Divergence-Free Iterative Expansion of the S Matrix in a Field Theory*

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A new method is proposed for evaluating the S-matrix as a series expansion in powers of the coupling constant. The method is applicable to field theories which in the usual formulation have ultraviolet divergences in self-energy and vertex parts and require self-energy, coupling constant, and wave-function renormalization. The procedure cannot be applied in its present form to theories which allow boson self-energy terms. In this new procedure the usual form of the Hamiltonian for the coupled system is retained. The theory results in an iterated solution in powers of the physical coupling constant and yields a series, each term of which is finite without subtractions or renormalization. It agrees up to all orders examined with the finite S-matrix elements obtained by renormalizing the old formulation of the scattering problem.

It is also shown that the nth order contribution to the iterative expansion of the S matrix, where n is any order, approaches 0 more rapidly, at high energy, than $E^{-(2-\delta)}$ where δ is any positive real number, no matter how small.

I. INTRODUCTION

THE procedure for evaluating cross sections in quantum field theory can be summarized as follows: A kinematical scattering formalism is established in which the following quantities appear: the Hamiltonian H, and a set of asymptotic states $\varphi(E_q,a)$. The Hamiltonian describes the temporal variation of Schrödinger states, and wave packets of the asymptotic states represent the totality of limits which scattering states can approach as $t\rightarrow\pm\infty$. These asymptotic states are not solutions of the equations $(H-E_q)\psi=0$, but rather represent the scattering particles in noninteracting configurations. Presumably packets of these asymptotic state functions correctly describe the particle configurations identified as "incident" and "scattered" when the particles involved are infinitely removed from each other.

In the usual formulation the Hamiltonian is divided into a "free field" part, H^0, and an "interaction Hamiltonian" H^1. It is then asserted that the asymptotic states $\varphi(E_q,a)$ obey the equation $(H^0-E_q)\varphi(E_q,a)=0$ so that they form an orthogonal complete set.[1] This assertion is either made directly[2] or is inferred from an adiabatic time dependence introduced into H^1 for precisely the purpose of forcing this behavior of the asymptotic states.[3] In this formalism the existence of a T matrix is established and the following can be derived:

$$\frac{d\sigma}{d\Omega}(a\rightarrow b)=\frac{2\pi}{v_a}|T(E_q,b;E_q,a)|^2\rho(E_q,b), \qquad (1a)$$

and

$$T(E_k,b;E_q,a)=(E_k,b|H^1|E_q,a)+\sum_c\int dE_l\frac{(E_k,b|H^1|E_l,c)T(E_l,c;E_q,a)}{E_q-E_l+i\eta}, \qquad (1b)$$

where $(H^0-E_q)|E_q,a)=0$, and the $|E_i,j)$ are asymptotic states.

In most applications to field theory it is necessary to use the iterative solution of (1b). For this purpose the operators S, T are defined by

$$T(E_k,b;E_q,a)=(E_k,b|T|E_q,a),$$

and

$$S(E_k,b;E_q,a)=(E_k,b|S|E_q,a).$$

Then

$$S(E_k,b;E_q,a)=(E_k,b|E_q,a)-2\pi i\delta(E_k-E_q)T(E_k,b;E_q,a),$$

and

$$T=\{H^1+H^1(E_q-H^0+i\eta)^{-1}H^1+H^1(E_q-H^0+i\eta)^{-1}\times H^1(E_q-H^0+i\eta)^{-1}H^1+\cdots+H^1(E_q-H^0+i\eta)^{-1}\times H^1\cdots(E_q-H^0+i\eta)^{-1}H^1+\cdots\}, \qquad (2)$$

within the radius of convergence of the iterative series. Equation (2) is trivially identical to

$$S=1+\sum\nolimits_n S_n,$$

where

$$S_n=(-i)^n(n!)^{-1}\int_{-\infty}^{+\infty}dt_1\cdots\int_{-\infty}^{+\infty}dt_n\times P[\mathcal{H}^1(t_1)\cdots\mathcal{H}^1(t_n)], \qquad (3b)$$

and where

$$\mathcal{H}^1(t)=\lim_{\epsilon\rightarrow 0}\{\exp[iH^0t]H^1\exp[-iH^0t]e^{-\epsilon|t|}\}.$$

Preliminary to the presentation of the new formalism we will discuss the application of the scattering formalism specified above (hereafter referred to as the "linear formalism") to a nonrelativistic field theory given by the Hamiltonian

$$H=H^0+H^1, \qquad (4a)$$

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[1] The orthogonality of degenerate states for different channels is trivially shown though not a consequence of $(H^0-E_q)\varphi(E_q,a)=0$.

[2] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[3] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

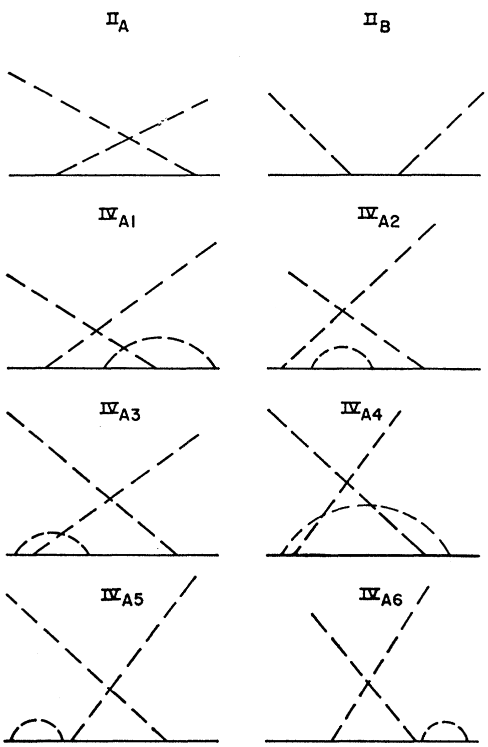

FIG. 1. Feynman graphs to fourth order for charge-symmetric scalar bosons scattered from a point source. For each diagram of type A there is a type-B diagram with which it is related by crossing symmetry. B diagrams will mostly not be shown.

where

$$H^0=\sum_{\mathbf{k},\alpha} a^\dagger_{\mathbf{k},\alpha}a_{\mathbf{k},\alpha}\omega_k, \tag{4b}$$

$$H^1=\sum_{\mathbf{k},\alpha}(a^\dagger_{\mathbf{k},\alpha}+a_{\mathbf{k},\alpha})V_{\mathbf{k},\alpha}, \tag{4c}$$

and where

$$V_{\mathbf{k},\alpha}=(2\omega_k)^{-\frac{1}{2}}g\tau_\alpha.$$

This Hamiltonian describes the charge-symmetric interaction of scalar bosons with a point source. When the S matrix is described to the fourth order by its corresponding Feynman diagrams, the graphs shown in Fig. 1 are obtained.

The S-matrix terms for II_A, II_B, are finite; those for $\mathrm{IV}_{A(1),A(2),A(3),A(5),A(6)}$, $\mathrm{IV}_{B(1),B(2),B(3),B(5),B(6)}$ are infinite; those for $\mathrm{IV}_{A(4)}$, $\mathrm{IV}_{B(4)}$ are finite. As in quantum field theory in general, in all orders above second most diagrams give rise to divergent integrals, which can be classified into self-energy, vertex, and improper or external self-energy graphs.[4] Boson self-energy graphs are not allowed by this Hamiltonian; however, the problem defined by the latter is sufficiently complex to preclude an exact solution.

It is of interest to inquire into the reason for the appearance of these divergences, which must be removed by renormalization procedures; it is also of interest to attempt a formulation of the problem in which divergences do not arise, but in which cross sections can be computed straightforwardly, at least as an iterative series, from the Hamiltonian which defines the problem.

In connection with these considerations, it is important to observe that one crucial assumption made in developing the linear scattering formalism, namely, that the asymptotic states obey the equation

$$(H^0-E_q)\varphi(E_q,a)=0,$$

is not satisfied in the case of field theories. For example, Van Hove has pointed out that in the case of theories with the persistent interactions characteristic of field theories, the eigenstates of the free-field Hamiltonian cannot be asymptotic states of the scattering system[5]; moreover, that quite probably[6] the exact and free-field states are orthogonal to each other.[7] In addition, analysis of the linear scattering formalism[8] demonstrates that it is valid only under conditions which in field theories either do not obtain, or at least cannot be shown to obtain. These conditions include the following: (1) The continuous spectrum of H and H^0 must coincide; and (2) the T matrix must be bounded and must have a bounded derivative on the energy shell (however, the left-hand and right-hand derivatives need not be identical). Although the continuous spectra of H and H^0 differ by the self-energy of the nucleon, condition (1) can be satisfied[9] by adding the self-energy to H^0 and subtracting it from H^1. However, it remains quite questionable whether the resulting T matrix is finite or not. If the requirement for a bounded $T(E_k,b;E_q,a)$, and $[\partial T(E_k,b;E_q,a)/\partial E_k]_{E_k=E_q}$ is not satisfied, then the theory leads to unphysical contributions of energy nonconserving parts even at infinite times and ceases to be a description of collision phenomena. These circumstances raise the interesting possibility that the appearance of divergences in field theory is related to the improper choice of eigenfunctions of H^0 as asymptotic states; and that a reformulation of the scattering theory in terms of packets of "physical" particles might remove this difficulty.

II. SCATTERING THEORY AND ASYMPTOTIC STATES

The linear scattering formalism previously described requires a set of asymptotic states which are eigenfunctions of a Hermitean operator, and are orthogonal and complete. Asymptotic states describing physical particles in noninteracting configurations are not orthogonal

[4] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955), Chap. 9.

[5] L. Van Hove, Physica **21**, 901 (1955).

[6] The theorem is proven for the neutral scalar field in (7). There is no reason to expect this difficulty not to arise in more complicated systems.

[7] L. Van Hove, Physica **18**, 145 (1952).

[8] H. E. Moses, Nuovo cimento **1**, 103 (1955); also Jauch and Rohrlich, reference 4, Chap. 7.

[9] A. Klein, Lectures on Pi-Meson Physics, University of Pennsylvania (unpublished).

and are not eigenfunctions of any Hermitean operator. Hence, to exclude reference to bare-particle eigenstates from the description of the collision process, a different scattering formalism, such as the one due to Ekstein,[10] is necessary. In this theory no dynamical requirements are made of the asymptotic states $\varphi(E_q,a)$, in particular they need not be eigenfunctions of any Hermitean operator and they need not be orthogonal or even linearly independent. Quantities $\chi(E_q,a)$ are defined by $(H-E_q)\varphi(E_q,a)=\chi(E_q,a)$ and a set of orthogonal steady-state solutions of the Schrödinger equation are derived from the fact that there are time-dependent solutions which tend to the $\varphi(E_q,a)$. These steady-state solutions are

$$\psi^{(\pm)}(E_q,a)=\varphi(E_q,a)-(H-E_q\mp i\eta)^{-1}\chi(E_q,a). \quad (5)$$

In this theory, the quantity which describes the scattering $(E_q,a)\rightarrow(E_q,b)$ is given by $\langle\chi(E_q,b)|\times\psi^{(+)}(E_q,a)\rangle$[11]; we will call this the R matrix. Although the R matrix has the same kinematic significance as the T matrix [it replaces the latter in Eq. (1a)], in general it obeys an entirely different integral equation. In the event that the asymptotic states of the scattering system really are eigenfunctions of H^0, the R matrix trivially reduces to the T matrix but when these asymptotic states do not satisfy an equation $(H^0-E_q)\times\varphi(E_q,a)=0$ then such is not the case.[12]

The absence of strong requirements on the asymptotic states in this formalism allows us to choose these states far more realistically than in the linear scattering theory. In the case of the Hamiltonian of Eq. (4) the wave function for the asymptotic states can be written as the product wave function of a "meson" and a "physical nucleon." Since the bosons in this theory do not have any persistent vacuum effects (the Hamiltonian does not allow boson self-energy graphs), the bare and the physical meson are identical. However, the nucleon has persistent vacuum effects so that the bare and the physical nucleon are not identical, the former obeying the equation $(H^0-E_0)|0)=0$, the latter $(H-E_0)|0\rangle=0$. (The isotopic spin index of the meson and the nucleon will be suppressed.) The asymptotic states in this theory can then be written: $a^\dagger_{\mathbf{q}}|0\rangle$, $(2)^{-\frac{1}{2}}a^\dagger_{\mathbf{q}(1)}a^\dagger_{\mathbf{q}(2)}|0\rangle$, $\cdots$, $(n!)^{-\frac{1}{2}}a^\dagger_{\mathbf{q}(1)}a^\dagger_{\mathbf{q}(2)}\cdots a^\dagger_{\mathbf{q}(n)}|0\rangle$. These state functions are not orthogonal to each other and, unlike the asymptotic states in the linear scattering theory, they form wave packets which have time-dependent behavior proper for state functions for particles in noninteracting configurations.[13]

III. DERIVATION OF THE INTEGRAL EQUATIONS

The R matrix for the scattering of a single meson $(E_q,a)\rightarrow(E_q,b)$ is $\langle\chi(E_k,b)|\psi(E_q,a)\rangle$, when the latter is evaluated on the energy shell. The asymptotic state $\varphi(E_q,a)$ is $a^\dagger_{\mathbf{q}}|0\rangle$. From the commutation relations of H with $a^\dagger_{\mathbf{k}}$, $\chi(E_k,b)$ can be shown to be $V_k|0\rangle$,[14] and the above expression becomes $\langle 0|V_k|\psi_{\mathbf{q}}\rangle$. In our notation this will be written $R_k(0;q)$.[15] Other matrix elements which occur in this theory are $\langle 0|V_k|\psi_{\mathbf{q}(1),\cdots\mathbf{q}(n)}\rangle$ which will be written as $R_k(0;{}_{q(1),q(2),\cdots q(n)})$, where $\psi_{\mathbf{q}(1),\cdots,\mathbf{q}(n)}$ is given by

$$\psi_{\mathbf{q}(1),\cdots,\mathbf{q}(n)}=(n!)^{-\frac{1}{2}}a^\dagger_{\mathbf{q}(1)}\cdots a^\dagger_{\mathbf{q}(n)}|0\rangle-(H-\omega_{q(1)}-\cdots-\omega_{q(n)}-i\eta)^{-1}\chi(\mathbf{q}(1),\cdots,\mathbf{q}(n)),$$

and also $\langle\psi_{\mathbf{p}(1),\cdots,\mathbf{p}(m)}|V_k|\psi_{\mathbf{q}(1),\cdots,\mathbf{q}(n)}\rangle$ which will be written $R_k(p(1),\cdots,p(m);q(1),\cdots,q(n))$. The $R_k(0;N)$ [N denotes $q(1),\cdots,q(n)$, collectively] are related to S-matrix elements for inelastic scattering. $R_k(M;N)$ are auxiliary quantities useful for writing the integral equations in tractable form.

Making use of the identity[14] $a_{\mathbf{q}}|0\rangle=-(H+\omega_q)^{-1}V_q|0\rangle$, $R_k(0;q)$ becomes

$$R_k(0;q)=-[\langle 0|V_q(H+\omega_q)^{-1}V_k|0\rangle+\langle 0|V_k(H-\omega_q-i\eta)^{-1}V_q|0\rangle]. \quad (6)$$

Making use of the fact that $|0\rangle$ is the only "bound" state and inserting a complete set of states, including $|0\rangle$ and the steady-state solutions ψ_N, this can be written

$$R_k(0;q)=-\sum_{N=0}^{\infty}\left[\frac{R_q(0;N)R_k^\dagger(0;N)}{E_N+\omega_q}+\frac{R_k(0;N)R_q^\dagger(0;N)}{E_N-\omega_q-i\eta}\right]. \quad (7)$$

$\sum_N$ includes summation over all meson numbers (including $N=0$, which denotes the state $|0\rangle$), and over $\boldsymbol{\kappa}$, the

[10] H. Ekstein, Phys. Rev. **101**, 880 (1956); a special case of this is given by G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).

[11] The superscript $^{(+)}$ denotes outgoing waves. Incoming wave states will not be used here, and the $^{(+)}$ will hereafter be understood.

[12] For example, the equation relating exact and asymptotic states in the linear theory, $\psi^{(\pm)}(E_q,a)=[1+(E_q-H\pm i\eta)^{-1}H^1]\times\varphi(E_q,a)$, is not satisfied when φ is identified as the physical-particle asymptotic state in the charge-symmetric scalar theory.

[13] G. C. Wick, reference 10; also, H. Ekstein, Nuovo cimento **4**, 1017 (1956). Other discussions of the use of the physical-particle representation of asymptotic states, besides the ones already cited, are Th. W. Ruijgrok, Physica **24**, 205 (1958); W. R. Frazer and L. Van Hove, Physica **24**, 137 (1958); R. Norton and A. Klein, Phys. Rev. **109**, 584 (1958); and H. Ekstein, J. Swihart, and K. Tanaka, Phys. Rev. **109**, 557 (1958). In the last of these, it should be noted that the integral equations (2.15) are not the same as the ones developed in Sec. III of this paper.

[14] G. Chew and F. Low, Phys. Rev. **101**, 1570 (1956).

[15] k and q denote E_k, β and E_q, α, respectively; the α and β denote the isotopic spin state of the bosons.

momentum and isotopic spin variables of each boson. This equation is identical to the Chew-Low equation for this Hamiltonian.[14,10] Since the iteration procedure is to be one in successive powers of the coupling constant, an infinite set of such coupled nonlinear integral equations must be generated and an iterative series developed. To do this, the following lemmas will be proven in the Appendix:

Lemma 1

$$\chi(\mathbf{q}_1,\cdots,\mathbf{q}_n)=n^{-\frac{1}{2}}\sum_i V_{q(i)}\varphi(\mathbf{q}_1,\cdots,\mathbf{q}_{(i-1)},\mathbf{q}_{(i+1)},\cdots,\mathbf{q}_n),$$

where

$$\varphi(\mathbf{q}_1,\cdots,\mathbf{q}_l)=(l!)^{-\frac{1}{2}}a^\dagger{}_{\mathbf{q}(1)}\cdots a^\dagger{}_{\mathbf{q}(l)}|0\rangle.$$

Lemma 2

$$\frac{1}{H+\lambda}a^\dagger{}_{\mathbf{q}(1)}\cdots a^\dagger{}_{\mathbf{q}(n)}=\sum_{0\le l\le n}(-1)^{n-l}\sum_{P(i;j)}a^\dagger{}_{\mathbf{q}[i,1]}\cdots a^\dagger{}_{\mathbf{q}[i,l]}\frac{1}{H+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}+\lambda}$$
$$\times V_{q[j,1]}\frac{1}{H+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}+\omega_{q[j,1]}+\lambda}\cdots V_{q[j,(n-l)]}\frac{1}{H+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}+\omega_{q[j,1]}+\cdots+\omega_{q[j,(n-l)]}+\lambda}$$

Here $[i,1]$, $\cdots$, $[i,l]$, $[j,1]$, $\cdots$, $[j,(n-l)]$ is a particular permutation of 1, $\cdots$, $n\cdot\sum_{P(i;j)}$ is the sum over the following permutations: all permutations of j's among themselves; all permutations involving interchange of an i and a j. It does not include any summation over permutations among any i's.

Lemma 3

$$\langle 0|a^\dagger{}_{\mathbf{q}(1)}\cdots a^\dagger{}_{\mathbf{q}(n)}\Omega|0\rangle=(-1)^n\sum_{P(i)}\left\langle 0\left|V_{q[i,1]}\frac{1}{H+\omega_{q[i,1]}}\cdots V_{q[i,l]}\frac{1}{H+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}}\right.\right.$$
$$\left.\left.\times\cdots V_{q[i,n]}\frac{1}{H+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}+\cdots+\omega_{q[i,n]}}\Omega\right|0\right\rangle$$

Here $\sum_{P(i)}$ is the sum over all permutations of i's among themselves, and Ω is any operator.

Lemma 4

$$a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}\frac{1}{H+\lambda}=\sum_{0\le l\le m}\sum_{P(i;j)}\frac{1}{H+\omega_{p[i,1]}+\cdots+\omega_{p[i,l]}+\omega_{p[j,1]}+\cdots+\omega_{p[j,(m-l)]}+\lambda}V_{p[j,1]}$$
$$\times\frac{1}{H+\omega_{p[i,1]}+\cdots+\omega_{p[i,l]}+\omega_{p[j,2]}+\cdots+\omega_{p[j,(m-l)]}+\lambda}\cdots V_{p[j,(m-l)]}$$
$$\times\frac{1}{H+\omega_{p[i,1]}+\cdots+\omega_{p[i,l]}+\lambda}a_{\mathbf{p}[i,1]}\cdots a_{\mathbf{p}[i,l]}.$$

Lemma 5

$$\langle 0|\Omega a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}|0\rangle=(-1)^m\sum_{P(i)}\left\langle 0\left|\Omega\frac{1}{H+\omega_{p[i,1]}+\cdots+\omega_{p[i,m]}}V_{p[i,1]}\frac{1}{H+\omega_{p[i,2]}+\cdots+\omega_{p[i,m]}}\right.\right.$$
$$\left.\left.\times\cdots V_{p[i,l]}\frac{1}{H+\omega_{p[i,(l+1)]}+\cdots+\omega_{p[i,m]}}\cdots V_{p[i,m]}\right|0\right\rangle.$$

Lemma 6

Before stating this lemma, the following notation will be defined: Consider a set $a^\dagger{}_{\mathbf{q}(1)}$, $\cdots$, $a^\dagger{}_{\mathbf{q}(n)}$; $a_{\mathbf{p}(1)}$, $\cdots$, $a_{\mathbf{p}(m)}$. The set $i(1)$, $\cdots$, $i(n)$ is a permutation of 1, $\cdots$, n. The set $j(1)$, $\cdots$, $j(m)$ is a permutation of 1, $\cdots$, m. Suppose $i(1)$ is omitted from $i(1)$, $\cdots$, $i(n)$, and a permutation of the remainder chosen. This permutation will be labeled $i[1;\bar{\imath}(1)]$, $i[2;\bar{\imath}(1)]$, $\cdots$, $i[(n-1);\bar{\imath}(1)]$. If $i(1)$, $i(2)$ are removed then a permutation of the remainder

is $i[1;\bar{\imath}(1),\bar{\imath}(2)]$, $\cdots$, $i[(n-2);\bar{\imath}(1),\bar{\imath}(2)]$, etc.; a similar notation applies to the j's. The statement of the lemma is as follows:

$$
\begin{aligned}
&a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}a^{\dagger}{}_{\mathbf{q}(1)}\cdots a^{\dagger}{}_{\mathbf{q}(n)} \\
&\quad = a^{\dagger}{}_{\mathbf{q}(1)}\cdots a^{\dagger}{}_{\mathbf{q}(n)}a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)} + \sum_{\substack{1\leq i(1)\leq n\\ 1\leq j(1)\leq m}} \delta(\mathbf{q}[i(1)],\mathbf{p}[j(1)])a^{\dagger}\{\mathbf{q}(i[1;\bar{\imath}(1)])\}\cdots a^{\dagger}\{\mathbf{q}(i[(n-1);\bar{\imath}(1)])\} \\
&\qquad \times a\{\mathbf{p}(j[1;\bar{\jmath}(1)])\}\cdots a\{\mathbf{p}(j[(m-1);\bar{\jmath}(1)])\} + \sum_{\substack{1\leq i(1)<i(2)\leq n\\ \left.\begin{smallmatrix}1\leq j(1)\leq m\\ 1\leq j(2)\leq m\end{smallmatrix}\right\} j(1)\neq j(2)}} \delta(\mathbf{q}[i(1)],\mathbf{p}[j(1)])\delta(\mathbf{q}[i(2)],\mathbf{p}[j(2)]) \\
&\qquad \times a^{\dagger}\{\mathbf{q}(i[1;\bar{\imath}(1),\bar{\imath}(2)])\}\cdots a^{\dagger}\{\mathbf{q}(i[(n-2);\bar{\imath}(1),\bar{\imath}(2)])\}a\{\mathbf{p}(j[1;\bar{\jmath}(1),\bar{\jmath}(2)])\}\cdots \\
&\qquad \times a\{\mathbf{p}(j[(m-2);\bar{\jmath}(1),\bar{\jmath}(2)])\} + \cdots + \sum_{\substack{1\leq i(1)<i(2)\cdots<i(l)\leq n\\ \left.\begin{smallmatrix}1\leq j(1)\leq m\\ 1\leq j(2)\leq m\\ \vdots\\ 1\leq j(l)\leq m\end{smallmatrix}\right\} j(1)\neq j(2)\cdots\neq j(l)}} \delta(\mathbf{q}[i(1)],\mathbf{p}[j(1)])\delta(\mathbf{q}[i(2)],\mathbf{p}[j(2)])\cdots \\
&\qquad \times \delta(\mathbf{q}[i(l)],\mathbf{p}[j(l)])a^{\dagger}\{\mathbf{q}(i[1;\bar{\imath}(1),\cdots,\bar{\imath}(l)])\}\cdots a^{\dagger}\{\mathbf{q}(i[(n-l);\bar{\imath}(1),\cdots,\bar{\imath}(l)])\} \\
&\qquad\qquad \times a\{\mathbf{p}(j[1;\bar{\jmath}(1),\cdots,\bar{\jmath}(l)])\}\cdots a\{\mathbf{p}(j[(m-l);\bar{\jmath}(1),\cdots,\bar{\jmath}(l)])\} + \cdots
\end{aligned}
$$

(series ends with term $l=n$ or $l=m$ whichever occurs first).

Using these lemmas, integral equations for $R_k(0;N)$ and $R_k(M;N)$ can be derived. In the case of the former we have

$$
\begin{aligned}
R_k(0;q_1,\cdots,q_n) = (n!)^{-\frac{1}{2}}[\langle 0|a^{\dagger}{}_{\mathbf{q}(1)}\cdots a^{\dagger}{}_{\mathbf{q}(n)}V_k|0\rangle - \langle 0|V_k(H-\omega_{q(1)}-\cdots-\omega_{q(n)}-i\eta)^{-1} \\
\times \textstyle\sum_{[i,l]} a^{\dagger}{}_{\mathbf{q}[i,1]}\cdots a^{\dagger}{}_{\mathbf{q}[i,(l-1)]}a^{\dagger}{}_{\mathbf{q}[i,(l+1)]}\cdots a^{\dagger}{}_{\mathbf{q}[n]}V_{q[i,l]}|0\rangle]. \qquad (8)
\end{aligned}
$$

By applying Lemma 3 to the first term and Lemmas 2 and 3 to the second, we obtain

$$
\begin{aligned}
R_k(0;q_1,\cdots,q_n) = \frac{(-1)^n}{(n!)^{\frac{1}{2}}} \sum_{0\leq l\leq n}\sum_{P(i)}\sum_{N_1,N_2,\cdots N_n} R_{q[i,1]}(0;N_1)\frac{1}{E_{N(1)}+\omega_{q[i,1]}}R_{q[i,2]}(N_1;N_2)\frac{1}{E_{N(2)}+\omega_{q[i,1]}+\omega_{q[i,2]}}\cdots \\
\times R_{q[i,l]}(N_{l-1};N_l)\frac{1}{E_{N(l)}+\omega_{q[i,1]}+\omega_{q[i,2]}+\cdots+\omega_{q[i,l]}}R_k(N_l;N_{l+1}) \\
\times \frac{1}{E_{N(l+1)}-\omega_{q[i,(l+1)]}-\omega_{q[i,(l+2)]}-\cdots-\omega_{q[i,n]}-i\eta}R_{q[i,(l+1)]}(N_{l+1};N_{l+2}) \\
\times \frac{1}{E_{N(l+2)}-\omega_{q[i,(l+2)]}-\cdots-\omega_{q[i,n]}-i\eta}R_{q[i,(l+2)]}(N_{l+2};N_{l+3}) \\
\times\cdots R_{q[i,(n-1)]}(N_{n-1};N_n)\frac{1}{E_{N(n)}-\omega_{q[i,n]}-i\eta}R_{q[i,n]}(N_n;0). \qquad (9)
\end{aligned}
$$

$R_k(M;N)$ can be written

$$
\begin{aligned}
&R_k(p_1,\cdots,p_m;q_1,\cdots,q_n) \\
&\quad = (n!\,m!)^{-\frac{1}{2}}[\langle 0|a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}V_k a^{\dagger}{}_{\mathbf{q}(1)}\cdots a^{\dagger}{}_{\mathbf{q}(n)}|0\rangle - (n!)^{\frac{1}{2}}\langle 0|a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}V_k(H-\omega_{q(1)}-\cdots-\omega_{q(n)}-i\eta)^{-1} \\
&\qquad \times\chi(\mathbf{q}_1,\cdots,\mathbf{q}_n) - (m!)^{\frac{1}{2}}\chi^{\dagger}(\mathbf{p}_1,\cdots,\mathbf{p}_m)(H-\omega_{p(1)}-\cdots-\omega_{p(m)}+i\eta)^{-1}V_k a^{\dagger}{}_{\mathbf{q}(1)}\cdots a^{\dagger}{}_{\mathbf{q}(n)}|0\rangle + (n!\,m!)^{\frac{1}{2}} \\
&\qquad \times\chi^{\dagger}(\mathbf{p}_1,\cdots,\mathbf{p}_m)(H-\omega_{p(1)}-\cdots-\omega_{p(m)}+i\eta)^{-1} \\
&\qquad\qquad \times V_k(H-\omega_{q(1)}-\cdots-\omega_{q(n)}-i\eta)^{-1}\chi(\mathbf{q}_1,\cdots,\mathbf{q}_n)]. \qquad (10)
\end{aligned}
$$

The various lemmas can be applied to systematically bring all creation operators to the left, annihilation operators to the right and to eliminate both from the matrix elements. The resulting integral equation is given in terms of the quantities $\bar{R}_k\{M;N\}$ which are

$$
\begin{aligned}
\bar{R}_k\{p_1,\cdots,p_r;q_1,\cdots,q_s\} = \sum_{P(l)}\sum_{N_1,\cdots,N_l} R_{l(1)}(0;N_1)[E_{N(1)}+\Lambda]^{-1}R_{l(2)}(N_1;N_2)[E_{N(2)}+\Lambda]^{-1}\cdots \\
\times R_{l(j)}(N_{(j-1)};N_{(j)})[E_{N(j)}+\Lambda]^{-1}\cdots[E_{N(r+s)}+\Lambda]^{-1}R_{l(r+s+1)}(N_{(r+s+1)};0), \qquad (11)
\end{aligned}
$$

where $l(1), l(2), \cdots, l(r+s+1)$ is a permutation of the set $p_1, \cdots, p_r, q_1, \cdots, q_s, k$;

$$\Lambda = \sum_{i=1}^{n} \sum_{j=1}^{m} (\omega_{q(i)}\xi_i + \omega_{p(j)}\zeta_j + \lambda i\epsilon),$$

where $\xi_i = 1$ for all q_i to the left of Λ if Λ is to the left of V_k; $\xi_i = 0$ for all q_i to the right of Λ if Λ is to the left of V_k; $\xi_i = -1$ for all q_i to the right of Λ if Λ is to the right of V_k; $\xi_i = 0$ for all q_i to the left of Λ if Λ is to the right of V_k; $\zeta_j = -1$ for all p_j to the left of Λ if Λ is to the left of V_k; $\zeta_j = 0$ for all p_j to the right of Λ if Λ is to the left of V_k; $\zeta_j = 1$ for all p_j to the right of Λ if Λ is to the right of V_k; $\zeta_j = 0$ for all p_j to the left of Λ if Λ is to the right of V_k; and $\lambda = -1$ if Λ is to the right of V_k, $\lambda = +1$ if Λ is to the left of V_k. The integral equation is

$$\begin{aligned} R_k(p_1,\cdots,p_m;q_1,\cdots,q_n) = \\ \frac{(-1)^{m-n}}{[(m!)(n!)]^{\frac{1}{2}}} \Big[\bar{R}_k\{p_1,\cdots,p_m;q_1,\cdots,q_n\} + \sum_{\substack{1\le i(1)\le n \\ 1\le j(1)\le m}} \delta(q[i(1)],p[j(1)]) \bar{R}_k\{p(j[1;\bar{\jmath}(1)]),\cdots,p(j[(m-1);\bar{\jmath}(1)]); \\ q(i[1;\bar{\imath}(1)]),\cdots,q(i[(n-1);\bar{\imath}(1)])\} + \cdots + \sum_{\substack{1\le i(1)<\cdots<i(l)\le n \\ \left.\begin{smallmatrix} 1\le j(1)\le m \\ \vdots \\ 1\le j(l)\le m \end{smallmatrix}\right\} j(1)\neq\cdots j(l)}} \delta(q[i(1)],p[j(1)])\cdots\delta(q[i(l)],p[j(l)]) \\ \bar{R}_k\{p(j[1;\bar{\jmath}(1),\cdots,\bar{\jmath}(l)]),\cdots p(j[(m-l);\bar{\jmath}(1),\cdots,\bar{\jmath}(l)]); \\ q(i[1;\bar{\imath}(1),\cdots,\bar{\imath}(l)]),\cdots q(i[(n-l);\bar{\imath}(1),\cdots,\bar{\imath}(l)])\}\Big] \end{aligned} \tag{12}$$

. . . etc. (series ends as in Lemma 6).

IV. ITERATIVE EXPANSION OF $R_k(0;q)$ TO SIXTH ORDER

From the integral equations, Eqs. (9) and (12), an iterative expansion of $R_k(0;q)$ can be performed. That this can be done is apparent from Eqs. (7), (9), and (12) and from the observation that $R_k(0;N)$ includes only powers of the coupling constant of order $(n+1)$ or higher. Therefore, to any order g^n, Eq. (7) can be rigorously terminated at some finite value N. Similarly, the other integral equations involved in the iteration can be rigorously cut at some finite point in the expansion in the boson numbers M, N. Thus the expression for $R_k^{(2)}(0;q)$ can be trivially seen to include contributions from $R_i(0;0)$ terms only. From Eq. (7) it is obvious that

$$R_k^{(2)}(0;q) = \tfrac{1}{2}g^2\omega_k^{-\frac{1}{2}}\omega_q^{-\frac{3}{2}} \times[\langle 0|\tau_\beta|0\rangle\langle 0|\tau_\alpha|0\rangle - \langle 0|\tau_\alpha|0\rangle\langle 0|\tau_\beta|0\rangle]. \tag{13}$$

τ is a T operator with respect to the total isotopic angular momentum,[16] and $g\langle 0|\tau_\alpha|0\rangle = g_\rho(0|\tau_\alpha|0)$ where g_ρ is a new constant called the "physical" or, in the more common terminology, the "renormalized" coupling constant. Equation (13) then becomes

$$R_k^{(2)}(0;q) = \tfrac{1}{2}g_\rho^2\omega_k^{-\frac{1}{2}}\omega_q^{-\frac{3}{2}}(\tau_\beta\tau_\alpha - \tau_\alpha\tau_\beta). \tag{13a}$$

In Eq. (7), $R_k(0;N)$ terms with $N \geqslant 2$ can contribute in no term lower than sixth, hence the only contributions to $R_k^{(4)}(0;q)$ are

$$R_k^{(4)}(0;q) = -\sum_{\kappa}\left[\frac{R_q^{(2)}(0;\kappa)R_k^{(2)\dagger}(0;\kappa)}{\omega_\kappa+\omega_q} - \frac{R_k^{(2)}(0;\kappa)R_q^{(2)\dagger}(0;\kappa)}{\omega_\kappa-\omega_q-i\eta}\right]. \tag{14}$$

Inserting (13a) into (14), we have

$$R_k^{(4)}(0;q) = \frac{(g_\rho)^4}{8\pi^2(\omega_k\omega_q)^{\frac{1}{2}}}\left[\int_0^\infty \frac{\kappa^2 d\kappa}{\omega_\kappa^3(\omega_\kappa+\omega_q)}\Gamma_A + \int_0^\infty \frac{\kappa^2 d\kappa}{\omega_\kappa^3(\omega_\kappa-\omega_q-i\eta)}\Gamma_B\right]. \tag{15}$$

Γ_A and Γ_B are sums of products of τ operators, and according to the sequence of τ's in a particular product it can be associated with a Feynman graph. Thus

$$\Gamma_A = \textstyle\sum_\lambda (\tau_\beta\tau_\lambda\tau_\lambda\tau_\alpha + \tau_\beta\tau_\lambda\tau_\alpha\tau_\lambda - \tau_\lambda\tau_\beta\tau_\alpha\tau_\lambda + \tau_\lambda\tau_\beta\tau_\lambda\tau_\alpha).$$

These component parts of Γ_A correspond to diagram IV_{A2}, IV_{A1}, IV_{A4}, and IV_{A3}, respectively. Similarly, Γ_B corresponds to equivalent B graphs. The values of Γ_A and Γ_B are

$$\Gamma_A = -2(3\tau_\alpha\tau_\beta + \tau_\beta\tau_\alpha), \tag{16a}$$

$$\Gamma_B = -2(3\tau_\beta\tau_\alpha + \tau_\alpha\tau_\beta). \tag{16b}$$

To compute $R_k^{(6)}(0;q)$ it is necessary to include the following contributions:

$$R_k^{(6)}(0;q) = -\sum_{\kappa}\frac{R_q^{(2)}(0;\kappa)R_k^{(4)\dagger}(0;\kappa)+R_q^{(4)}(0;\kappa)R_k^{(2)\dagger}(0;\kappa)}{\omega_\kappa+\omega_q} - \sum_{\kappa,\kappa'}\frac{R_q^{(3)}(0;\kappa,\kappa')R_k^{(3)\dagger}(0;\kappa,\kappa')}{\omega_\kappa+\omega_{\kappa'}+\omega_q} + \text{C.S.}, \tag{17}$$

[16] E. U. Condon and G. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1935), Chap. 3.

where C.S. stands for the terms related to the ones explicitly written by the crossing operation.

The "two-meson" contributions are in turn iterated to third order from Eq. (9); only the lowest order iterative terms, $R_i(0;0)$, can contribute to the right side of (9) in this case.

The expression for $R_k^{(6)}(0;q)$ is a sum of terms each of which is a product of an integral and a coefficient which can be regarded as a sum of hexalinear products of τ operators. These coefficients correspond to 6th order Feynman graphs in precisely the same fashion in which the sums of tetralinear products correspond to 4th order graphs. There are terms in the iteration corresponding to 60 graphs. (The 6th order in the linear scattering theory gives 90 graphs, but in the iteration here developed, the 30 graphs, which in the old theory correspond to the ones called "improper" or external self-energy graphs, never arise.) Of these 60 graphs one half are related to the other half by the crossing symmetry; of the 30 graphs in which emission precedes absorption, some are related to others by reflection in a line perpendicular to the nucleon propagator, and correspond to identical matrix elements. There remain then 19 independent graphs. The integrals in terms of which they are given are

TABLE I. Contributions to $R_k^{(6)}(0;q)$.

Graph	Contribution
VI-1	$-\tau_\alpha\tau_\beta\mathfrak{A}_1$
VI-2	$-3\tau_\alpha\tau_\beta\mathfrak{A}_1$
VI-3	$-\tau_\alpha\tau_\beta(\mathfrak{A}_1+\mathfrak{B}_2)$
VI-4	$-\frac{3}{2}\tau_\alpha\tau_\beta\mathfrak{B}_1$
VI-5	$-\tau_\alpha\tau_\beta(\mathfrak{A}_2-\frac{1}{2}\mathfrak{B}_1)$
VI-6	$-3\tau_\alpha\tau_\beta\mathfrak{A}_2$
VI-7	$5\tau_\alpha\tau_\beta(\mathfrak{A}_1-\mathfrak{B}_1+2\mathfrak{B}_2)$
VI-8	$-\tau_\alpha\tau_\beta\mathfrak{A}_2$
VI-9	$-(\tau_\alpha\tau_\beta+2\tau_\beta\tau_\alpha)\mathfrak{A}_1$
VI-10	$-3\tau_\alpha\tau_\beta\mathfrak{A}_1$
VI-11	$-(\tau_\alpha\tau_\beta+2\tau_\beta\tau_\alpha)(\mathfrak{A}_1+\frac{1}{2}\mathfrak{B}_1)$
VI-12	$3(\tau_\alpha\tau_\beta+2\tau_\beta\tau_\alpha)\mathfrak{B}_1$
VI-13	$(\tau_\alpha\tau_\beta-2\tau_\beta\tau_\alpha)(\mathfrak{A}_1-\frac{1}{2}\mathfrak{B}_2)$
VI-14	$-(5\tau_\alpha\tau_\beta+4\tau_\beta\tau_\alpha)(\mathfrak{B}_1-\frac{1}{2}\mathfrak{B}_2)$
VI-15	$-(\tau_\alpha\tau_\beta-4\tau_\beta\tau_\alpha)\mathfrak{B}_2$
VI-16	$-(\tau_\alpha\tau_\beta+4\tau_\beta\tau_\alpha)(\mathfrak{A}_2-\mathfrak{B}_1)$
VI-17	$9\tau_\alpha\tau_\beta(\mathfrak{B}_1-2\mathfrak{B}_2)$
VI-18	$-3\tau_\alpha\tau_\beta(\mathfrak{A}_2-\mathfrak{B}_2)$
VI-19	$-9\tau_\alpha\tau_\beta\mathfrak{A}_2$

$$\mathfrak{A}_1=\frac{(g_\rho)^6}{32\pi^4(\omega_q\omega_k)^{\frac{1}{2}}}\int_0^\infty\int_0^\infty\frac{\kappa^2d\kappa(\kappa')^2d\kappa'}{\omega_\kappa{}^2\omega_{\kappa'}{}^3(\omega_\kappa+\omega_{\kappa'})(\omega_\kappa+\omega_q)},$$

$$\mathfrak{A}_2=\frac{(g_\rho)^6}{32\pi^4}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}\left[\int_0^\infty\frac{\kappa^2d\kappa}{\omega_\kappa{}^3(\omega_\kappa+\omega_q)}\right]^2,$$

$$\mathfrak{B}_1=\frac{(g_\rho)^6}{32\pi^4(\omega_q\omega_k)^{\frac{1}{2}}}\int_0^\infty\int_0^\infty\frac{\kappa^2d\kappa(\kappa')^2d\kappa'}{\omega_\kappa{}^3\omega_{\kappa'}{}^3(\omega_\kappa+\omega_{\kappa'}+\omega_q)},$$

$$\mathfrak{B}_2=\frac{(g_\rho)^6}{32\pi^4(\omega_q\omega_k)^{\frac{1}{2}}}\int_0^\infty\int\frac{\kappa^2d\kappa(\kappa')^2d\kappa'}{\omega_\kappa{}^2\omega_{\kappa'}{}^2(\omega_\kappa+\omega_{\kappa'})^2(\omega_\kappa+\omega_{\kappa'}+\omega_q)}.$$

[The $\mathfrak{A}$ integrals arise from one-meson, the $\mathfrak{B}$ integrals from two-meson contributions to $R_k(0;q)$.] The contributions to the various graphs are given in Table I and Fig. 2.

Iterations to higher order become more tedious, though simple in principle. It is, incidentally, worth noticing that the iteration is far less tedious in this formalism than in the old linear scattering theory. Although the analysis of contributions in terms of Feynman graphs can be given, it is not a necessary or even a helpful step in the iteration procedure. A much more straightforward procedure is to sum over the τ operators in each order of the iteration, so that the identification of contributions as originating from specific graphs is lost. Not only does this greatly simplify the formal iteration procedure, but the number of independent terms is far smaller and increases far less rapidly in successive iterative orders than when the renormalized linear scattering theory is applied. The number of terms in 4th order in the linear theory is 12, whereas here there are 2. In 6th order there are 90 diagrams in the linear theory, whereas here there are 8 independent terms. The problem of determining a lower bound for the radius of convergence of the series seems therefore much less forbidding than in the old linear theory,

V. COMPARISON WITH THE RENORMALIZED LINEAR SCATTERING THEORY

When the series expansion of the R matrix is written in terms of contributions from specific graphs, it is simple to compare it with the results of renormalizing the T matrix in the linear scattering theory. The procedure for this latter calculation is as follows[17]: In each superficially convergent graph (such as II_A, IV_{A4}, VI-14), the vertex factors and the self-energy factors are replaced by the finite, renormalized vertex and self-energy parts. These, in turn, are generated from irreducible parts, which are obtained, in the case of vertex and the so-called "internal vertex" parts, from

[17] A. Lenard (unpublished). G. Chew, Phys. Rev. **94**, 1749 (1954); Jauch and Rohrlich, reference 4.

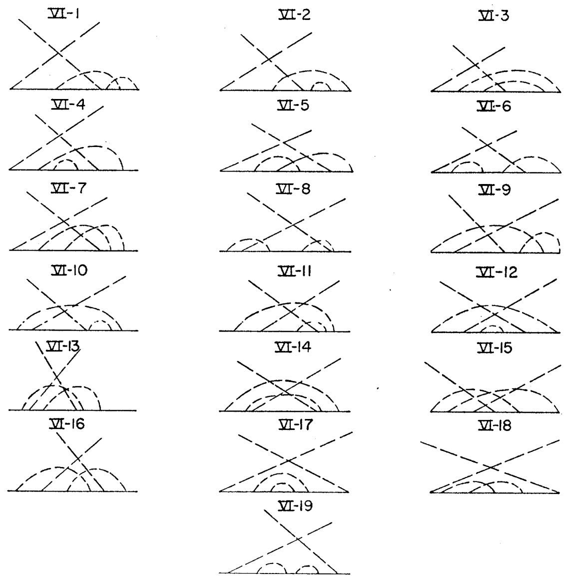

FIG. 2. Diagrammatic representation of sixth-order iteration of $R_k(0;q)$. Graphs related to the ones shown by reflection in a line perpendicular to nucleon propagator or by crossing symmetry are not shown. Note the absence of "external self-energy" graphs.

the primitive graphs by subtracting out the zero-energy and zero energy-transfer parts. The reducible diagrams are constructed by inserting the finite higher-order vertex and self-energy parts into irreducible graphs and performing the same subtractions again. Self-energy and internal vertex parts are related by Ward's identity; the connection between external self-energy graphs (Figs. $\mathrm{IV}_{A5,6}$) and wave-function renormalization follows from the imposition of unitarity on the renormalized T matrix. The procedure and proof of its validity are essentially identical to the case of quantum electrodynamics and will, therefore, not be discussed here in detail.

The renormalization of this theory to sixth order was performed with the following results: On the energy shell, the renormalized T matrix is identical with the R matrix to the same order; i.e., a table constructed on the basis of the standard renormalization procedure is identical, in detail, to Table I, and similarly for tables for $(g_\rho)^2$, $(g_\rho)^4$.

VI. DEMONSTRATION THAT $R_k^{(n)}(0;q)$ IS A CONVERGENT INTEGRAL

Let us consider the integral equation, Eq. (7), and use Eq. (9) to make the E_n dependence of the former explicit; this can be done since the $\omega_{q(i)}$ dependence of matrix elements $R_{q(i)}(M;N)$ is trivial.[18] The typical integral that results when the indicated summation in Eq. (7) is performed is

$$
\begin{aligned}
I(E_1,\cdots,E_n;E_1'\cdots,E_n',\omega_p)
&=\int_1^\infty\cdots\int_1^\infty d\omega_1\cdots d\omega_n(\omega_1^2-1)^{\frac{1}{2}}\cdots(\omega_n^2-1)^{\frac{1}{2}}[(\omega_{\alpha(1)}+E_1)\cdots(\omega_{\alpha(1)}+\cdots+\omega_{\alpha(l)}+E_l)\\
&\quad\times(\omega_{\alpha(l+1)}+\cdots+\omega_{\alpha(n)}-E_{(l+1)}-i\eta)\cdots(\omega_{\alpha(n)}-E_n-i\eta)(\omega_{\beta(1)}+E_1')\cdots(\omega_{\beta(1)}+\cdots+\omega_{\beta(l')}+E_{l'}')\\
&\qquad\times(\omega_{\beta(l'+1)}+\cdots+\omega_{\beta(n)}-E'_{(l'+1)}+i\eta)(\omega_{\beta(n)}-E_n'+i\eta)(\omega_1+\cdots+\omega_n+\omega_p)]^{-1},
\end{aligned}\tag{18}
$$

where $\alpha_1,\cdots,\alpha_n$ and $\beta_1,\cdots,\beta_n$ are permutations of $1,\cdots,n$, and where $0\leqslant l\leqslant n$, $0\leqslant l'\leqslant n$. Examination of this integral shows that each of the factors in the denominator contains a linear combination of ω_α's or ω_β's, which is linearly independent of all other linear combinations of the ω_α's or ω_β's, respectively. We can write

$$I=\int_1^\infty\cdots\int_1^\infty d\omega_1\cdots d\omega_n\,F(\omega_1,\cdots,\omega_n,\omega_p)G(\omega_1,\cdots,\omega_n).\tag{18a}$$

Here

$$
\begin{aligned}
F&=[\omega_1\cdots\omega_n(\omega_1+\cdots+\omega_n+\omega_p)]^{-1},\\
G&=(1-1/\omega_1^2)^{\frac{1}{2}}\cdots(1-1/\omega_n^2)^{\frac{1}{2}}\mathfrak{G}_a\mathfrak{G}_b^*,
\end{aligned}
$$

and the $\mathfrak{G}_l$ are given by

$$
\begin{aligned}
\mathfrak{G}_l=\Bigg\{&\left(1+\frac{E_1}{\omega_{\gamma(1)}}\right)\cdots\left(1+\frac{\omega_{\gamma(1)}+\omega_{\gamma(2)}+\cdots+\omega_{\gamma(l-1)}+E_l}{\omega_{\gamma(l)}}\right)\\
&\times\left(1+\frac{\omega_{\gamma(l+2)}+\cdots+\omega_{\gamma(n)}-E_{(l+1)}-i\eta}{\omega_{\gamma(l+1)}}\right)\cdots\left(1-\frac{E_n-i\eta}{\omega_{\gamma(n)}}\right)\Bigg\}^{-1}.
\end{aligned}
$$

It can be shown that I is a convergent integral by demonstrating that I_0, where

$$I_0=\int_1^\infty\cdots\int_1^\infty d\omega_1\cdots d\omega_n\,F(\omega_1,\cdots,\omega_n,\omega_p),$$

is convergent and by subsequently showing that the behavior of $\mathfrak{G}$ cannot interfere with the convergence of I_0. The convergence of I_0 is easily demonstrable.

[18] $R_{q(i)}(0;N)$ and $R_{q(i)}(M;0)$ are special cases of $R_{q(i)}(M';N')$ as examination of Eqs. (9) and (12) demonstrate.

To demonstrate that $\mathfrak{G}$ cannot make I diverge, we rewrite the entire integral in the variables $\mathfrak{R}$, z_1, z_2, $\cdots$, $z_{(n-1)}$ where

$$\begin{aligned}\omega_1 &= \mathfrak{R}z_1,\\ &\vdots\\ \omega_{(n-1)} &= \mathfrak{R}z_{(n-1)},\\ \omega_n &= \mathfrak{R}(1-z_1^2-\cdots-z_{(n-1)}^2)^{\frac{1}{2}}.\end{aligned}$$

When $\mathfrak{G}$ is written in these variables it is obvious that $\mathfrak{G}$ is bounded as $\mathfrak{R}\to\infty$. As functions of the linearly independent sums of ω_i, $\mathfrak{G}_a$ and $\mathfrak{G}_b$ have poles which, because of the aforementioned linear independence of the factors in the energy denominators of I are at most simple poles, not coincident with the axes of the ω hyperspace along which F is unbounded.[19] G, therefore, cannot inhibit the convergence of I, except perhaps for terms like $(W-E-i\eta)^{-1}(W-E'+i\eta)^{-1}$, where W is a linear combination of ω's identical in the α and β permutations. In such a case the following factor appears in the integral:

$$(E-E'+i\eta)^{-1}[(W-E-i\epsilon_1)^{-1}-(W-E'+i\epsilon_2)^{-1}].$$

The integration over the ω's leaves

$$(E-E'+i\eta)^{-1}[\Pi(E)+\Delta(E)-\Pi(E')+\Delta(E')],$$

where the Π's are contributions from the principal-value parts and the Δ's from the δ-function parts of integral. Since the Π's and Δ's are well behaved, the subsequent integration over E in the next iteration will again lead to finite results.

After Eq. (9) has been substituted into Eq. (7) the latter contains terms like $R_{q(\alpha)}(N';N)R_{q(\beta)}(N;N'')$ with appropriate energy denominators, and calls for integration over all the energy variables E_n. Iteration of $R_k(0;q)$ is performed by repeated substitution for the $R_{q(\gamma)}(M;N)$ from Eq. (12), always to the appropriate order in the coupling constant. It will be shown that all the integrals that appear in this iterative process are finite and that the integrals that arise in any iterative order are such that the subsequent integrations over the energy variables in the next order of iteration are again finite.

The typical integral that appears when the iteration of $R_k(0;q)$ is carried out is

$$J^\lambda(E_1,\cdots,E_{2(n+m)})=\int_1^\infty\cdots\int_1^\infty d\omega_{q(1)}\cdots d\omega_{q(n)}\,\mathfrak{F}^{(\lambda-1)}\mathcal{G}, \tag{19a}$$

where

$$\mathcal{G}=(1-\omega_{q(1)}^{-2})^{\frac{1}{2}}\cdots(1-\omega_{q(n)}^{-2})^{\frac{1}{2}}\gamma_A\gamma_B, \tag{19a}$$

and

$$\begin{aligned}\gamma_L=\omega_{q(1)}\cdots\omega_{q(n)}\Big[&(\omega_{q[1,1]}+E_1)\cdots(\omega_{q[1,1]}+\cdots+\omega_{q[1,r(1)]}+E_{r(1)})(\omega_{q[1,1]}+\cdots+\omega_{q[1,r(1)]}-\omega_{p[1,1]}+E_{r(1)+1}-i\eta)\cdots\\ &\times(\omega_{q[1,1]}+\cdots+\omega_{q[1,r(1)]}-\omega_{p[1,1]}-\cdots-\omega_{p[1,s(1)]}+E_{r(1)+s(1)}-i\eta)\cdots(\omega_{q[1,1]}+\cdots+\omega_{q[1,r(1)]}\\ &+\cdots+\omega_{q[l,1]}+\cdots+\omega_{q[l,r(l)]}-\omega_{p[1,1]}-\cdots-\omega_{p[1,s(1)]}-\cdots-\omega_{p[l',1]}-\cdots-\omega_{p[l',s(l')]}+E_{\Sigma\{r(i)+s(j)\}}-i\eta)\\ &\times(\omega_{p[l'+1,1]}+\cdots+\omega_{p[l'+1,s(l'+1)]}+\cdots+\omega_m-\omega_{q[l+1,1]}-\cdots-\omega_{q[l+1,r(l+1)]}-\cdots-\omega_{q(n)}+E_{\Sigma\{r(i)+s(j)\}}+i\eta)\cdots\\ &\times\left(\left\{\begin{matrix}+\omega_{p(m)}\\-\omega_{q(n)}\end{matrix}\right\}+E_{n+m}+i\eta\right)\Big]^{-1},\end{aligned} \tag{19b}$$

$$\mathfrak{F}^{(\lambda-1)}=[\omega_{q(1)}\cdots\omega_{q(n)}]^{-1}J^{(\lambda-1)}(\omega_{q(1)},\cdots,\omega_{q(n)},\omega_k), \tag{19c}$$

and $J^{(\lambda-1)}$ originates from an earlier iteration. In the special case of the first iteration $J^{(\lambda-1)}=I$.

Here again, if the integral

$$J_0{}^\lambda=\int_1^\infty\cdots\int_1^\infty d\omega_1\cdots d\omega_n\,\mathfrak{F}^{(\lambda-1)}(\omega_1,\cdots,\omega_n,\omega_p)$$

converges then it is easy to show that $\mathcal{G}$ cannot hinder the convergence of J. The argument is a simple extension of the one made before and will not be repeated.

In order to study the high-frequency behavior of J^λ, the fact that $\int_a^\infty f(x)x^{-(1-\epsilon)}dx$ is finite if and only if $f(x)\to 0$

[19] The limits of integration are such that the ω_i axes are outside tne domain of integration, thongh for sufficiently large R they can be arbitrarily close to $z_i=0,1$.

more rapidly than $x^{-\epsilon}$ for large x will be made use of. If we define

$$K_J{}^\lambda=\int_1^\infty\cdots\int_1^\infty J^\lambda(E_1,\cdots,E_N)E_1{}^{-[1-\epsilon(1)]}\cdots E_N{}^{-[1-\epsilon(N)]}dE_1\cdots dE_N,\tag{20}$$

then if $K_J{}^\lambda$ is finite, $J^\lambda\to 0$ at least as rapidly as $(E_1{}^{\alpha(1)}\cdots E_n{}^{\alpha(n)})^{-1}$ where each $\alpha_i>\epsilon_i$. We will examine $K_J{}^\lambda$ for one of the previously defined J^λ. The integration over the E space will be performed by integrating to an upper limit ρ and then later letting $\rho\to\infty$ so that the integrations over the E_i and the ω_i can be interchanged.

$K_J{}^\lambda$ will then be written

$$K_J{}^\lambda=\lim_{\rho\to\infty}\int_1^\infty\cdots\int_1^\infty d\omega_1\cdots d\omega_n\,\mathfrak{F}^{\lambda-1)}\mathfrak{L}(\rho),\tag{20a}$$

where

$$\mathfrak{L}=\int_1^\rho\cdots\int_1^\rho dE_1\cdots dE_n[E_1{}^{[1-\epsilon(1)]}\cdots E_n{}^{[1-\epsilon(n)]}]^{-1}\mathfrak{G}(\omega_1,\cdots,\omega_n;E_1,\cdots,E_n),\tag{21}$$

with $\rho\gg 1$. The integrations over E space in Eq. (21) consists of evaluating a set of integrals like

$$\mu=\int_1^\rho dE[E^{(\epsilon-1)}(\Lambda\mp E)^{-1}],\tag{22}$$

where Λ is a linear combination of ω's. Equation (22) can be written

$$\mu=\Lambda^{-(1-\epsilon)}\nu_\pm,\tag{22a}$$

where

$$\nu_\pm=\epsilon^{-1}\int_{\Lambda^{-1/\epsilon}}^{\rho'}\frac{du}{(u^{1/\epsilon}\pm 1)},\tag{22b}$$

and where ν_- is understood as the principal value (since the contributions from the pole cannot affect the validity of this argument).

We can without loss of generality let $\epsilon=1/n$ where n is an even integer. Then

$$\nu_+=2\sum_{l=0}^{(n-2j)/4}\left\{\tfrac{1}{2}(\cos\alpha_l)\ln\left[\frac{u^2-2u\cos\alpha_l+1}{u^2+2u\cos\alpha_l+1}\right]-(\sin\alpha_l)\left[\tan^{-1}\left(\frac{u-\cos\alpha_l}{\sin\alpha_l}\right)+\tan^{-1}\left(\frac{u+\cos\alpha_l}{\sin\alpha_l}\right)\right]\right\}\Bigg|_{u=\Lambda^{-n}}^{u=\rho'},\tag{23}$$

where $j=1$ if $n/2$ is odd, $j=0$ if $n/2$ is even, and $\alpha_l=[(2l+1)/n]\pi$, and

$$\nu_-=2\sum_{l=1}^{(n-2j)/4}\left\{\tfrac{1}{2}(\cos\beta_l)\ln\left[\frac{u^2-2u\cos\beta_l+1}{u^2+2u\cos\beta_l+1}\right]-(\sin\beta_l)\left[\tan^{-1}\left(\frac{u-\cos\beta_l}{\sin\beta_l}\right)+\tan^{-1}\left(\frac{u+\cos\beta_l}{\sin\beta_l}\right)\right]\right\}\Bigg|_{u=\Lambda^{-n}}^{u=\rho'},\tag{24}$$

where $\beta_l=[2l/n]\pi$. The $\nu_\pm$ are nonsingular functions of Λ which approach a constant as $\Lambda\to\infty$. As $\rho\to\infty$ they approach a finite limit smoothly, independently of their Λ dependence, demonstrating the interchangeability of the E and ω integrations.

In order to demonstrate that $K_J{}^\lambda$ converges and therefore, that $J^\lambda\to 0$ at least as quickly as $[E_1{}^{\epsilon(1)}\cdots E_n{}^{\epsilon(n)}]^{-1}$ where all $\epsilon_i>0$, we can write

$$K_J{}^\lambda=\int_1^\infty\cdots\int_1^\infty d\omega_1\cdots d\omega_n\,f^{(\lambda-1)}g,$$

where

$$f^\mu=\Lambda_1{}^{[\epsilon(1)-1]}\cdots\Lambda_n{}^{[\epsilon(n)-1]}J^\mu(\omega_1,\cdots,\omega_n),$$

and where g is a function that cannot impede the convergence of $K_J{}^\lambda$. Hence, it will suffice to show that

$$(K_J{}^\lambda)_0=\int_1^\infty\cdots\int_1^\infty d\omega_1\cdots d\omega_n\,f^{(\lambda-1)}$$

converges.

The function f^{μ} is given by $f^{\mu}=\phi(q,p)\phi(q,p')J^{\mu}$, where

$$\begin{aligned}\phi(q,p)=(\omega_{q(1)}\cdots\omega_{q(n)})^{\frac{1}{2}}\Big[&(\omega_{q[1,1]})^{[1-\epsilon(1)]}\cdots(\omega_{q[1,1]}+\cdots+\omega_{q[1,r(1)]})^{[1-\epsilon(r_1)]}(\omega_{q[1,1]}+\cdots+\omega_{q[1,r(1)]}\\&-\omega_{p[1,1]}-i\eta)^{[1-\epsilon(r_1+1)]}\cdots(\omega_{q[1,1]}+\cdots+\omega_{q[1,r(1)]}-\omega_{p[1,1]}-\cdots-\omega_{p[1,s(1)]}-i\eta)^{[1-\epsilon(r_1+s_1)]}\cdots\\&\times(\omega_{q[1,1]}+\cdots+\omega_{q[1,r(1)]}+\cdots+\omega_{q[l,1]}+\cdots+\omega_{q[l,r(l)]}-\omega_{p[1,1]}-\cdots\omega_{p[1,s(1)]}-\cdots\\&-\omega_{p[l',s(l')]}-i\eta)^{[1-\epsilon(\Sigma(r_i+s_j))]}\cdots(\omega_{p[l'+1,1]}+\cdots+\omega_{p[l'+1,s(l'+1)]}+\cdots+\omega_{p[m]}-\omega_{q[l+1,1]}-\cdots\\&-\omega_{q[l+1,r(l+1)]}-\cdots-\omega_{q[n]}+i\eta)^{[1-\epsilon(\Sigma(r_i+s_j))]}\cdots\left(\begin{Bmatrix}+\omega_{p[m]}\\-\omega_{q[n]}\end{Bmatrix}+i\eta\right)^{[1-\epsilon(n+m)]}\Big]^{-1},\end{aligned}$$

so that $(K_J{}^{\lambda})_0$ can be written

$$(K_J{}^{\lambda})_0=\int_1^{\infty}\cdots\int_1^{\infty}d\omega_1\cdots d\omega_n\,\mathfrak{f}^{(\lambda-1)}\mathfrak{g},$$

where

$$\mathfrak{f}^{\mu}=\omega_1{}^{[\delta(1)-1]}\cdots\omega_n{}^{[\delta(n)-1]}J^{\mu}(\omega_1,\cdots,\omega_n),$$

and where $\mathfrak{g}$ is a function which cannot inhibit the convergence of $(K_J{}^{\lambda})_0$; $\delta_1,\cdots,\delta_n$ is a set of numbers such that all the $\delta_i>0$ if all the $\epsilon_i>0$, and vice versa.

We can now compare $\int_1^{\infty}\cdots\int_1^{\infty}d\omega_1\cdots d\omega_n\,\mathfrak{f}^{(\lambda-1)}$ with $J_0{}^{\lambda}$ and see that the condition that $J^{\lambda}\rightarrow 0$ more rapidly than $[E_1{}^{\epsilon(1)}\cdots E_n{}^{\epsilon(n)}]^{-1}$ is more stringent than the one for the convergence of J^{λ}. We therefore need to demonstrate only that the former is satisfied.

We note that the integral

$$U=\int_1^{\infty}\cdots\int_1^{\infty}\omega_1{}^{[\delta(1)-1]}\cdots\omega_n{}^{[\delta(n)-1]}(\omega_1\cdots\omega_n)^{-\alpha}d\omega_1\cdots d\omega_n$$

converges if $\delta_1+\cdots+\delta_n<\alpha$ where all $\delta_i\geqslant 0$. This can be shown by an induction on n, and by transforming each integral as in Eq. (22), (22b).

It is now easy to see that I_0 is convergent, since I_0 is given by U with all $\delta_i=0$ and $\alpha=1$. Since we can also easily choose a set of δ_i such that all $\delta_i>0$ and such that $\sum_i\delta_i<1$, there is a set of positive ϵ_i such that any $I\rightarrow 0$ at least as rapidly as $[E_i{}^{\epsilon(1)}\cdots E_n{}^{\epsilon(n)}]^{-1}$. Since we can then, in the next iteration, certainly again find a set of δ_j' such that all $\delta_j'>0$ and $\sum_j\delta_j'<\delta_i$ for any δ_i, there is a set of ϵ_j' such that any $J^{(1)}\rightarrow 0$ at least as rapidly as $[E_1{}^{\epsilon'(1)}\cdots E_n{}^{\epsilon'(n)}]^{-1}$.

Clearly, this argument can be repeated indefinitely often so that in general J^{λ} converges and $J^{\lambda}\rightarrow 0$ sufficiently fast to make any $J^{(\lambda+1)}$ from the next iteration converge. Hence all $R_k{}^{(n)}(0;q)$ are finite.

It is of interest to note that the condition $\alpha=1$ in I_0 is not necessary to demonstrate convergence of $R_k{}^{(n)}(0;q)$. Any value $\alpha>0$ would have sufficed. This circumstance permits us to set a bound on the high-frequency behavior of $R_k{}^{(n)}(0;q)$. If we evaluate

$$\mathcal{G}=\int_1^{\infty}F(\omega_1,\cdots,\omega_n,\omega_p)\omega_p{}^{-\gamma}d\omega_p,$$

we find that $\mathcal{G}=[\omega_1\cdots\omega_n(\omega_1+\cdots+\omega_n)^{\gamma}]^{-1}u(\omega_1+\cdots+\omega_n)$, where $u(x)\rightarrow C$ as $x\rightarrow\infty$ (and where C is a constant). Since, for any $\gamma>0$ the entire previous argument with $\alpha=\gamma$ goes through, $I\rightarrow 0$ more rapidly than $\omega_p{}^{(\gamma-1)}$ for large ω_p. Since $R_k{}^{(n)}(0;q)$ contains an extra power $\omega_q{}^{-1}$ from the trivial energy dependence of $R_q(0;N)$ and $R_k(N;0)$, all $R_k{}^{(n)}(0;q)\rightarrow 0$, for high energies, E, faster than $E^{-(2-\delta)}$ where δ is any number >0.

For example, the high-frequency dependence of $R_k{}^{(2)}(0;q)$ is $O[1/\omega_q)^2]$, of $R_k{}^{(4)}(0;q)$ is $O[\ln\omega_q/(\omega_q)^2]$, and of $R_k{}^{(6)}(0;q)$ contains terms of $O[(\ln\omega_q/\omega_q)^2]$.

VII. DISCUSSION

We have shown that the formulation of the scattering problem in terms of a kinematical formalism, which permits the use of the nonorthogonal physical-particle asymptotic states, leads to the following: an iterative perturbation procedure in which the S-matrix expansion is finite to all orders without any renormalization procedures, and in which it involves physical-particle parameters only. The theory is in agreement, at least up to sixth order, with the old renormalized linear scattering theory. Moreover, it was possible to demonstrate that at high energies, to all orders, $R_k{}^{(n)}(0;q)\rightarrow 0$

more rapidly than $\omega_q^{-(2-\gamma)}$ where γ is any number >0; thus, if $\sum_n R_k^{(n)}(0;q)$ is a convergent series, the cross section will approach 0, at high energies, more rapidly than $\omega_q^{-(2-\gamma)}$.

The same iteration procedure can also be applied to the nonrenormalizable gradient coupling theory

$$[V_{\mathbf{k},\alpha}=ig(2\omega_k)^{-\frac{1}{2}}\boldsymbol{\sigma}\cdot\mathbf{k}\tau_\alpha].$$

In this case, the individual matrix elements are still infinite, but diverge less badly, by one order, than the corresponding ones in the old unrenormalized theory. Quite clearly, the divergences which remain in this theory have a significance that the divergences which this theory avoids do not share. The latter have no physical basis, and reflect an inappropriate choice of asymptotic scattering states in the linear scattering formalism. The former, on the other hand, are symptomatic of more serious difficulties in the Hamiltonian. For example, in the case of the gradient coupling theory, such a difficulty may stem from the abandonment of the fermion pair states in the pseudoscalar theory with γ_5 coupling.

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APPENDIX

The proof of Lemmas 1–6 will be supplied here. The proofs will be based on the theorem of complete induction and in some cases on the following modification of the latter: If the statement is true for $N=1$, and if the statement for all $n<N$ implies the statement for N, then the statement is true.

Lemma 1

For $n=1$, the statement of this lemma is

$$(H-\omega_q)a^\dagger_{\mathbf{q}}|0\rangle=V_q|0\rangle. \tag{A-1}$$

This equation can be shown to be a trivial consequence of the commutation relations. These same commutation relations also imply

$$\begin{aligned}(H-\omega_{q(1)}-\cdots-\omega_{q(n+1)})&\varphi(\mathbf{q}_1,\cdots,\mathbf{q}_{(n+1)})\\ =(n+1)^{-\frac{1}{2}}[a^\dagger_{\mathbf{q}(n+1)}&(H-\omega_{q(1)}-\cdots-\omega_{q(n)})\\ &\times\varphi(\mathbf{q}_1,\cdots,\mathbf{q}_n)+V_{q(n+1)}\varphi(\mathbf{q}_1,\cdots,\mathbf{q}_n)].\end{aligned} \tag{A-2}$$

Rewriting $\varphi(\mathbf{q}_1,\cdots,\mathbf{q}_n)$ under the assumption that Lemma 1 is true for n, transforms Eq. (A-2) into a statement of the lemma for $(n+1)$, and the lemma is proven.

Lemma 2

For $n=1$, the statement of lemma 2 is

$$\begin{aligned}(H+\lambda)^{-1}a^\dagger_{\mathbf{q}}=a^\dagger_{\mathbf{q}}&(H+\lambda+\omega_q)^{-1}\\ &-(H+\lambda)^{-1}V_q(H+\lambda+\omega_q)^{-1}.\end{aligned} \tag{A-3}$$

The validity of Eq. (A-3) follows from a formal power series expansion of $(H+\lambda)^{-1}$ in powers of H/λ, commutation of $a^\dagger_{\mathbf{q}}$ and H, and regrouping of terms. Application of Eq. (A-3) also leads to the following:

$$\begin{aligned}(H+\lambda)^{-1}&a^\dagger_{\mathbf{q}(1)}\cdots a^\dagger_{\mathbf{q}(n+1)}\\ &=a^\dagger_{\mathbf{q}(n+1)}(H+\lambda+\omega_{q(n+1)})^{-1}a^\dagger_{\mathbf{q}(1)}\cdots a^\dagger_{\mathbf{q}(n)}\\ &\quad-(H+\lambda)^{-1}V_{q(n+1)}(H+\lambda+\omega_{q(n+1)})^{-1}\\ &\qquad\times a^\dagger_{\mathbf{q}(1)}\cdots a^\dagger_{\mathbf{q}(n)}.\end{aligned} \tag{A-4}$$

Repeated application of the statement of the lemma to cases in which the product of the $a^\dagger$'s to the right of $(H+\mu)^{-1}$ contains fewer factors than $(n+1)$, (μ is any C number) leads to

$$\begin{aligned}(H+\lambda)^{-1}&a^\dagger_{\mathbf{q}(1)}\cdots a^\dagger_{\mathbf{q}(n+1)}\\ =&\sum_{0\le l\le n}(-1)^{n-l}\sum_{P(i;j)}a^\dagger_{\mathbf{q}[i,1]}\cdots a^\dagger_{\mathbf{q}[i,l]}a^\dagger_{\mathbf{q}[n+1]}(H+\lambda+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}+\omega_{q[n+1]})^{-1}V_{q[j,1]}\\ &\times(H+\lambda+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}+\omega_{q[j,1]}+\omega_{q[n+1]})^{-1}V_{q[j,2]}\cdots V_{q[j,(n-l)]}(H+\lambda+\omega_{q(1)}+\cdots+\omega_{q(n+1)})^{-1}\\ &-\sum_{0\le l\le n}(-1)^{n-l}\sum_{P(i;j)}\Big[\sum_{0\le t\le l}(-1)^{l-t}\sum_{P(r;s)}a^\dagger_{\mathbf{q}[r,1]}\cdots a^\dagger_{\mathbf{q}[r,t]}(H+\lambda+\omega_{q[r,1]}+\cdots+\omega_{q[r,t]})^{-1}V_{q[s,1]}\\ &\times(H+\lambda+\omega_{q[r,1]}+\cdots+\omega_{q[r,t]}+\omega_{q[s,1]})^{-1}V_{q[s,2]}\cdots V_{q[s,(l-t)]}(H+\lambda+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]})^{-1}\Big]\\ &\times V_{q[n+1]}(H+\lambda+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}+\omega_{q[n+1]})^{-1}V_{q[j,1]}(H+\lambda+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}+\omega_{q[j,1]}+\omega_{q[n+1]})^{-1}\cdots\\ &\times V_{q[j,(n-l)]}(H+\lambda+\omega_{q(1)}+\cdots+\omega_{q(n+1)})^{-1}.\end{aligned} \tag{A-5}$$

Here $r_1,\cdots,r_t,s_1,\cdots,s_{(l-t)}$ is a permutation of $i_1,\cdots,i_l$. This, in turn, can be rewritten:

$$\begin{aligned}(H+\lambda)^{-1}a^\dagger_{\mathbf{q}(1)}\cdots a^\dagger_{\mathbf{q}(n+1)}=&\sum_{0\le l\le n+1}(1)^{n-l+1}\sum_{P(i;j)}a^\dagger_{\mathbf{q}[i,1]}\cdots a^\dagger_{\mathbf{q}[i,l]}(H+\lambda+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]})^{-1}\\ &\times V_{q[j,1]}(H+\lambda+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]}+\omega_{q[j,1]})^{-1}\cdots V_{q[j,(n+1-l)]}(H+\lambda+\omega_{q(1)}+\cdots+\omega_{q(n+1)})^{-1},\end{aligned} \tag{A-6}$$

and the lemma is proven.

Lemma 3

For $n=1$, the statement of the lemma is

$$\langle 0|a^\dagger{}_{\mathbf{q}}\Omega|0\rangle = -\langle 0|V_q(H+\omega_q)^{-1}\Omega|0\rangle. \tag{A-7}$$

This equation follows from Eq. (19), reference 14. Application of this equation and of Lemma 2 gives

$$\langle 0|a^\dagger{}_{\mathbf{q}(1)}\cdots a^\dagger{}_{\mathbf{q}(n+1)}\Omega|0\rangle = -\sum_{0\leq l\leq n}(-1)^{n-l}\sum_{P(i;j)}\langle 0|a^\dagger{}_{\mathbf{q}[i,1]}\cdots a^\dagger{}_{\mathbf{q}[i,l]}V_{q[n+1]}(H+\omega_{q[i,1]}+\cdots+\omega_{q[i,l]} +\omega_{q[n+1]})^{-1}\cdots V_{q[j,(n-l)]}(H+\omega_{q(1)}+\cdots+\omega_{q(n+1)})^{-1}\Omega|0\rangle. \tag{A-8}$$

Everything that appears to the right of $a^\dagger{}_{\mathbf{q}[i,l]}$ can be regarded as an operator Ω'; then the statement of the lemma can be applied to the right-hand side of Eq. (A-8), remembering that $l<n+1$. This leads to the statement of the lemma for the $(n+1)$ case and the lemma is proven.

Lemmas 4, 5

The proofs proceed identically to those for Lemmas 2 and 3, respectively.

Lemma 6

This will be proven by an induction on n. For $n=1$, the statement of the lemma is:

$$a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}a^\dagger{}_{\mathbf{q}(1)} = \sum_{1\leq j(1)\leq m}\delta(\mathbf{q}(1),\mathbf{p}[j,1])a\{\mathbf{p}(j[1;\bar{j}(1))\}\cdots \times a\{\mathbf{p}(j[(m-1);\bar{j}(1)])+a^\dagger{}_{\mathbf{q}(1)}a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}. \tag{A-9}$$

The validity of Eq. (A-9) follows simply from successive commutation of $a^\dagger{}_{\mathbf{q}(1)}$ and $a_{\mathbf{p}}$'s until $a^\dagger{}_{\mathbf{q}(1)}$ is absent or at the extreme left of the expression. Equation (A-9) also leads to the equation

$$a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}a^\dagger{}_{\mathbf{q}(1)}\cdots a^\dagger{}_{\mathbf{q}(n+1)} = a^\dagger{}_{\mathbf{q}(n+1)}a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}a^\dagger{}_{\mathbf{q}(1)}\cdots a^\dagger{}_{\mathbf{q}(n)} + \sum_{1\leq j(1)\leq m}\delta(\mathbf{q}(n+1),\mathbf{p}[j,1]) a\{\mathbf{p}(j[1;\bar{j}(1)])\}\cdots a\{\mathbf{p}(j[(m-1);\bar{j}(1)])\}a^\dagger{}_{\mathbf{q}(1)}\cdots a^\dagger{}_{\mathbf{q}(n)}. \tag{A-10}$$

Application of the statement of the lemma to products like $a_{\mathbf{p}(1)}\cdots a_{\mathbf{p}(m)}a^\dagger{}_{\mathbf{q}(1)}\cdots a^\dagger{}_{\mathbf{q}(l)}$ (with $l<n+1$), leads to a statement of the lemma for $(n+1)$, thereby proving the lemma. This can be seen in the following fashion: If we look at that part of the right-hand side of Eq. (A-10) which is α-linear in the $a^\dagger$'s and β-linear in the a's, we see that the contribution of the first term on the right-hand side of Eq. (A-10) contains only elements in which the $q(n+1)$ never occurs as an argument of a δ function; in contributions from the second term, however, $q(n+1)$ always appears only as an argument of a δ function. It is easy to see that the sum of the two terms just suffices to make Eq. (A-10) a statement of the lemma for $(n+1)$.