Remarks on Schrödinger's Model of de Sitter Space

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Schrödinger in his book "Expanding Universes" developed a "reduced model" of de Sitter space-time consisting of a certain hyperboloid of one sheet. He showed, *inter alia*, that timelike sections of the hyperboloid by planes through its center correspond to free paths in de Sitter space-time. The main purpose of the present note is to show that timelike sections by arbitrary planes correspond to paths of uniformly accelerated particles, and then to deduce some simple properties of such paths. What is here called Schrödinger's model seems to have been first proposed by H. Weyl in Physik. Z. **24**, 230 (1923) and was further discussed by H. P. Robertson in Phil. Mag. **5**, 835 (1928) and Revs. Modern Phys. **5**, 62 (1933). Schrödinger's discussion, however, is the fullest.

IN a recent paper[1] (hereafter referred to as paper I) I showed that a timelike world-line of constant curvature and zero torsion vector in a general space-time represents the analogue of "hyperbolic" motion in flat space-time and thus belongs to a "uniformly accelerated" particle; if the curvature of the world-line is $i\alpha$ the proper acceleration of the particle is α; and as α tends to zero or infinity such a path becomes a geodesic, non-null or null, respectively. In de Sitter space-time referred to the usual metric

$$ds^2=-e^{2t/R}\{dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\}+dt^2, \qquad (1)$$

where R is the reciprocal of Hubble's constant, the world-line of any particle moving in a purely radial direction ($\theta,\phi=$ constant) automatically has zero torsion vector and will correspond to a uniformly accelerated particle if it has constant curvature. Schrödinger[2] has described a useful realization (or model) of a subspace $\mathcal{S}$ of (1) corresponding to $\theta,\phi=$ constant. This consists of a certain hyperboloid on which is defined a certain metric. Schrödinger showed that all timelike sections of the hyperboloid by planes through its center represent free paths in de Sitter space-time and that the generators represent light-paths. The main purpose of this note is to show that timelike sections by *arbitrary* planes represent paths of uniformly accelerated particles, and then to deduce some simple properties of such paths.

Following Schrödinger, we construct in ordinary Euclidean 3-space the hyperboloid

$$x^2+y^2-z^2=R^2, \qquad (2)$$

(see Fig. 1) which we shall call $\mathcal{H}$. Onto $\mathcal{H}$ we now map the space $\mathcal{S}$ by the transformation

$$r=\frac{Rx}{y+z}, \quad t=R\ln\frac{y+z}{R}. \qquad (3)$$

Although this mapping has many pleasant properties, visible symmetry is not one of them. Thus, for example, the contemporary spaces $t=$ constant correspond, by (3)(ii), to the (parabolic) sections of $\mathcal{H}$ by the planes

$$y+z=Re^{t/R} \quad (t=\text{constant}).$$

(See Figs. 1 and 2.) Hence one half of $\mathcal{H}$, namely that lying above the plane $y+z=0$, suffices for the mapping of all the events of $\mathcal{S}$. (This suggests the so-called elliptic map in which antipodes of $\mathcal{H}$ are identified.) Another asymmetry: only one of the meridian hyperbolae, namely the section of $\mathcal{H}$ by the plane $x=0$, corresponds to the world-line of a fundamental particle, in this case the origin-particle with $r=0$. The world-line of a general fundamental particle characterized by $r=$ constant corresponds, by (3)(i), to the section of $\mathcal{H}$ by the plane

$$Rx-r(y+z)=0 \quad (r=\text{constant}).$$

Each such plane contains the line $x=0=y+z$, and consequently intersects $\mathcal{H}$ in a hyperbola with center at the origin. The projection of this hyperbola on to the (x,y) plane (see Figs. 3 and 4) is evidently another hyperbola with center at the origin. Its vertex lies on the waist-circle $x^2+y^2=R^2$ and it is easily found that its asymptotes are the lines

$$x=0, \quad x=y\tan2\chi \quad (\tan\chi=r/R). \qquad (4)$$

It may be noted that the x coordinate in $\mathcal{H}$ has a simple physical significance: it corresponds to the proper distance l from the origin-particle $r=0$. This follows at once from (3) since $l=re^{t/R}$.[3]

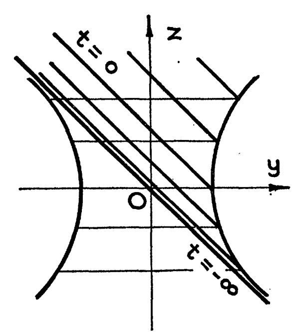

FIG. 1. This figure shows $\mathcal{H}$ from the positive x direction, with some level curves of z, and 45° plane sections corresponding to contemporary spaces.

[1] W. Rindler, Phys. Rev. **119**, 2082 (1960).
[2] E. Schrödinger, *Expanding Universes* (Cambridge University Press, Cambridge, England, 1956), Chap. I.
[3] See reference (1), formula (42).

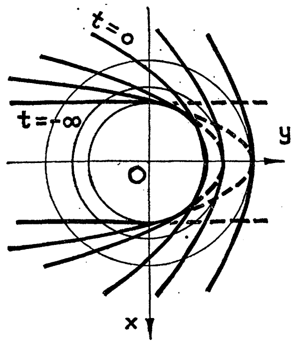

FIG. 2. This figure is a projection of the same configuration onto the (x,y) plane.

The most important feature of Schrödinger's model, however, is the simple representation of interval. By a straightforward calculation it can be verified that the substitution (3), restricted by Eq. (2), transforms the metric of $\mathcal{S}$ [namely metric (1) with $d\theta=d\phi=0$] as follows:

$$-ds^2=d\mathfrak{s}^2=dx^2+dy^2-dz^2. \tag{5}$$

We have here defined a quantity $d\mathfrak{s}^2$ which will be of use below. Evidently we can eliminate dz^2 from (5) by means of (2), and thus we obtain

$$-ds^2=d\mathfrak{s}^2=dx^2+dy^2+\frac{(xdx+ydy)^2}{R^2-x^2-y^2}. \tag{6}$$

But this is formally identical with the line element induced on a sphere

$$x^2+y^2+z^2=R^2 \tag{7}$$

by the metric

$$ds^2=dx^2+dy^2+dz^2 \tag{8}$$

of the embedding Euclidean 3-space. Consequently the surface $\mathfrak{H}$, structured by the "distance" $d\mathfrak{s}$ as in (5) and (6), is formally isometric with the ordinary sphere (7) structured by ordinary distance. We say "formally" isometric, since the ranges of the coordinates are mutually exclusive: for real points on the sphere, $|x|$ and $|y|$ are $\leqslant R$ while on $\mathfrak{H}$ they are $\geqslant R$.

We are now in a position to prove the following *theorem*: any section of $\mathfrak{H}$ by a plane

$$ax+by+cz=d \tag{9}$$

represents a uniformly accelerated particle moving with constant proper acceleration α in de Sitter space-time, where

$$\alpha=\frac{\eta}{R(R^2-\eta^2)^{\frac{1}{2}}},\quad \eta=\frac{|d|}{(a^2+b^2-c^2)^{\frac{1}{2}}}. \tag{10}$$

[If α is real and nonzero the corresponding world-line is timelike and represents a real particle; if α is imaginary the corresponding world-line is spacelike and represents no real particle; if α is indeterminate ($\eta=0/0$ or $\eta=R$) the world-line is null and represents a photon; and if α is zero the world-line is either a spacelike or a timelike geodesic and represents a real particle only in the latter case.]

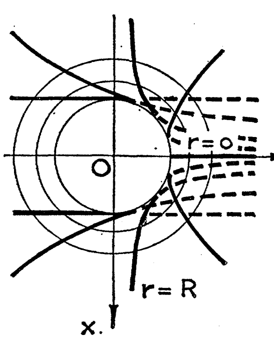

FIG. 3. This figure shows the projections of the fundamental particle paths.

We could, of course, prove this theorem by direct computation. But it is both simpler and more illuminating to make use of the isometric sphere. Clearly any plane (9) cuts the sphere (7) in a real or imaginary circle. If it is real, it follows from simple geometry that such a circle (and only such a circle) is a line of constant geodesic curvature[4] on the sphere, and that this curvature is, in fact, given by

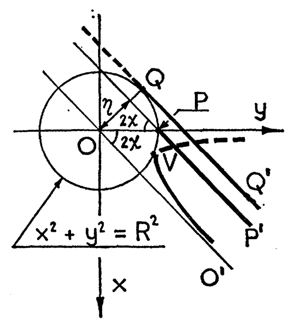

FIG. 4. This figure shows the projection yVO' of the path of a fundamental particle, the projection PP' of the path of a uniformly accelerated particle, and the projection QQ' of the path of a photon, all meeting in the infinite future.

$$K_g=\frac{\xi}{R(R^2-\xi^2)^{\frac{1}{2}}},\quad \xi=\frac{|d|}{(a^2+b^2+c^2)^{\frac{1}{2}}}. \tag{11}$$

If the circle is imaginary, an analytic calculation must formally yield the same result also. The parametric equations of any such circle, using z as a parameter,[5] can be found from (7) and (9); let us write them formally as

$$x=f(z;\, a,b,c,d),\quad y=g(z;\, a,b,c,d). \tag{12}$$

The geodesic curvature of (12) could be calculated directly by use of the line element (6). But we already know the result of this calculation: it is (11). Now if we set

$$z=i\tilde{z},\quad c=-i\tilde{c}, \tag{13}$$

[4] D. J. Struik, *Classical Differential Geometry* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950), Sec. 4-1. Equation (11) is most simply proved by interpreting K_g as the projection onto the tangent plane of the curvature vector of the curve.

[5] If the circle is parallel to the (x,y) plane we can use an arbitrary parameter and apply the same argument.

the intersection of (2) and (9) has the same equation in $\tilde{z}$, a, b, $\tilde{c}$, d, as the intersection of (7) and (9) has in z, a, b, c, d. Consequently its parametric equations are

$$x=f(\tilde{z};\, a,b,\tilde{c},d), \quad y=g(\tilde{z};\, a,b,\tilde{c},d), \tag{14}$$

and its geodesic curvature could now be computed from the same line element (6) as that of (12). Evidently the two computations would be formally identical, and so for the result we need merely set $\tilde{z}$, $\tilde{c}$, for z, c in (11) (actually z does not occur). Thus we find that the geodesic curvature of the curve (14) relative to the metric $d\tilde{s}^2$ of (6) is given by

$$K_g=\frac{\eta}{R(R^2-\eta^2)^{\frac{1}{2}}}, \quad \eta=\frac{|d|}{(a^2+b^2-c^2)^{\frac{1}{2}}}. \tag{15}$$

Our theorem will now follow from the remark that introduced this paper, provided we can demonstrate the following two facts: (i) if K_g is evaluated relative to a metric $d\tilde{s}^2$ instead of $ds^2=-d\tilde{s}^2$ we obtain the same numerical value multiplied by i; (ii) a curve having constant geodesic curvature K_g in $\mathcal{S}$ corresponds to a curve having the same constant curvature κ in the full de Sitter space-time. The general expression for K_g on a 2-surface is[6]

$$K_g{}^2=g_{ij}\left(\frac{d^2x^i}{ds^2}+\Gamma_{ab}{}^i\frac{dx^a}{ds}\frac{dx^b}{ds}\right)\left(\frac{d^2x^j}{ds^2}+\Gamma_{cd}{}^j\frac{dx^c}{ds}\frac{dx^d}{ds}\right), \tag{16}$$

the suffixes taking values 1 and 2 only. Since the Christoffel symbols $\Gamma_{ab}{}^i$ remain unchanged when we replace g_{ij} by $-g_{ij}$, each of the three factors in (16) becomes minus the function of $i\tilde{s}$ that it was of s, and (i) follows. The formula for the curvature κ of a curve

$$x^i=x^i(s) \tag{17}$$

in full de Sitter space-time is given[7] by an expression formally identical with the right member of (16), except that the suffixes now range from 1 to 4. For a curve in the subspace $\mathcal{S}$, two of the four equations (17) reduce to $\theta=$ constant, $\phi=$ constant, and it is easily seen that those Christoffel symbols having no suffix corresponding to θ or ϕ below, have none above; consequently all suffixes are summed only over the remaining two values, and (ii) follows. This completes the proof of our main theorem; we proceed to prove the statements in the bracket following the theorem.

Sections of $\mathcal{H}$ by planes subtending an angle $<45°$ with the horizontal [i.e., with the (x,y) plane] are ellipses. They cannot contain a line element subtending an angle $\geqslant 45°$ with the horizontal and are consequently spacelike ($ds^2<0$). Such sections are characterized by $a^2+b^2-c^2<0$ and thus make α imaginary (or zero if the plane contains the origin). Sections by planes subtending an angle of 45° with the horizontal are parabolae, except those by planes through the origin which are pairs of parallel generators. The former are evidently spacelike and make α imaginary [in fact, Eq. (10) gives $\alpha=i/R$], while the latter are null and make η, and consequently α, indeterminate. Sections by planes subtending an angle $>45°$ with the horizontal are hyperbolae whose asymptotes are inclined at exactly 45° to the horizontal, as can be seen by a simple projective argument. These sections fall into two main classes, according as to whether (i) the real axis or (ii) the conjugate axis is horizontal. It is clear that sections of class (i) are timelike ($ds^2>0$) since their slope everywhere exceeds that of their asymptotes, while sections of class (ii) are spacelike since their slope is everywhere less than that of their asymptotes. Class (i) is characterized by $\eta^2<R^2$, class (ii) by $\eta^2>R^2$, and these classes are separated by the class (iii) of sections by tangent planes, which have $\eta^2=R^2$ and consist of a pair of generators which are null. Consequently, by (10), class (i) has α real and nonzero, class (ii) has α imaginary and class (iii) has α indeterminate. Since we have Schrödinger's result that all sections of $\mathcal{H}$ by planes through the origin represent geodesic world-lines and that the generators of $\mathcal{H}$ represent light-paths, all statements in the bracket have now been established.

Not all uniformly accelerated motions in de Sitter space-time are purely radial, even if we allow the origin-particle to be chosen arbitrarily.[8] Thus, although the model $\mathcal{H}$ is sufficient for the description of the most general "free" motion in de Sitter space-time, it can be used to describe only a subclass of uniformly accelerated motions, viz., those which are radial with respect to *some* fundamental particle. This is, however, the most important case in practice. Let us, therefore, now consider an arbitrary timelike section of $\mathcal{H}$ by a plane π. By a suitable rotation of axes about the z axis, π can be made perpendicular to the (y,z) plane; by a Lorentz transformation of y and z it can then be made perpendicular to the (x,y) plane; and a second rotation about the z axis can finally bring the origin-event $(0,R,0)$ to lie on π. None of these transformations has altered the shape of $\mathcal{H}$ or its metric. The conclusion from all this is the intuitively obvious fact that coordinates can be chosen so that the most general uniformly accelerated particle confined to $\mathcal{S}$ moves from rest at $r=t=0$. Figure 4 illustrates the motion of such a particle. The plane π is perpendicular to the plane of the diagram, which it cuts along the line PP'; it makes an angle 2χ with the y axis and its distance from the origin is consequently given by

$$\eta=R\sin 2\chi. \tag{18}$$

This is the η of Eq. (10) since $c=0$ in our case. It is clear from the diagram that the accelerating particle corresponding to π will asymptotically approach, but never overtake, the fundamental particle whose asymptote projects into OO', where the angle POO' is 2χ. From

[6] J. L. Synge and A. Schild, *Tensor Calculus* (University of Toronto Press, Toronto, Canada, 1949), Sec. 5.223.

[7] See reference (6), Sec. 2.704.

[8] See reference (1), footnote 21.

(4) we see that the radial coordinate r_∞ of this particle (the suffix denotes that the meeting takes place in the infinite future) is given by

$$r_\infty = R\tan\chi. \tag{19}$$

The acceleration α corresponding to an η as in (18) is found, from (10), to be

$$\alpha = R^{-1}\tan 2\chi, \tag{20}$$

and this allows us to eliminate χ from (19). Thus we easily find

$$r_\infty = \alpha^{-1}\{(1+\alpha^2R^2)^{\frac{1}{2}}-1\}. \tag{21}$$

The same result was obtained analytically in paper I, which should be consulted for the connection between r_∞ and the "α horizon." If α is infinite, the "accelerating" particle is a photon. This corresponds, by (19) and (20), to $r_\infty = R$. Hence even a light-signal does not overtake all fundamental particles on its line of motion, which is, of course, a well-known fact.

It is also clear from the diagram that there exists a photon which, sent after the accelerating particle, only intercepts it in the infinite future. Its projection on to the (x,y) plane is QQ', parallel to PP' and tangent to the waist-circle. The coordinates of Q are evidently $(0, R\operatorname{cosec}2\chi, 0)$, while the coordinates of the point on $\mathcal{H}$ whose projection is Q are $(0, R\operatorname{cosec}2\chi, R\cot 2\chi)$. By 3 (ii), this corresponds to a time

$$t_{\text{crit}} = R\ln(\operatorname{cosec}2\chi + \cot 2\chi),$$

which, by (20), is equivalent to

$$t_{\text{crit}} = R\ln[\alpha^{-1}R^{-1}\{(1+\alpha^2R^2)^{\frac{1}{2}}+1\}]. \tag{22}$$

The subscript denotes that this is a critical time for photons sent from the origin-particle ($r=0$) to intercept an accelerating particle that was released there from rest at $t=0$. Photons emitted earlier intercept the receding particle, while photons emitted later do not This result too was obtained analytically in paper I.

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Absence of Bound States in a Gravitational Field*

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The Klein-Gordon and Dirac equations for a particle in the gravitational field of a point mass are investigated. It is shown that the geometrical properties of the Schwarzschild metric prevent the normalization of any bound-state solutions.

CALLAWAY[1] has recently investigated the general relativistic Klein-Gordon and Dirac equations for an attractive center of mass M and charge e, in the case $M<e$.[2]

The opposite case, $M>e$, displays wholly different features: Equation (C2) shows that the metric is singular at $r=M\pm(M^2-e^2)^{\frac{1}{2}}$. We shall take here $e=0$, since this does not essentially differ from $0<e<M$.

The Klein-Gordon equation (C4) is singular at $r=2M$. The indicial equation at this point shows that the solutions behave there as $(r-2M)^{\pm 2iEM}$. Since the current density is

$$J^k = i(-g)^{\frac{1}{2}}g^{kn}(\bar{\psi}\psi_{,n}-\bar{\psi}_{,n}\psi)/2m,$$

then it follows from (C1) that $\int J^0 drd\theta d\phi$ diverges at $r=2M$.

In the case of the Dirac equation, one has[1]

$$J^0=\bar{\psi}(-g)^{\frac{1}{2}}\gamma^0\psi \sim (-g)^{\frac{1}{2}}(F^2+G^2)/r^2g_{00}.$$

Expansion of (C7) about $r=2M$ shows that the leading terms are

$$(r-2M)F'+2MEG=0 \quad \text{and} \quad (r-2M)G'-2MEF=0.$$

It follows that F and G behave as $(r-2M)^{\pm 2iEM}$, so that here also the wave function is not normalizable.

Thus, there are no bound states in the gravitational field of a point mass.

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[1] J. Callaway, Phys. Rev. **112**, 290 (1958), hereafter referred to as C. There is a misprint in Eq. (C7); see D. Brill and J. A. Wheeler, Revs. Modern Phys. **29**, 465 (1957).

[2] Natural units are used: $c=G=\hbar=1$.