of the salient features by means of two distinct interactions. It is assumed that the actual $N-\bar{N}$ annihilation occurs between the cores, producing pions, and in such a short time that the pion clouds are unaffected. The resulting "unattached" clouds then disperse, producing additional pions. By assigning an effective core radius of $2/3\mu$, they are able to fit the total and absorptive cross sections as well as the pion multiplicity fairly well. However, it is difficult to understand the available energy-spectra data on this basis. Thus if we assume that all the pions emitted are in S states, relative to the barycentric system of the $N-\bar{N}$ system, we have $kR\geqslant 0$, where k is the wave number and R is the radius of interaction. Accordingly, we should expect for the maximum contribution, $kR\sim 2$ or $k\sim 400$ Mev/c.[11] On this basis we would obtain a momentum spectra peaked around 400 Mev/c. Although this is in agreement with an average value of $E_0/5$, it is rather large with respect to the experimental value, viz., $k\sim 300$ Mev/c. Koba and Takeda point out that interactions in the annihilation region can change the energy spectra and that their actual numerical value for the effective core radius is not to be taken seriously, but one wonders whether any reasonable effective core radius, producing half of the pions, will lead to the correct spectra. This is particularly true if the cross-section data are still to be satisfied.

The approach of Ball and Chew, although along the same lines, eliminates this problem. They treat only the problem of cross sections and assume that the "cores" annihilate in the sense that an ingoing-wave boundary condition is present to represent the large probability of annihilation if the particles come close together. On this basis they are able to obtain the low-energy experimental data for the cross sections. Further, the results of their calculations, as expected in the considered energy range, are very insensitive to the location of this boundary.

Considering the success of Ball and Chew, it is not unreasonable to hope that the pion multiplicity can also be treated without attaching special characteristics to the nuclear core and pion cloud. However, attempts along this line of reasoning, e.g., by the Fermi model and modifications to it,[12–14] have not led to favorable

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results. Present results show, nevertheless, that a similar approach will reproduce the experimental data if one includes (a) the approximate energy dependence of the matrix elements, which are neglected in the Fermi model, and (b) the results of the calculation of Ball and Chew with respect to the partial waves involved in annihilation. The present model cannot be considered a statistical model in the Fermi sense, but under certain restrictions reduces to it.

II. THE INTERACTION MODEL

A. Formulation of the Proposed Model

In this section we will develop a model, the interaction model, for treating complicated production problems in a relatively simple way. As the essential physical approximation, we assume that the primary features of a given process are produced by an interaction confined to a small volume in coordinate space and further characterized by a parameter giving the coupling strength. In order to implement this approximation, we begin by writing the scattering amplitude in the coordinate representation. The resulting integral equation is rewritten in terms of some complete set of states, these states being chosen for their convenience in describing the process. The coefficients of these states, the partial-wave scattering amplitudes, are coordinate integrals over the interaction operator, and we introduce the above approximation by restricting the limits of the integrals to a small volume of space. Of course, these contributions must be such that the appropriate quantum numbers are conserved. This physical approximation is closely related to that in the Fermi model, and in fact we shall show that under certain restrictions the interaction model reduces, essentially, to the Fermi model.

We begin by constructing the state to describe a given system of particles. Proceeding in the usual way, we describe physical one-particle states by a complete orthonormal set of one-particle state vectors in a Hilbert subspace. The subspace describing a state of n particles of different types, i.e., nucleons, pions, etc., is given by the direct product of n one-particle subspaces corresponding to the appropriate type of particle. The total Hilbert space is given by the sum of all such subspaces. We may choose as the basis set for the one-particle state vectors the coordinate eigenvectors, and the n-particle subspace basis set for particle types $i, \cdots, j$, is therefore

$$|r_1{}^i\cdots r_n{}^j\rangle = |r_1{}^i\rangle\cdots|r_n{}^j\rangle.$$

Of course, depending on the nature of the particles, we must symmetrize the state vectors appropriately.

In the following, we shall confine ourselves to nucleons and pions for which we will use $|r\rangle$ and $|\xi\rangle$, respectively. A system of nucleons and pions is thus given as

$$|N\pi\rangle = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r_1\cdots}\sum_{\xi_1\cdots}\psi_{np}(r_1,\cdots,r_n;\xi_1,\cdots,\xi_p) \times|r_1\cdots r_n;\xi_1\cdots\xi_p\rangle, \quad (1)$$

where $\psi_{np}(r_1\cdots r_n;\xi_1\cdots\xi_p)$ represents the probability amplitude for finding n nucleons and p pions at their respective positions, $r_1, \cdots, r_n; \xi_1, \cdots, \xi_p$. If $|N\pi\rangle$ represents a system of noninteracting particles, then, of course, we have $\psi_{np}(r_1\cdots\xi_p)=\psi(r_1)\cdots\psi(r_n)\phi(\xi_1)\cdots\phi(\xi_p)$, where $\psi(r_i)$ and $\phi(\xi_j)$ represent the appropriate one-particle wave functions for describing nucleons and pions, respectively.

In the Schrödinger picture, the time development of such a system of particles is given by the Schrödinger equation,

$$(\mathbf{H}_0+\mathbf{V})|\psi\rangle = i(\partial/\partial t)|\psi\rangle,$$

where $|\psi\rangle$ is the state vector for the system and $\mathbf{H}_0$ is the Hamiltonian operator for a system of noninteracting nucleons and pions. The interaction operator, $\mathbf{V}$, contains all of the interaction and can include the creation and destruction of particles.

If we consider this as a stationary-state scattering problem in which there is a continuous incoming and outgoing flux of particles, we have

$$|\psi\rangle = e^{-iE_0t}|\psi\rangle,$$

which gives

$$(E_0-\mathbf{H}_0)|\psi\rangle = \mathbf{V}|\psi\rangle,$$

and we apply the usual scattering formalism. We thus have the formal solution,

$$|\psi\rangle = |\phi_0\rangle + \mathbf{g}_0(E_0)\mathbf{V}|\psi\rangle, \quad (2)$$

where $|\phi_0\rangle$ and $\mathbf{g}_0(E_0)$ are defined by the equations,

$$(E_0-\mathbf{H}_0)|\phi_0\rangle = 0, \quad (3)$$

and

$$(E_0-\mathbf{H}_0)\mathbf{g}_0(E_0) = \mathbf{I}, \quad (4)$$

and $\mathbf{I}$ is the identity operator. As seen from Eq. (3), the state vector $|\phi_0\rangle$ represents a free-particle state of nucleons and pions with a total energy E_0, and from Eq. (4), $\mathbf{g}_0(E_0)$ is the free many-particle Green's operator for the parameter E_0.

For definiteness, we can take $|\phi_0\rangle$ to represent a state of n' nucleons and p' pions for which we have

$$E_0 = E(N_1)+\cdots+E(N_{n'})+W(\pi_1)+\cdots+W(\pi_{p'}).$$

Thus the probability amplitude for finding a state of n nucleons and p pions is given by the scalar product of $|\psi\rangle$ with the subspace describing the system of n nucleons and p pions. If we use the coordinate basis set for this projection, then we find, from Eq. (2),

$$\langle r_1\cdots r_n\xi_1\cdots\xi_p|\psi\rangle = \langle r_1\cdots\xi_p|\phi_{0n'p'}\rangle + \langle r_1\cdots\xi_p|\mathbf{g}_0(E_0)\mathbf{V}|\psi\rangle, \quad (5)$$

or

$$\psi(r_1\cdots\xi_p)=\psi(r_1)\cdots\phi(\xi_p)\delta_{nn'}\delta_{pp'}+\int d^3r_1'\cdots d^3\xi_p' \times \mathbf{g}_0(r_1\cdots\xi_p;r_1'\cdots\xi_p';E_0)\langle r_1'\cdots\xi_p'|\mathbf{V}|\psi\rangle,$$

which is an integral equation to determine $|\psi\rangle$.

If we confine ourselves to the problem of particle production in collisions and annihilations, then Eq. (5) becomes

$$\psi(r_1\cdots r_n,\xi_1\cdots\xi_p)=\int d^3r_1'\cdots d^3\xi_p' \times \mathbf{g}_0(r_1\cdots\xi_p;r_1'\cdots\xi_p';E_0)\langle r_1'\cdots\xi_p'|\mathbf{V}|\psi\rangle. \quad (6)$$

We now introduce a complete set of states in terms of the identity operator; these states are energy eigenfunctions with additional quantum numbers chosen for their convenience in treating a particular problem. Representing these states by ν and η for the nucleons and pions, respectively, from Eq. (6) and using

$$\mathbf{E}|\nu_i\rangle=E_i|\nu_i\rangle \quad\text{and}\quad \mathbf{E}|\eta_i\rangle=\omega_i|\eta_i\rangle,$$

we have

$$\psi(r_1\cdots\xi_p)=\sum_{\nu,\eta}\langle r_1\cdots\xi_p|\nu_1\cdots\eta_p\rangle\left\{\frac{1}{E_0-E_1-\cdots-\omega_p}\times\int d^3r_1'\cdots d^3\xi_p'\langle\nu_1\cdots\eta_p|r_1'\cdots\xi_p'\rangle\langle r_1'\cdots\xi_p'|\mathbf{V}|\psi\rangle\right\}.$$

To be precise, we must, of course, define the Green's function properly, but we will take the point of view that the primary effect is to impose energy conservation. That is, we know

$$T_{ab}=-2\pi i\delta(E_a-E_b)T_{ab} \quad\text{and}\quad T_{ab}=(\phi_a,V\psi_b^-).$$

Additional factors will occur, but in the actual calculations we will work only with ratios for which we may assume that these factors balance out. Thus we replace the Green's function by a delta function (Kronecker or Dirac, depending on spectra) and write

$$\psi(r_1\cdots\xi_p)=\sum_{\nu,\eta} r_1\cdots\xi_p\nu_1\cdots\eta_p \times\delta(E_0-E_1-\cdots-\omega_p)\psi(\nu_1\cdots\eta_p), \quad (7)$$

where[15]

$$\psi(\nu_1\cdots\eta_p)=\int d^3r_1'\cdots d^3\xi_p'\langle\nu_1\cdots\eta_p|r_1'\cdots\xi_p'\rangle \times\langle r_1'\cdots\xi_p'|\mathbf{V}|\psi\rangle. \quad (8)$$

Of course, $\psi(\nu_1\cdots\eta_p)$ is just the scattered wave amplitude in terms of another complete set. From Eq. (7) we have, for the total probability,

$$P_{np}=\sum_{\nu,\eta}\delta(E_0-E_1-\cdots-\omega_p)|\psi(\nu_1\cdots\eta_p)|^2. \quad (9)$$

[15] Although not explicitly written out, such integrals are always considered to include sums over the various spin spaces.

In the present development we shall propose an approximate method of calculating $\psi(\nu_1\cdots\eta_p)$. Rewriting Eq. (8), we have

$$\psi(\nu_1\cdots\eta_p)=\int d^3r_1'\cdots d^3\xi_p'\,\psi_{\nu(1)}^*(r_1')\cdots\phi_{\eta(p)}^*(\xi_p') \times\langle r_1'\cdots\xi_p'|\mathbf{V}|\psi\rangle. \quad (10)$$

This is an overlap integral in the usual fashion if we interpret $\langle r_1'\cdots\xi_p'|\mathbf{V}|\psi\rangle$ as an initial state. We will adopt this viewpoint in the sense that $\langle r_1'\cdots\xi_p'|\mathbf{V}|\psi\rangle$ specifies the conserved dynamical variables, remembering that the effect of the interaction operator must also be included. In order to calculate $\psi(\nu_1\cdots\eta_p)$ we approximate the effect of $\langle r_1'\cdots\xi_p'|\mathbf{V}|\psi\rangle$ by assuming that the essential features of the problem are given by considering only the contributions from a small volume of the coordinate space. We will always consider ourselves to be working in the barycentric system, and then we will take

$$\begin{aligned}\langle r_1'\cdots\xi_p'|V|\psi\rangle&=G_N{}^nG_\pi{}^p\Psi(r_1'\cdots\xi_p')\\&=G_N{}^nG_\pi{}^pC\theta(R-r_1')\cdots\theta(R-\xi_p')\\&\quad\times Y(r_1'\cdots\xi_p')S(N)I(N\pi),\end{aligned} \quad (11)$$

where G_N and G_π are of the nature of dimensionless coupling constants, C is a constant determined by our interpretation of the initial state, i.e.,

$$\int|\Psi(r_1'\cdots\xi_p')|^2d^3r_1'\cdots d^3\xi_p'=1, \quad (12)$$

and $Y(r_1',\cdots,\xi_p')$, $S(N)$, and $I(N\pi)$ are angular, mechanical spin, and isotopic spin functions, respectively. We take G_N, G_π, and R as characterizing the interaction leading to a given final state, $\psi(\nu_1\cdots\eta_p)$. In order to distinguish this approximation from the form in Eq. (10), we will write

$$S(\nu_1\cdots\eta_p)=G_N{}^nG_\pi{}^p\int d^3r_1'\cdots d^3\xi_p'\,\psi_{\nu(1)}^*(r_1')\cdots \times\phi_{\eta(p)}^*(\xi_p')\Psi(r_1'\cdots\xi_p'). \quad (13)$$

Although we represent the effect of the interaction by the factors G_N and G_π and we interpret $\Psi(r_1'\cdots\xi_p')$ as an initial state, $\Psi(r_1'\cdots\xi_p')$ can possibly have other properties that are related to the interaction. From a field-theoretic viewpoint, we would expect $\mathbf{V}$ itself to be composed of particle fields combined to produce current terms. It is not unreasonable thus to expect $S(\nu_1\cdots\eta_p)$ to have additional energy dependence. In general, this is unknown, and is suppressed in Eq. (13). However, since pions are involved in the process, pion field operators should occur in $\mathbf{V}$. We can at least include the normalization of this field and write, instead of

Eq. (13),

$$S(\nu_1\cdots\eta_p)=G_N{}^nG_\pi{}^p d^3r_1'\cdots d^3\xi_p'\psi_{\nu(1)}{}^*(r_1')\cdots \times\phi_{\eta(p)}{}^*(\xi_p')\frac{\Psi(r_1'\cdots\xi_p')}{(2\omega_1R\cdots2\omega_pR)^{\frac{1}{2}}}, \quad (14)$$

where we include the factor R for dimensional reasons. Instead of Eq. (9) we have

$$P_{np}=\sum_{\nu,\eta}\delta(E_0-E_1-\cdots-\omega_p)|S(\nu_1\cdots\eta_p)|^2. \quad (15)$$

In the above, $\Psi(r_1'\cdots\xi_p')$ has been taken to be a step function in the radial coordinates, but this is not necessary and other forms could be chosen. However, if this model is at all meaningful, the results should be practically independent of such variations. In the present case involving nucleons and pions, we should expect from other considerations. e.g., Fermi model or static model, that R should be at most of the order of a pion Compton wavelength. If the nucleons are considered nonrelativistic, then the free-particle wave functions do not have violent changes for $r<R$, and we would not expect the results obtained by using a step function to vary appreciably from those resulting from another choice.

B. Relation to the Fermi Model

Before considering the problem of $N-\bar{N}$ annihilations, we shall show that, with additional restrictions, the interaction model essentially reduces to the Fermi model.[6] Consider the case where the complete set consists of plane waves. Then from Eq (13), we write

$$S_{np}(\mathbf{K}_1,\cdots,\mathbf{K}_n;\mathbf{k}_1,\cdots,\mathbf{k}_p)=G_N{}^nF_\pi{}^p\int d^3r_1'\cdots d^3\xi_p \times\frac{\exp(-i\mathbf{K}_1\cdot\mathbf{r}_1')}{\sqrt{V}}\cdots\frac{\exp(-i\mathbf{k}_p\cdot\boldsymbol{\xi}_p')}{\sqrt{V}}\Psi(r_1'\cdots\xi_p').$$

We neglect the possible mechanical and isotopic spin dependence of $\Psi(r_1'\cdots\xi_p')$ and take an isotropic angular distribution. Then from Eq. (11) we have

$$\Psi(r_1'\cdots\xi_p')=C\theta(R-r_1')\cdots\theta(R-\xi_p'),$$

where C, determined from Eq. (12), is

$$C=(\tfrac{4}{3}\pi R^3)^{-(n+p)/2}.$$

Typical integrals are of the form

$$I(\beta)=\int d^3\rho\exp(-i\boldsymbol{\beta}\cdot\boldsymbol{\rho})\theta(R-\rho)=4\pi R^3\frac{j_1(\beta R)}{\beta R},$$

where $j_l(\beta R)$ is the usual spherical Bessel function. Thus, we can write

$$S(\mathbf{K}_1\cdots\mathbf{k}_p)=G_N{}^nG_\pi{}^p\left(\frac{4\pi}{3}R^3\right)^{-(n+p)/2} \times\frac{(4\pi R^3)^{n+p}}{V^{(n+p)/2}}\frac{j_1(K_1R)}{K_1R}\cdots\frac{j_1(k_pR)}{k_pR},$$

and

$$P_{np}=G_N{}^{2n}G_\pi{}^{2p}\sum_{\mathbf{K},\mathbf{k}}\delta(E_0-E_1-\cdots-\omega_p) \times\frac{(4\pi R^3)^{n+p}3^{n+p}}{V^{n+p}}\frac{j_1{}^2(K_1R)}{(K_1R)^2}\cdots\frac{j_1{}^2(k_pR)}{(k_pR)^2}, \quad (16)$$

with the condition that the momentum values must be such that momentum is conserved.

Now let us impose the condition that the kinetic energy available to the particles is small so that $K_iR\ll1$ and $k_iR\ll1$ for all i. Using the relation,

$$j_1(\beta R)\xrightarrow{\beta R\to0}\frac{\beta R}{3},$$

we have

$$P_{np}\xrightarrow{KR\to0}G_N{}^{2n}G_\pi{}^{2p}\sum_{\mathbf{K},\mathbf{k}}\delta(E_0-E_1-\cdots-\omega_p)\frac{(\frac{4}{3}\pi R^3)^{n+p}}{V^{n+p}},$$

and the matrix element is independent of the momenta. Passing to a continuous spectrum and setting $G_N=G_\pi=1$, we obtain Fermi's result,

$$P_{np}=\left(\frac{\frac{4}{3}\pi R^3}{(2\pi)^3}\right)^{n+p}\int d^3K_1\cdots d^3k_p\,\delta(E_0-E_1-\cdots-\omega_p),$$

with the additional constraint that momentum be conserved. Of course, one could certainly take the point of view that in the Fermi model only the product $\frac{4}{3}\pi G^2R^3$ is important. Thus if $kR\ll1$, then a two-parameter model, the interaction model, reduces to a one-parameter model, the Fermi model.

If we had made the reduction in terms of angular momenta, the result would not have been as clear, but since we take $kR\ll1$, it follows that only S waves will contribute, and thus the matrix element is independent of the momenta.

As pointed out in the introduction, in the case of $N-\bar{N}$ annihilation, the Fermi model yields a surprisingly large value for the interaction radius to give the correct pion multiplicity. The above remarks show that this is not surprising, for in $N-\bar{N}$ annihilation, if we chose $R\sim1/\mu$, we would have $kR\sim1$ to 7, and the restriction $kR\ll1$ would certainly not be satisfied. However, it is still possible to use a very crude approximation in the case of plane waves to arrive at a rough idea of the interaction radius. We notice from Eq. (16) that we are summing (or integrating) over a product of

similar functions. One should expect that the main contribution to the sum should occur when these functions are all near their maximum values, i.e., $kR\sim 2$. Thus we consider the average value of k in a given process, $\bar{k}$, and estimate R. In $N-\bar{N}$ annihilations, we can write

$$\bar{\omega}=(\bar{k}^2+\mu^2)^{\frac{1}{2}}=E_0/M,$$

where M is the experimental multiplicity. Considering the relevant quantities, this gives $R\sim 0.7/\mu$. This number does not include the effects of momentum conservation or of the wave-function symmetrization, and a somewhat better estimate leads to $R\sim 0.9/\mu$.

III. APPLICATION TO NUCLEON-ANTINUCLEON ANNIHILATION

A. Analysis in Terms of Angular Momenta

In this section we will treat the problem of multiple meson production in $N-\bar{N}$ annihilations. In order to accomplish this, it is convenient to use as our complete set of states those which are eigenfunctions of the energy and angular momenta. Ball and Chew[9] have shown that only certain partial waves in the $N-\bar{N}$ system annihilate to produce mesons, thus we wish to impose the condition that the meson system form only these angular momentum states. This is most easily done by using angular-momentum eigenfunctions.

We are interested in the scattered-wave amplitudes for a system of p pions (we shall consider K-particle production later), and from Eq. (14) we have

$$S(\eta_1\cdots\eta_p)=G_\pi{}^p\int d^3\xi_1\cdots d^3\xi_p\,\phi_{\eta(1)}{}^*(\xi_1)\cdots\times\phi_{\eta(p)}{}^*(\xi_p)\frac{\Psi(\xi_1\cdots\xi_p)}{(2\omega_1R\cdots 2\omega_pR)^{\frac{1}{2}}}.$$

Since we wish to use a complete set of angular momentum states, we take

$$\phi_{\eta(i)}(\xi_i)=\left(\frac{2k_i^2}{D}\right)^{\frac{1}{2}} j_{l(i)}(k_i\xi_i)Y_{l(i)}{}^{m(i)}(\theta_i,\phi_i)\chi_1{}^{\beta(i)}(i),$$

where $\chi_1{}^{\beta(i)}(i)$ is an isotopic-spin wave function, and D is the radius of the normalization volume. Thus we write

$$\begin{aligned}S(\eta_1\cdots\eta_p)&=G_\pi{}^p\left(\frac{2k_1{}^2\cdots 2k_p{}^2}{D^p}\right)^{\frac{1}{2}}\int d^3\xi_1\cdots d^3\xi_p\prod_i j_{l(i)}(k_i\xi_i)\\&\quad\times Y_{l(i)}{}^{m(i)*}(\theta_i,\phi_i)\chi_1{}^{\beta(i)*}(i)\frac{\Psi(\xi_i\cdots\xi_p)}{(2\omega_1R\cdots 2\omega_pR)^{\frac{1}{2}}} \qquad (17)\\&=\left(\frac{2k_1{}^2\cdots 2k_p{}^2}{D^p}\right)^{\frac{1}{2}} S'(\eta_1\cdots\eta_p). \qquad (18)\end{aligned}$$

For the probability we have, from Eqs. (15) and (18),

$$P_p=\sum_{\eta_1\cdots\eta_p}\delta(E_0-\omega_1-\cdots-\omega_p)|S(\eta_1\cdots\eta_p)|^2,$$

which becomes, in the limit $D\rightarrow\infty$,

$$P_p=\left(\frac{2}{\pi}\right)^p\sum_l\sum_m\sum_\beta\int k_1{}^2dk_1\cdots k_p{}^2dk_p\times\delta(E_0-\omega_1-\cdots-\omega_p)|S'(\eta_1\cdots\eta_p)|^2. \qquad (19)$$

Thus, in order to determine the probability for a state of p pions, it is necessary to determine the $S'(\eta_1\cdots\eta_p)$. However, before proceeding to that calculation, we must first recognize that the pions are bosons and symmetrize the wave functions. We accomplish this approximately in the usual way by just including the normalization factor. Using Eqs. (17) and (18), we find

$$\bar{S}(\eta_1\cdots\eta_p)=G_\pi{}^p\left(\frac{\prod_\eta(N_\eta!)}{p!}\right)^{\frac{1}{2}}\int d^3\xi_1\cdots d^3\xi_p \prod_i j_{l(i)}(k_i\xi_i)Y_{l(i)}{}^{m(i)*}(\theta_i,\phi_i)\chi_1{}^{\beta(i)*}(i)\times\frac{\Psi(\xi_i\cdots\xi_p)}{(2\omega_1R\cdots 2\omega_pR)}, \qquad (20)$$

where N_η is the number of times a particular state, $(klm\beta)$, occurs. This approximation has the effect of neglecting possible cross terms in $|S'|^2$, but again since we will always work with ratios we may assume that this effect is small.

Now consider $\Psi(\xi_1\cdots\xi_p)$. From Eq. (11) we have

$$\Psi(\xi_1\cdots\xi_p)=C\theta(R-\xi_1)\cdots\theta(R-\xi_p)\times Y(\xi_1\cdots\xi_p)I(\pi), \qquad (11')$$

where $Y(\xi_1\cdots\xi_p)$ gives the angular dependence and is thus a superposition of angular momentum states. Thus, let us write,

$$Y(\xi_1\cdots\xi_p)=\sum_{J,M}a_J{}^MY_J{}^M(\xi_1\cdots\xi_p). \qquad (21)$$

If we wish, we may also write

$$Y_J{}^M(\xi_1\cdots\xi_p)=\sum_{j(1)\cdots j(p)}{}^J A_J{}^M{}_{j(1)\cdots j(p)}\times Y_J{}^M{}_{j(1)\cdots j(p)}(\xi_1\cdots\xi_p), \qquad (22)$$

where $Y_J{}^M{}_{j(1)\cdots j(p)}(\xi_1\cdots\xi_p)$ is an eigenfunction of J, M, $j(1)$, $\cdots$, $j(p)$ and $\sum^J$ signifies sums over possible angular momenta leading to an angular momentum J. We can form $Y_J{}^M{}_{j(1)\cdots j(p)}$ in the usual way as

$$Y_J{}^M{}_{j(1)\cdots j(p)}(\xi_1\cdots\xi_p)=\sum_{\mu(1)\cdots\mu(p)}\langle J,M;j(1)\cdots j(p)\times|j(1)\cdots j(p),\mu(1)\cdots\mu(p)\rangle\times Y_{j(1)}{}^{\mu(1)}(\xi_1)\cdots Y_{j(p)}{}^{\mu(p)}(\xi_p). \qquad (23)$$

Equations (21), (22), and (23) allow us to express the amount of various partial waves in the $N-\bar{N}$ system contributing to the annihilation, the possible values of the pion angular momenta, and their m-value contributions. We write the isotopic spin part in an analogous way and obtain, after the angular and isotopic-spin integration,

$$S'(\eta_1\cdots\eta_p)=C\left(\frac{\prod_\eta(N_\eta!)}{p!}\right)^{\frac{1}{2}}F_{l(1)\cdots l(p)}(k_1\cdots k_p)$$
$$\times(\sum_{J,M}a_J{}^MA_J{}^M{}_{l(1)\cdots l(p)}$$
$$\times\langle J,M;l_1\cdots l_p|l_1\cdots l_p;m_1\cdots m_p\rangle)$$
$$\times(\sum_{I,I(z)}b_I{}^{I(z)}B_I{}^{I(z)}{}_{1,\cdots,1}$$
$$\times\langle I,I_z;1\cdots1|1\cdots1;\beta_1\cdots\beta_p\rangle),\quad(24)$$

where we have set

$$F_{l(1)\cdots l(p)}(k_1\cdots k_p)=\prod_i\left(\frac{G_\pi}{(2\omega_iR)^{\frac{1}{2}}}\int\xi_i{}^2d\xi_i\,j_{l(i)}(k_i\xi_i)\theta(R-\xi_i)\right).\quad(25)$$

Thus, from Eqs. (19) and (24), we find

$$P_p=C^2\left(\frac{2}{\pi}\right)^p\int k_1{}^2dk_1\cdots k_p{}^2dk_p\,\delta(E_0-\omega_1-\cdots-\omega_p)$$
$$\times\sum_{J,M}\sum_{I,I(z)}\sum_{l(1)\cdots l(p)}{}^J|a_J{}^M|^2$$
$$\times|A_J{}^M{}_{l(1)\cdots l(p)}|^2|b_I{}^{I(z)}|^2|B_I{}^{I(z)}{}_{1,\cdots,1}|^2$$
$$\times\frac{\prod_\eta(N_\eta!)}{p!}|F_{l(1)\cdots l(p)}(k_1\cdots k_p)|^2,\quad(26)$$

where we have used the relation,

$$\sum_{m(1)\cdots m(p)}\langle J,M;l_1\cdots l_p|l_1\cdots l_pm_1\cdots m_p\rangle\times\langle J',M';l_1\cdots l_p|l_1\cdots l_pm_1\cdots m_p\rangle=\delta_{JJ'}\delta_{MM'}.$$

At large distances we take the $N-\bar{N}$ system to be a plane wave; thus only $M=0$ values will contribute. Further, since we consider an averaging over the $N-\bar{N}$ states, we consider the A and B terms to be independent of the various variables and set them equal to one. Finally, we impose the results of the calculations of Ball and Chew and take a_J and b_I to be one or zero corresponding to whether a given $N-\bar{N}$ state, (J,I), will or will not annihilate. The initial charge state, of course, determines I_z. Because of the normalization of the angular functions, we take $C^2=(R^3/3)^{-p}$, and thus from Eq. (26) we have

$$P_p=(6/\pi R^3)^p\int k_1{}^2dk_1\cdots k_p{}^2dk_p\,\delta(E_0-\omega_1-\cdots-\omega_p)$$
$$\times\sum_{J,I}{}'\sum_{l(1)\cdots l(p)}{}^J\frac{\prod_\eta(N_\eta!)}{p!}|F_{l(1)\cdots l(p)}(k_1\cdots k_p)|^2,\quad(27)$$

where $\sum'_{J,I}$ means the sum over the states (J,I) leading to annihilations.

This can be cast into a more convenient form by considering summations of the $l(1)\cdots l(p)$ on $F_{l(1)\cdots l(p)}(k_1\cdots k_p)$ in Eq. (27). Since the integrals are independent (except for the energy delta function) and we integrate over all the momenta, we see that F is invariant under such permutations. Thus for a particular selection of the $l_1,\cdots,l_p$, which we shall call the set $\{l_1\cdots l_p\}$, F gives the same contribution. Since there are $p!/P_\eta(N_\eta!)$ possible permutations, we have

$$\sum_{l(1)\cdots l(p)}{}^J|F_{l(1)\cdots l(p)}(k_1\cdots k_p)|^2=\sum_{\text{sets}}{}^J\frac{p!}{\prod_\eta(N_\eta!)}|F_{\{l(1)\cdots l(p)\}}(k_1\cdots k_p)|^2,$$

and thus arrive at

$$P_p=(6/\pi R^3)^p\int k_1{}^2dk_1\cdots k_p{}^2dk_p\,\delta(E_0-\omega_1-\cdots-\omega_p)$$
$$\times\sum_{J,I}{}'\sum_{\text{sets}}{}^J|F_{\{l(1)\cdots l(p)\}}(k_1\cdots k_p)|^2.\quad(28)$$

B. Calculation of the Probabilities

In order to calculate the probabilities, it is convenient to investigate the properties of the quantities $F_{\{l(1)\cdots l(p)\}}(k_1\cdots k_p)$. From the previous section we know

$$F_{\{l(1)\cdots l(p)\}}=\frac{G_\pi{}^p}{(2\omega_1R\cdots2\omega_pR)^{\frac{1}{2}}}\int_0^\infty\xi_1{}^2d\xi_1\cdots\times\int_0^\infty\xi_p{}^2d\xi_p\,j_{l(1)}(k_1\xi_1)\cdots\times j_{l(p)}(k_p\xi_p)\theta(R-\xi_1)\cdots\theta(R-\xi_p),\quad(25')$$

and thus we have products of integrals of the form

$$\int_0^\infty\xi_i{}^2d\xi_i\,j_{l(i)}(k_i\xi_i)\theta(R-\xi_i)=\frac{1}{k_i{}^3}\int_0^{k_iR}p_i{}^2j_{l(i)}(p_i)dp_i=\frac{K_{l(i)}(k_iR)}{k_i{}^3}.\quad(29)$$

The quantities $K_{l(i)}(k_iR)$ have been plotted in Fig. 1

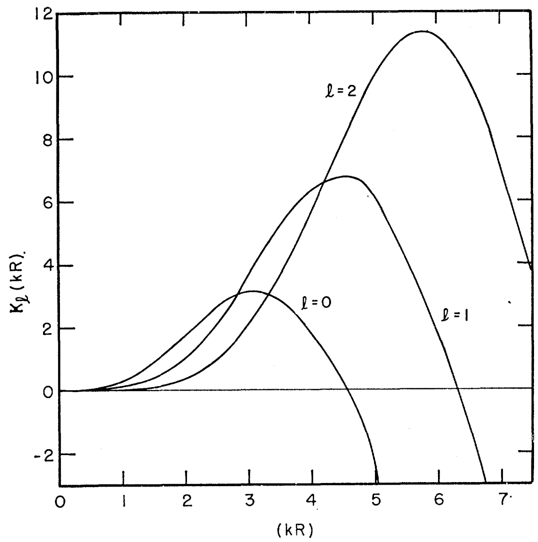

FIG. 1. $K_l(kR)$ vs kR for $l=0$, 1, and 2.

for $l=0$, 1, 2. From Eqs. (28) and (29), we can write

$$P_p=(6/\pi)^p R G_\pi^{2p}\int d(k_1R)\cdots\times\int d(k_pR)\delta(E_0R-\omega_1R-\cdots-\omega_pR)\times\sum_{J,I}{}'\sum_{\text{sets}}^{J}\frac{K_{l(1)}{}^2(k_1R)}{2\omega_1R(k_1R)^4}\cdots\frac{K_{l(p)}{}^2(k_pR)}{2\omega_pR(k_pR)^4}.$$

This becomes,[16] after introduction of the variable $z_i^2=k_i^2R^2+\mu^2R^2$,

$$P_p=(6/\pi)^pG_\pi^{2p}\sum_{J,I}{}'\sum_{\text{sets}}^{J}\int_{\mu R}^{\infty}dz_1\cdots\times\int_{\mu R}^{\infty}dz_p\,\delta(E_0R-z_1-\cdots-z_p)\times A_{l(1)}(z_1;\mu R)\cdots A_{l(p)}(z_p;\mu R),\quad(30)$$

where

$$A_{l(i)}(z_i;\mu R)=\frac{K_{l(i)}{}^2[(z_i^2-\mu^2R^2)^{\frac{1}{2}}]}{2[z_i^2-\mu^2R^2]^{\frac{5}{2}}}.\quad(31)$$

In Figs. 2(a) and 2(b) we have plotted $A_l(z;\mu R)$ for $R=1/\mu$ and $R=1/2\mu$, respectively, for various values of l.

Although it is possible to perform the integrations in Eq. (30) explicitly, it is more convenient to approximate the functions $A_{l(i)}(z_i;\mu R)$ by Gaussians. Thus, let us take

$$A_l(z;\mu R)=H_l\exp\{-[(z-z_l')/a_l]^2\},\quad(32)$$

where $H_l(\mu R)$, $z_l'(\mu R)$, and $a_l(\mu R)$ are constants independent of z, and we impose the condition

$$\int_{\mu R}^{\infty}dz\,H_l\exp\left\{-\left[\frac{z-z_l'}{a_l}\right]^2\right\}=\int_{\mu R}^{\infty}A_l(z;\mu R)dz=A_l(\mu R).\quad(33)$$

However, we notice

$$\int_{\mu R}^{\infty}dz\,H_l\exp[-(z-z_l')^2/a_l^2]\simeq\int_{-\infty}^{\infty}dz\,H_l\exp(-z^2/a_l^2)=\pi^{\frac{1}{2}}H_la_l,$$

and thus, from Eq. (33), we obtain

$$a_l(\mu R)=\frac{1}{\sqrt{\pi}}\frac{A_l(\mu R)}{H_l(\mu R)},$$

where $A_l(\mu R)$ is determined by measuring the area under the curves in Fig. 2 with a planimeter, and we take

$$H_l(\mu R)=\max\{A_l(z;\mu R)\},$$

which occurs for $z=z_l'$. Thus $H_l(\mu R)$, $z_l'(\mu R)$, and $a_l(\mu R)$ are determined, and $A_l(z;\mu R)$ is represented by Gaussian forms. To show that this approximation is not a bad one, we have plotted the appropriate Gaussian for $l=0$ and $l=3$ in Fig. 2(a) as dashed curves.

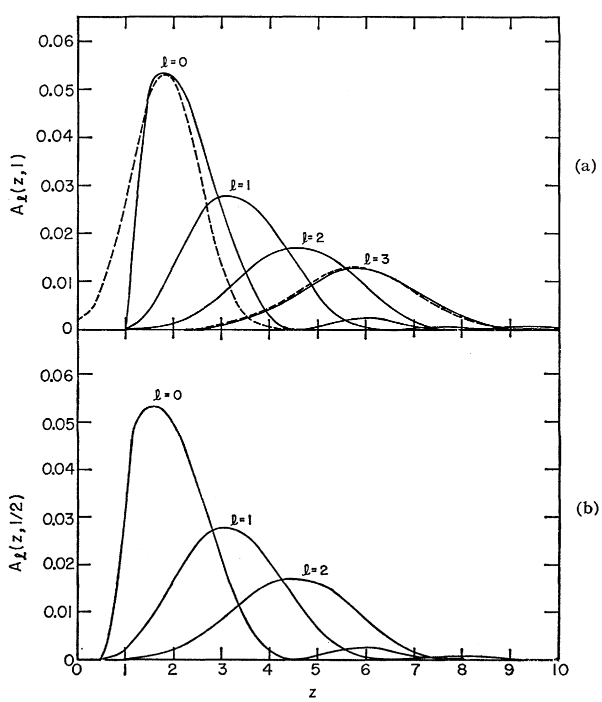

FIG. 2. (a) $A_l(z,1)$ vs z for $l=0$, 1, 2, and 3 (solid curve) with comparison to Gaussian approximation for $l=0$ and 3 (dashed curve). (b) $A_l(z,\frac{1}{2})$ vs z for $l=0$, 1, and 2.

[16] Again, since we will work with ratios, we have dropped a factor of R.

From Eqs. (30) and (32) we thus write

$$P_p=(6/\pi)^p(G_\pi^2)^p\sum_{J,I}{}'\sum_{\text{sets}}{}^J I(E_0R,\mu R;l_1\cdots l_p),\qquad(34)$$

where

$$I(E_0R,\,\mu R;\,l_l\cdots l_p)$$
$$=\int_{\mu R}^{\infty}dz_1\cdots\int_{\mu R}^{\infty}dz_p\,\delta(E_0R-z_1-\cdots-z_p)\times\prod_{i=1}^{p}H_{l(i)}\exp\left[-\left\{\frac{(z_i-z_i')}{a_i}\right\}^2\right].$$

This integral can be well approximated (see Appendix) and gives

$$I(E_0R,\mu R;l_1\cdots l_p)=\frac{A_{l(1)}(\mu R)\cdots A_{l(p)}(\mu R)}{\sqrt{\pi}}\times\exp\left[-\left\{\frac{RE_0-z_1'-\cdots-z_p'}{a_1^2+\cdots+a_p^2}\right\}^2\right]\Big/(a_1^2+\cdots+a_p^2)^{\frac{1}{2}}.$$

Table I lists the appropriate quantities for $R=1/\mu$ and $1/2\mu$.

The value of $I(E_0R,\mu R;l_1\cdots l_p)$ was calculated on the IBM-650 computer for various values of l for $R=1/(3\mu)$, $1/(2\mu)$, $3/(4\mu)$, and $1/\mu$. Further, these calculations were done for an incident antiproton with a laboratory kinetic energy of 200 Mev, and all following results are for this energy.

To arrive at the probabilities, we must impose the various conservation laws. For a given J and I of the system, only certain sets of l values are allowed, these l values being determined by the selection rules. The selection rules have been discussed by many authors[17,18] and are given in Table II. Further, as pointed out earlier, Ball and Chew have shown that not all combinations and values of J and I will contribute to the annihilation. In Table II only those states with a transmission coefficient of one will contribute. Using the results from the IBM 650 and the selection rules as modified by Ball and Chew, we can now calculate the probabilities. How the states are selected will depend on the initial $N-\bar{N}$ state. The following results are for the case of $p-\bar{p}$ annihilations. The probabilities are given in Table III for the cases when $R=1/(2\mu)$, $3/(4\mu)$, and $1/\mu$, where G_π is adjusted to give a multiplicity of 4.90.

At this point no mention of momentum conservation has been made, but the quantities in Table III include it in an approximate way. Fermi has calculated the statistical weight for n outgoing particles without momentum conservation, and Lepore and Stuart have calculated the same quantity with momentum conservation.[19] Call these $S_{n\mathrm{F}}$ and $S_{n\mathrm{LS}}$, respectively. By including momentum conservation, the possible final states are restricted, and this, of course, leads to a reduction of the statistical weight. Although the statistical weights have little meaning in themselves, their ratio, $S_{n\mathrm{LS}}/S_{n\mathrm{F}}$, should give the fractional reduc-

TABLE I. Parameters for the Gaussian approximation.

l	$A_l(1)$	$R=1/\mu$ $H_l(1)$	$a_l(1)$	$z_l'(1)$
0	0.0939	0.0533	0.992	1.82
1	0.0706	0.0277	1.438	3.10
2	0.0520	0.0170	1.725	4.62
3	0.0403	0.0130	1.750	5.80
4	0.0320	0.0090	2.007	7.07
5	0.0258	0.0063	2.310	8.20
6	0.0225	0.0046	2.753	9.40
7	0.0210	0.0037	3.205	10.50
8	0.0205	0.0035	3.302	11.70
9	0.0205	0.0035	3.302	12.90
l	$A_l(1/2)$	$R=1/2\mu$ $H_l(1/2)$	$a_l(1/2)$	$z_l'(1/2)$
0	0.1030	0.0533	1.085	1.60
1	0.0746	0.0277	1.513	3.05
2	0.0550	0.0170	1.812	4.60
3	0.0420	0.0130	1.901	5.80
4	0.0325	0.0090	2.029	7.07
5	0.0258	0.0063	2.310	8.20
6	0.0225	0.0046	2.753	9.40
7	0.0210	0.0037	3.205	10.50
8	0.0205	0.0035	3.302	11.70
9	0.0205	0.0035	3.302	12.90

TABLE II. Table of selection rules.[a]

n \ J		0	1		2		3
	S	${}_1{}^1S_0{}^1$	${}_3{}^3S_1{}^1$				
even[b]	P	${}_1{}^3P_1{}^0$	${}_1{}^3P_1{}^0$	${}_3{}^1P_1{}^1$	${}_1{}^3P_2{}^1$		
	D		${}_3{}^3D_1{}^0$		${}_3{}^3D_2{}^0$	${}_1{}^1D_2{}^0$	${}_3{}^3D_3{}^0$
odd[c]	S	${}_3{}^1S_0{}^1$	${}_1{}^3S_1{}^1$				
	P	${}_3{}^3P_0{}^0$	${}_1{}^1P_1{}^0$	${}_3{}^3P_1{}^1$	${}_3{}^3P_2{}^1$		
	D		${}_1{}^3D_1{}^0$		${}_1{}^3D_2{}^0$	${}_3{}^1D_2{}^0$	${}_1{}^3D_3{}^1$

[a] The notation employed for ${}_I{}^SL_J{}^T$ is as follows: S is the spin state; I, the isotopic spin state; L, the orbital angular momentum; T, the transmission coefficient; J, the total angular momentum.
[b] For $n=2$, the diagonal terms are removed.
[c] For $n=3$, ${}_3{}^3P_0$ does not contribute.

TABLE III. Probabilities for finding a certain number, p, of pions.

p	P_p, $R=1/(2\mu)$	P_p, $R=3/(4\mu)$	P_p, $R=1/\mu$
2	0.024	0.016	0.014
3	0.151	0.221	0.236
4	0.260	0.213	0.190
5	0.220	0.208	0.235
6	0.174	0.179	0.144
7	0.091	0.100	0.104
8	0.046	0.046	0.055
9	0.018	0.014	0.017
10	0.008	0.004	0.005

[17] Charles Goebel, Phys. Rev. **103**, 258 (1956).
[18] T. D. Lee and C. N. Yang, Nuovo cimento **3**, 749 (1956).
[19] J. Lepore and R. N. Stuart, Phys. Rev. **94**, 1724 (1954); see also Richard H. Milburn, Revs. Modern Phys. **27**, 1 (1955).

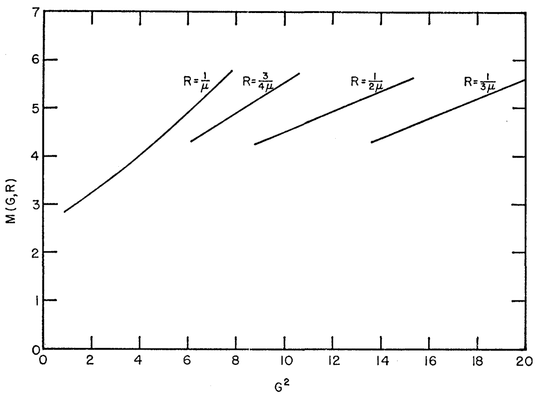

FIG. 3. Multiplicity vs G^2 for constant R values.

tion of the statistical weight in an absolute sense. Since the probabilities are proportional to the statistical weights, this ratio should give a good approximate evaluation of the reduction of the probability ratios. Even though the Fermi model is not as general as the present model, the fractional reduction produced should be compatible with the spirit of the present calculation.

In order to calculate this fractional reduction, it is convenient to use the explicit formulas of Lepore and Stuart which apply only to extremely relativistic particles. Of course, in the case of annihilation, all of the pions are not relativistic, but on the other hand, the effect is most pronounced for small numbers of pions, in which case the particles are at least relativistic. Further for larger multiplicities, the probabilities are small and do not greatly influence the value of the multiplicity. The explicit formulas are

$$S_{n\mathrm{F}}=\frac{\Omega^{n-1}E^{3n-1}}{\pi^{2n}(3n-1)!},$$

and

$$S_{n\mathrm{LS}}=\frac{\Omega^{n-1}(4n-4)!E^{3n-4}}{\pi^{2n-2}2^{4n-4}(2n-1)!(2n-2)!(3n-4)!}.$$

According to the previous remarks, we write,

$$P_n{}^w=P_n{}^{wo}\frac{S_{n\mathrm{LS}}}{S_{n\mathrm{F}}},$$

where $P_n{}^w$ and $P_n{}^{wo}$ represent the probability for finding n particles with and without including momentum conservation, respectively. It is convenient to work with

$$\frac{P_n{}^w}{P_2{}^w}=R_2{}^n\frac{P_n{}^{wo}}{P_2{}^{wo}},$$

where

$$R_2{}^n=\frac{S_{n\mathrm{LS}}}{S_{n\mathrm{F}}}\frac{S_{2\mathrm{F}}}{S_{2\mathrm{LS}}}.$$

We find

$$R_2{}^2=1,\qquad R_2{}^7=11.31,$$
$$R_2{}^3=2.45,\qquad R_2{}^8=14.09,$$
$$R_2{}^4=4.26,\qquad R_2{}^9=17.14,$$
$$R_2{}^5=6.19,\qquad R_2{}^{10}=20.28.$$
$$R_2{}^6=8.72,$$

C. Multiplicity

As seen in Sec. IIIB, the probabilities depend on two parameters, R and G, which in Table III have been adjusted to give a multiplicity of 4.9. In order for the model to have any physical meaning, it should satisfy at least two sets of data. In the case of $p-\bar{p}$ annihilation, these two sets are taken to be the pion multiplicity and energy spectra.

The total pion multiplicity, M, is given as

$$M(G,R)=\sum_{n=2}^{N}n\frac{P_n}{P_2}\Big/\sum_{n=2}^{N}\frac{P_n}{P_2},$$

where N is the maximum number of pions that can be produced compatible with the energy, E_0. In Fig. 3, the results of the calculations are given by plotting the multiplicity against G^2 for curves of constant R.

Since only charged annihilation events are measured experimentally, it is of interest to give probabilities for various numbers of prongs and for the charged, $M^{\pm}$, and uncharged, M^0, multiplicity as well as the total multiplicity. In Table IV we give the results for the case when $M=4.9$ with $R=1/\mu$ and $1/(2\mu)$, where G is adjusted appropriately. Table IV A gives the probability of finding a given number of prongs resulting from an annihilation with a certain total number of final particles, n. Table IV B gives the same probability weighted to give the probability of finding a certain total number of final particles. At the bottom of Table IV B, we find the probabilities for finding a particular number of prongs. The probabilities given for various numbers of prongs are in agreement with the present experimental results, although the two-prong annihilation events seem to be somewhat high. The fractions of charged particles resulting from a particular mode are given in Table IV C. These fractions lead to the final result,

for $R=1/(2\mu)$,

$$M^{\pm}=3.24,\quad M^0=1.66,$$

and for $R=1/\mu$,

$$M^{\pm}=3.21,\quad M^0=1.69.$$

This is in agreement with the present experimental results. Given the total number of charged particles resulting from a group of annihilations, Table IV D gives the fraction contributed by annihilations resulting in n final particles. These are to be compared with the

TABLE IV. Probabilities for various prong multiplicities.

Table IV A.[a] Probabilities for given numbers of prongs for a total of n final particles.

n	0	2	4	6	8	10
2	0.167	0.833	0	0	0	0
3	0.100	0.900	0	0	0	0
4	0.033	0.567	0.400	0	0	0
5	0.014	0.338	0.648	0	0	0
6	0.005	0.176	0.635	0.184	0	0
7	0.002	0.086	0.503	0.410	0	0
8	0	0.10	0.10	0.700	0.10	0
9	0	0	0.05	0.500	0.450	0
10	0	0	0	0.20	0.60	0.20

Table IV B. Probability for given numbers of prongs for a total of n particles, weighted according to the probabilities of n.

	$R=1/(2\mu)$						$R=1/\mu$					
n	0	2	4	6	8	10	0	2	4	6	8	10
2	0.004	0.020	0	0	0	0	0.002	0.012	0	0	0	0
3	0.015	0.136	0	0	0	0	0.024	0.212	0	0	0	0
4	0.009	0.147	0.104	0	0	0	0.006	0.108	0.076	0	0	0
5	0.003	0.074	0.143	0	0	0	0.003	0.079	0.152	0	0	0
6	0.001	0.031	0.110	0.032	0	0	0.001	0.025	0.091	0.026	0	0
7	0	0.008	0.046	0.037	0	0	0	0.009	0.052	0.043	0	0
8	0	0.005	0.005	0.032	0.005	0	0	0.006	0.006	0.037	0.006	0
9	0	0	0.001	0.009	0.008	0	0	0	0.001	0.009	0.007	0
10	0	0	0	0.001	0.005	0.001	0	0	0	0.001	0.003	0.001
Total	0.032	0.421	0.409	0.111	0.018	0.001	0.036	0.451	0.378	0.116	0.016	0.001

Table IV C. Fractions of charged particles from particular modes.

	$R=1/(2\mu)$						$R=1/\mu$					
n	0	2	4	6	8	10	0	2	4	6	8	10
2	0	0.020	0	0	0	0	0	0.012	0	0	0	0
3	0	0.091	0	0	0	0	0	0.141	0	0	0	0
4	0	0.074	0.104	0	0	0	0	0.054	0.076	0	0	0
5	0	0.030	0.115	0	0	0	0	0.032	0.122	0	0	0
6	0	0.010	0.073	0.032	0	0	0	0.008	0.060	0.026	0	0
7	0	0.002	0.026	0.032	0	0	0	0.003	0.030	0.037	0	0
8	0	0.001	0.003	0.024	0.005	0	0	0.002	0.003	0.028	0	0
9	0	0	0	0.006	0.008	0	0	0	0	0.006	0.007	0
10	0	0	0	0.001	0.004	0.001	0	0	0	0	0	0
Total	0	0.228	0.321	0.095	0.017	0.001	0	0.252	0.291	0.097	0.015	0

Table IV D. Fractions of charged particles contributed by modes with given values n of total numbers of final particles.

n	$R=1/(2\mu)$	$R=1/\mu$
2	0.030	0.018
3	0.138	0.216
4	0.269	0.198
5	0.219	0.236
6	0.174	0.144
7	0.091	0.107
8	0.050	0.060
9	0.021	0.020
10	0.009	0

[a] I am indebted to B. Desai, Lawrence Radiation Laboratory, University of California, Berkeley, for supplying these numbers. Those for $n=8$, 9, and 10 are estimates.

probabilities given in Sec. IIIB, where a small difference is observed.

D. Energy Spectra

We have seen in Sec. IIIC that it is possible to fit the experimental data for the pion multiplicity by an appropriate choice of both G and R. In this section we shall show that the data available on the energy spectra can also be described, and leads to definite values of G and R.

The energy distribution of the pions is

$$P_p(E_0R,\mu R;z_1)dz_1=(6/\pi)^p G_\pi^{2p} \sum_{J,I}{}' \sum_{l(1)\cdots l(p)} \frac{p!}{\prod_\eta N_\eta!} \times A_{l(1)}(z_1;\mu R) I(E_0R-z_1,\mu R;l_2\cdots l_p)dz_1. \quad (35)$$

We have seen from Fig. 2 that $A_{l(1)}(z_1;\mu R)$ is effectively zero except in a given region; thus as an approximation we take

$$I(E_0R-z_1,\, \mu R;\, l_2\cdots l_p) \simeq I(E_0R-z_1',\, \mu R;\, l_2\cdots l_p).$$

This allows us to calculate these quantities on the IBM-650 computer in exactly the same way as was done for the actual probabilities. In this case we obtain the energy spectra as a superposition of the energy dependences of the various angular momentum functions. The coefficients are determined primarily by the values of the $I(E_0R-z_1',\mu R;l_2\cdots l_p)$ and by the fact that we

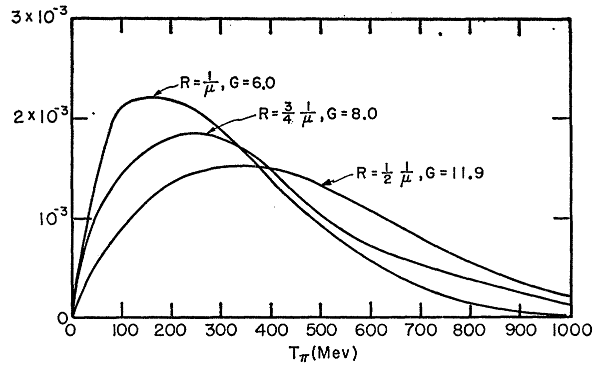

FIG. 4. Energy spectra as functions of the barycentric pion kinetic energy for $R=1/\mu$, $3/(4\mu)$, $1/(2\mu)$, with G values chosen to give a multiplicity of 4.9.

have

$$P_p(E_0R;\mu R)=\int_{\mu R}^{\infty} P_p(E_0R,\mu R;z_1)dz_1.$$

The resulting kinetic-energy distributions, $P(E_0R, \mu R;\omega)$, are plotted in Fig. 4 for $R=1/(2\mu)$, $3/(4\mu)$, and $1/\mu$. The calculations become quite tedious for $R>3/(4\mu)$ and the curve for $R=1/\mu$ is estimated by assuming that the increase in higher angular momentum states is linear. Although this is not correct, the curve for $R=1/\mu$ should not be too different from that calculated explicitly. For the total spectra, the actual contributions are

$$P(E_0/2\mu,\tfrac{1}{2};\omega)d\omega =\frac{1}{2\mu}[7.59A_0(\omega;\tfrac{1}{2})+2.29A_1(\omega;\tfrac{1}{2}) +0.27A_2(\omega;\tfrac{1}{2})+0.01A_3(\omega;\tfrac{1}{2})]d\omega,$$

$$P(3E_0/4\mu,\tfrac{3}{4};\omega)d\omega =\frac{3}{4\mu}[5.58A_0(\omega;\tfrac{3}{4})+4.16A_1(\omega;\tfrac{3}{4})+2.14A_2(\omega;\tfrac{3}{4}) +0.69A_3(\omega;\tfrac{3}{4})+0.11A_4(\omega;\tfrac{3}{4})]d\omega,$$

and

$$P(E_0/\mu,1;\omega)d\omega=\frac{1}{\mu}[4.41A_0(\omega;1)+5.23A_1(\omega;1) +3.22A_2(\omega;1)+1.09A_3(\omega;1)+0.17A_4(\omega;1)]d\omega,$$

which are normalized as $\int P(E_0R,\mu R;\omega)d\omega=1$.

These distributions are to be compared with the recent experimental data of Horwitz *et al.*,[20] Agnew *et al.*,[21] and Chamberlain *et al.*[5] in Fig. 5. Although the experimental data have a range in incident energy of from 0 to 500 Mev, the fractional change in the total energy in the barycentric system is small, and it is reasonable to compare our calculations with their data. It should be pointed out that the data presented are the charged-pion spectra, while the curves in Fig. 4 are for both charged and uncharged pions. However, as mentioned in Sec. IIIC, the probabilities for charged and uncharged pions do not differ appreciably, and thus, within the framework of the present model, we can expect their spectra to be essentially the same. The data of Chamberlain *et al.* include only 4- and 6-prong events, but the effect of the 2-prong events should be small, just raising slightly the intermediate section of the histogram (300 to 600 Mev). The comparison shows that the results of the choice $R=1/\mu$ give a good representation of the data, particularly for those of Chamberlain *et al.*, where the statistics are best.

It is interesting to note, however, that the characteristic feature of the lower energy data is found in the height of the distribution at the most probable kinetic energy 175 Mev. If this has statistical significance, it can possibly be related to the selection rules in the angular momentum states. It has been pointed out that for annihilations at rest the predominant angular momentum states contributing are S or P.[22] Thus we should expect a relative increase in S- and P-state pions which will have the effect in the theoretical curves of raising the maximum while reducing the intermediate and tail regions.

IV. CONCLUSIONS

As pointed out in the introduction, the nucleon-antinucleon interaction shows two primary effects, viz., an "anomalously" large total cross section and pion multiplicity. However, the data on the total and elastic cross sections can be understood, as shown by the work of Ball and Chew,[9] in much the same way as the nucleon-nucleon interaction in the energy range where the use of the WKB approximation is justified. The present paper shows that it is also possible to understand the data available concerning the pion multiplicity as well as the data concerning the energy spectra of the emitted pions, while still maintaining that the interaction takes place in a volume characterized by a radius of one pion Compton wavelength. This is accomplished by making two physical assumptions. The first of these is that only those partial waves with a transmission coefficient of one, as obtained from the results of the calculations of Ball and Chew, will contribute to the production of pions. This has the effect of restricting the number of possible initial states as seen in Table II. The other consists of including, in an approximate way, the momentum dependence of the matrix elements involved in the annihilation reaction. This momentum dependence strongly influences the probability of finding a particular number of pions with assigned values of the angular momentum,

[20] N. Horwitz, D. Miller, J. Murray, and R. Tripp, Phys. Rev. **115**, 472 (1959).

[21] L. E. Agnew, Jr., T. Elioff, W. B. Fowler, R. L. Lander, W. M. Powell, E. Segrè, H. M. Steiner, H. S. White, C. Wiegand, and T. Ypsilantis, Phys. Rev. **118**, 1371 (1960).

[22] T. B. Day, G. A. Snow, and J. Sucher, Phys. Rev. Letters **3**, 61 (1959).

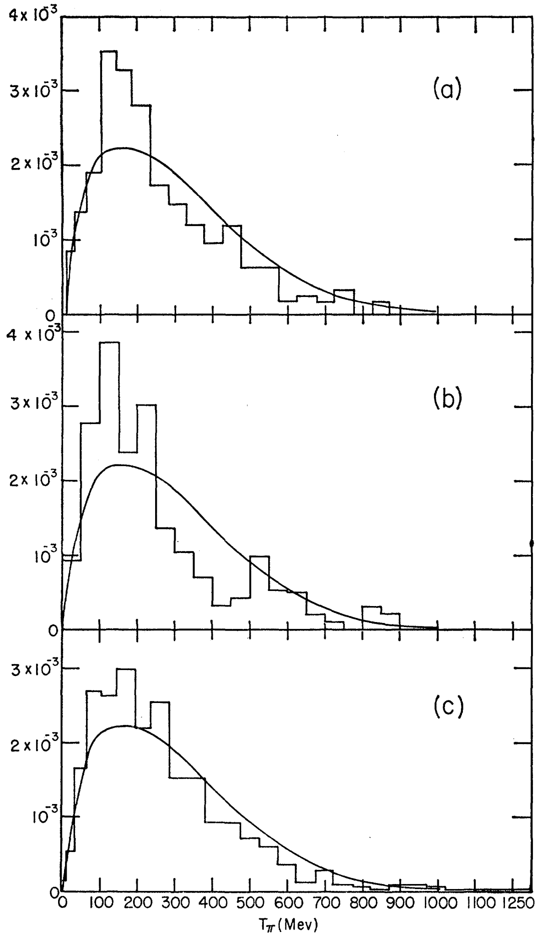

FIG. 5. (a) Spectral data of Horwitz *et al.*[20]; 6 events at an average incident energy of 50 Mev and 75 events at rest. (b) Spectral data of Agnew *et al.*[21]; 100 events at an average incident energy of 80 Mev and 30 events at rest. (c) Four- and six-prong spectral data of Chamberlain *et al.*[5]; 450 events at an average incident energy of 450 Mev and essentially no events at rest.

and thus is the essential feature in obtaining the theoretical energy spectra.

In connection with these remarks it is of some interest to estimate the relative importance of these two physical characteristics, viz., the matrix element is momentum-dependent, and only certain partial waves are annihilated. If we ignore the momentum dependence, then in accordance with the remarks in Sec. II we have a one-parameter model with an effective volume of $\Omega(R,G)=G^2(\frac{4}{3}\pi R^3)$. However, using the curves in Fig. 3, we see that this effective volume varies; in fact, to produce a multiplicity of 4.9, we have for $\Omega(R,G)$

$$\Omega(1/\mu,6)=6\Omega_0, \qquad \Omega(1/2\mu,12)=1.3\Omega_0,$$
$$\Omega(3/4\mu,8)=4.2\Omega_0, \quad \Omega(1/3\mu,19)=0.4\Omega_0,$$

where $\Omega_0=\frac{4}{3}\pi(1/\mu)^3$. It seems reasonable to assume that these differences are primarily due to the restrictions on the annihilating partial waves, since the momentum dependence has been removed. However, using the Fermi model to give a multiplicity of approximately five, we must have $\Omega\sim 10\Omega_0$. Because we have taken into account approximately the effect of the partial waves, it appears that the discrepancy between $10\Omega_0$ and the figures given above is due to the inclusion of the momentum dependence of the matrix element. Therefore, we see that for $R\sim 1/\mu$ the two physical characteristics both produce about the same effect, i.e., to reduce the effective volume, but for $R\leqslant 3/(4\mu)$ the restrictions on the annihilating partial waves become the predominant feature.

There remain two problems in meson-producing annihilations which should be investigated. Recently, measurements have been made on the angular correlation of the pions; in particular, the angles between the emitted pions have been measured in two charge combinations.[23] On the basis of momentum conservation, it is possible to obtain the angular distribution of these angles, neglecting the charges. In the pairing by unlike charges one finds approximate agreement with the theoretical curves, but for the like charges a marked disagreement appears. It has been suggested that this phenomena may be the result of a final-state pion-pion interaction, but it has also been pointed out that the effect of the Bose statistics should not be overlooked. This question is being investigated in order to determine whether or not there is evidence for a $\pi-\pi$ interaction. Our calculations do not include such an effect, but one should expect that even a final-state interaction would not change the results in Sec. III in an essential way since the correlation functions given by Kalogeropoulos[24] are not very different for the two charge combinations.

The second problem to be investigated is that of strange-particle production. The present experimental data show that approximately 5% of the annihilations involve K particles, this factor depending slightly on the incident energy. Although the interaction model can treat K-particle production, the results would not be definitive and have not been treated in this paper. At present, essentially no information is available concerning the energy spectra of pions produced with K particles in annihilation. Thus we have only one piece of data to fit two parameters. If it were possible to relate the coupling strength, G, to the various coupling constants in field theory, then it would be possible to estimate the effect on the basis of present data. For example, if one were to assume some relation between the G values and the field-theory coupling constants, and that the form of the $\pi-N$ and $K-N$ interactions were the same, then G_K would be approxi-

[23] G. Goldhaber, W. B. Fowler, S. Goldhaber, T. F. Hoang, T. E. Kalogeropoulos, and W. M. Powell, Phys. Rev. Letters 3, 181 (1959).

[24] Theordore Kalogeropoulos, thesis, University of California Radiation Laboratory Report UCRL-8677, March 6, 1959 (unpublished).

mately known, and the radius for K-particle production would then be chosen to agree with the experimental data.

As indicated in Sec. II, the present model can be used to treat any production problem, and it may be of some interest to examine the data available on pion production in $N-N$ collisions. It is known that the Fermi model does not give remarkably good agreement with the experimental data, and it is to be hoped that the same parameters would satisfy the $N-N$ data as have satisfied the $N-\bar{N}$ data.

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APPENDIX

We wish to evaluate

$$I=I(E_0R,\mu R;l_1\cdots l_p)=\int_{\mu R}^{\infty}dz_1\cdots\int_{\mu R}^{\infty}dz_p\,\delta(E_0R-z_1-\cdots-z_p)$$
$$\times H_{l(1)}\cdots H_{l(p)}\exp\left\{-\left[\frac{z_1-z_1'}{a_1}\right]^2\right\}\cdots\exp\left\{-\left[\frac{z_p-z_p'}{a_p}\right]^2\right\}.$$

We begin by noticing

$$I\sim\int_{-\infty}^{\infty}dz_1\cdots\int_{-\infty}^{\infty}dz_p\,H_{l(1)}\exp\left\{-\left[\frac{z_1-z_1'}{a_1}\right]^2\right\}\cdots H_{l(p)}\exp\left\{-\left[\frac{z_p-z_p'}{a_p}\right]^2\right\}\delta(E_0R-z_1-\cdots-z_p),$$

and thus we have

$$I=\frac{1}{2\pi}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha\int_{-\infty}^{\infty}dz_1\cdots\int_{-\infty}^{\infty}dz_p\exp[i\alpha(RE_0-z_1-\cdots z_p)]H_{l(1)}\exp\left\{-\left[\frac{z_1-z_1'}{a_1}\right]^2\right\}\cdots H_{l(p)}\exp\left\{-\left[\frac{z_p-z_p'}{a_p}\right]^2\right\};$$
$$\epsilon>0,$$

$$=\frac{1}{2\pi}H_{l(1)}\cdots H_{l(p)}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha\exp(i\alpha RE_0)\int_{-\infty}^{\infty}dz_1\exp[-(z_1/a_1)^2-i\alpha(z_1+z_1')]\cdots$$
$$\times\int_{-\infty}^{\infty}dz_p\exp[-(z_p/a_p)^2-i\alpha(z_p+z_p')]$$

$$=\frac{1}{2\pi}H_{l(1)}\cdots H_{l(p)}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha\exp[-i\alpha(RE_0-z_1'-\cdots-z_p')]\exp[-(\alpha a_1/2)^2-\cdots-(\alpha a_p/2)^2]$$
$$\times\int_{-\infty}^{\infty}dz_1\exp\left[-\left(\frac{z_1}{a_1}+\frac{i\alpha a_1}{2}\right)^2\right]\cdots\int_{-\infty}^{\infty}dz_p\exp\left[-\left(\frac{z_p}{a_p}+\frac{i\alpha a_p}{2}\right)^2\right]$$

$$=\frac{1}{2\pi}H_{l(1)}\cdots H_{l(p)}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha\exp[-i\alpha(RE_0-z_1'-\cdots-z_p'-(\alpha^2/4)(a_1^2+\cdots+a_p^2)]\pi^{p/2}a_1\cdots a_p$$

$$=\frac{\pi^{p/2}}{2\pi}H_{l(1)}a_1\cdots H_{l(p)}a_p\exp\left[\frac{-(RE_0-z_1'-\cdots-z_p')^2}{a_1^2+\cdots+a_p^2}\right]$$
$$\times\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha\exp\left\{-\left[\frac{\alpha}{2}(a_1^2+\cdots+a_p^2)^{\frac{1}{2}}+\frac{i(RE_0-z_1'-\cdots-z_p')}{(a_1^2+\cdots+a_p^2)^{\frac{1}{2}}}\right]^2\right\}$$

$$=\frac{1}{2\pi}\pi^{p/2}H_{l(1)}a_1\cdots H_{l(p)}a_p\exp\left[\frac{-(RE_0-z_1'-\cdots-z_p')^2}{a_1^2+\cdots+a_p^2}\right]\frac{2\sqrt{\pi}}{(z_1^2+\cdots+a_p^2)^{\frac{1}{2}}}$$

$$=\frac{A_{l(1)}(\mu R)\cdots A_{l(p)}(\mu R)}{\sqrt{\pi}}\frac{\exp\{-[(RE_0-z_1'-\cdots-z_p')/(a_1^2+\cdots+a_p^2)^{\frac{1}{2}}]^2\}}{(a_1^2+\cdots+a_p^2)^{\frac{1}{2}}}.$$

which is the desired result.