Generalized Angular Momentum in Many-Body Collisions*

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With short-range forces, initial and final states in a classical 3-body collision are straight-line trajectories into and out of a region where all three particles are close together at the same time. Using six coordinates, three describing the relative position of a pair of particles, and three the relative position of the third particle and the center of mass of the pair, the condition for simultaneous togetherness can be expressed with the help of the 6×6 *grand angular momentum tensor*, $\mathbf{\Lambda}$, whose components are $\Lambda_{ij}=(m_i/m_j)^{\frac{1}{2}}x_ip_j-(m_j/m_i)^{\frac{1}{2}}x_jp_i$. For a close 3-body collision $\Lambda^2=\frac{1}{2}\sum_{i,j}\Lambda_{ij}^2$ must be small. Λ^2 commutes with the ordinary angular momentum operators and with the kinetic energy; its eigenvalues are $\lambda(\lambda+4)\hbar^2$, with integral λ, and its eigenfunctions hyperspherical harmonics. Initial and final 3-body states can be described quantally by the total energy E, Λ^2, and a commuting set of ordinary angular momenta; this description has the same relation to a momentum representation as the ordinary angular momentum analysis has for a 2-body collision. A collision of $(N+1)$ particles can be described by using a hierarchy of operators $\Lambda_n^2(2\leqslant n\leqslant N)$; their eigenvalues are $\lambda_n(\lambda_n+3n-2)\hbar^2$.

I. INTRODUCTION

Angular momentum and rotational symmetry in 3-dimensional space are intimately and indissolubly connected. But the value of analysis in terms of angular momentum, its conjugate angular coordinates, and its eigenfunctions transcends the limits of exact rotational symmetry, providing expansions, tools for computation, and insights in a galaxy of problems. In 2-body collisions, for instance, it provides a classification of initial and final states in terms of the particles' spins and the collisional angular momentum even though the separate conservation of some of these quantities breaks down in the region of close collision.

In problems involving three or more interacting bodies, it will often be found that important parts of the problem involve operators which are formally symmetric with respect to rotations in a space of six or more dimensions. Such operators are, for instance, the kinetic energy for the 3-body problem, and even the Hamiltonian for that problem in the absence of any interaction. True, the latter case appears trivial—but the extended symmetry of the problem leads to a generalization of angular momentum that provides a description for initial and final states in a 3-body collision with short-range interactions even though the quantity concerned is not conserved while the particles are close to one another. And it may be expected that this generalized angular momentum, together with its conjugate angular coordinates and its eigenfunctions, will lead to new insight and computational methods in other problems where the full symmetry does not persist.

Collisions involving 3 or more particles are often described formally in a momentum (plane-wave) representation.[1] Such a description would be directly applicable to the unscattered beams in an experiment where two collimated beams impinge on a relatively stationary gas. That experiment is rare, and it is much more usual to encounter 3-body collisions in statistical assemblages like a chemically reacting gas or a recombining plasma; the inverse process of 3-body or N-body breakup is important in many places, including ionization or dissociation by electronic, atomic, or molecular collisions, nuclear reactions, and high-energy events. These events can be discussed as wave packets in the plane-wave representation, but Delves[2] has shown that another representation, involving a new, "unphysical," angular coordinate and a new quantum number λ, is much more convenient. Similar coordinates have been invoked before,[3] notably in the problems of the helium atom[4,5] and of the threshold law for ionization by electron impact.[6] It is one of my aims in this paper to show that the quantum number λ in Delves's representation of 3-body states arises naturally from the generalization of angular momentum, and to give it further physical and intuitive significance.

In considering the possible collision of 3 bodies moving in space, it is natural to ask: If the particles continued undisturbed on their initial straight trajectories, how close would they come to colliding simultaneously at a point? This question arises most simply when the 3 particles are constrained to move along a line. Its answer can be found in quantities like $\Lambda_{ij}=(m_i/m_j)^{\frac{1}{2}}x_ip_j-(m_j/m_i)^{\frac{1}{2}}x_jp_i$ (where the coordinates are measured from the center of mass), which must vanish if the trajectories lead to an exact 3-body collision. A condition for a close (if not exact) 3-body collision course is that Λ_{ij}, or, more generally, $\Lambda^2=\frac{1}{2}\sum_{ij}(\Lambda_{ij})^2$ be small.

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[1] See E. Gerjuoy, Ann. Phys. **5**, 58 (1958) and literature cited therein; also J. M. Jauch, Helv. Phys. Acta **31**, 661 (1958), and I. I. Zinnes, Suppl. Nuovo cimento **12**, 87 (1959).

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[3] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, New York, 1953), p. 1730.

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[5] V. Fock, Izvest. Akad. Nauk S.S.S.R. Ser. Fiz. **18**, 161 (1954) [translation: Kgl. Norske Videnskab. Selskabs Forh. **31**, 138, 145 (1958).

[6] G. H. Wannier, Phys. Rev. **90**, 817 (1953).

Related to Λ^2 is a characteristic distance, analogous to the impact parameter for a 2-body collision. These relations persist for 3-body collisions in space, and even for collisions involving N particles, and there results an antisymmetric *grand angular momentum tensor* $\mathbf{\Lambda}$ with $3N-3$ rows and columns, embracing the ordinary 2-body angular momenta among its elements.

In quantum mechanics, one can construct from the elements of $\mathbf{\Lambda}$ a set of operators which commute with each other and the kinetic energy. They include the familiar angular momentum operators, and one or more new operators of the general type of Λ^2, which lead to a class of hyperspherical harmonics as their eigenfunctions.

In this paper, I shall develop the basic physical ideas and formalism of the grand angular momentum tensor, first in classical mechanics and then in quantum mechanics, and apply them to the description of 3-body collisions. The argument of the first section opens with a discussion of the motion of 3 particles on a line, including the classification of possible collisions among them and the first appearance of grand angular momentum, in a situation where the ordinary angular momentum is zero. There follows the treatment in full 3-dimensional space. An important preparatory point is the systematic use of normalized center-of-mass coordinates, which exhibit the symmetry of the kinetic energy and encourage its exploitation. Once the proper generalization of angular momentum has been found in the antisymmetric tensor $\mathbf{\Lambda}$, the development of its formal properties is straightforward.

In Sec. II, this development is continued with the quantal commutation rules and the construction of commuting sets of operators suitable for describing the 3-body system in regions where the interaction is negligible. This description has to the more familiar momentum representation the same relation as the angular momentum description has for 2-body collisions. It has various advantages, concentrating the focus on true 3-body collisions (small Λ^2), and providing solutions that are normalized in the same way as, and orthogonal to, 2-body solutions, so that they are particularly suited to describing processes like $A+BC\rightarrow A+B+C$.

The key notions leading to the concept of grand angular momentum seem to me to be the following: first, the use of a symmetric, normalized coordinate system; second, focusing attention on simultaneous closeness in a three-body collision (which is assisted by a position rather than a momentum representation); and third, the generalization of angular momentum as an antisymmetric tensor.

I. CLASSICAL MECHANICS

A. Classification of 3-Body Collisions

Three-body collisions involving short-range forces can be conceptually dissected into three stages: the approach, when the particles are moving without interaction; the collision proper, when the interaction influences the motion strongly; and the retreat. If a trajectory is thought of as a path in the 9-dimensional space defined by the coordinates of the 3 particles—or in the 6-dimensional space remaining when the motion of the center of mass of the system is eliminated—the approach and retreat trajectories are represented by straight lines. The collision, or interaction, stage need not be examined in detail here. We need only know that it converts an approach trajectory into some retreat trajectory, and that it involves one or more of these processes:

(*a*) No collision—approach and retreat trajectories the same;

(*b*) A 2-body collision only;

(*c*) Successive 2-body collisions, separated by a segment of straight trajectory without interaction;

(*d*) A 2-body collision between a stable compound and another particle;

(*e*) Complex 2-body collisions—an initial 2-body collision forms a metastable collision complex that survives long enough to collide with a third particle;

(*f*) A pure 3-body collision—the approach trajectory brings the three particles together directly into a region where all three are subject to forces of interaction.

Some of these events are illustrated in Fig. 1. The first three processes, (*a*), (*b*), (*c*), involve no true 3-body events. True 3-body collisions have trajectories passing through the central region of simultaneous 3-body interaction; entry and departure may each occur by any of the processes (*d*), (*e*), and (*f*)—in a rearrangement collision, for instance, both entry and departure occur by process (*d*).

If we ignore the process (*d*) and start with three separate particles, it is obvious that the chance of entering the region of 3-body interaction is the greater, the closer the initial trajectory is aimed at the origin in Fig. 1. For the initial trajectory (*e*) in the figure, the distance of closest approach, R_1, is a generalized impact parameter which must be small if a 3-body event is to be likely. In fact, R will be an appropriate parameter for classifying 3-body collision trajectories.

B. Normalized Center-of-Mass Coordinates

It is necessary here to specify the coordinate system in more detail. The positions of three particles in space are fixed by nine coordinates x_i^α, where $\alpha(=1, 2, 3)$ labels the particles, and $i(=1, 2, 3)$ the directions in a Cartesian coordinate space. $\mathbf{x}^\alpha$ is an ordinary 3-component vector, while $\bar{x}_i$ has the three components, x_i^1, x_i^2, x_i^3. With both affixes omitted, the symbol $\mathbf{x}$ represents a 9-component column vector, whose components may be represented by x_j, where the single suffix runs from 1 to 9 $[j=i+3(\alpha-1),\ x_{i+3(\alpha-1)}\equiv x_i^\alpha]$. The masses

of the particles are m^α, and their momenta are (classically) $\mathbf{p}^\alpha = m^\alpha d\mathbf{x}^\alpha/dt$.

It is convenient to make a transformation to a center-of-mass coordinate system. This can be done so that the volume element is unchanged, and the kinetic energy matrix becomes a diagonal form with a common reduced mass for all the internal coordinates of the center-of-mass system. Such a transformation leads to coordinates ξ_i^α:

$$\bar{\xi}_i = U\bar{x}_i, \quad U = \begin{Bmatrix} 0 & 1/d & -1/d \\ -d & \dfrac{m_2 d}{m_2+m_3} & \dfrac{m_3 d}{m_2+m_3} \\ m_1/M & m_2/M & m_3/M \end{Bmatrix}, \tag{1}$$

where

$$M = \textstyle\sum_\alpha m^\alpha, \quad M\mu^2 = \prod_\alpha m^\alpha, \quad \text{and} \quad m_2 m_3 d^2 = \mu(m_2+m_3).$$

The momenta transform to

$$\bar{\pi}_i = \tilde{U}^{-1}\bar{p}_i, \quad U^{-1} = \begin{Bmatrix} 0 & -d\mu/m_1 & 1 \\ \dfrac{\mu}{m_2 d} & \dfrac{d\mu}{m_2+m_3} & 1 \\ \dfrac{-\mu}{m_3 d} & \dfrac{d\mu}{m_2+m_3} & 1 \end{Bmatrix}, \tag{2}$$

and the kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}\sum_{i,\alpha} \frac{1}{m^\alpha}(p_i^\alpha)^2 \\ &= \tfrac{1}{2}\sum_{i=1}^{3} \left\{ \frac{1}{\mu}[(\pi_i^1)^2 + (\pi_i^2)^2] + \frac{1}{M}(\pi_i^3)^2 \right\}. \end{aligned} \tag{3}$$

Since the motion of the center of mass can always be separated out, we can henceforth assume that $\pi_i^3 = 0$ and $\xi_i^3 = 0$, and write simply

$$T = \frac{1}{2\mu}\sum_{j=1}^{6} (\pi_j)^2 = E. \tag{4}$$

In these normalized coordinates the kinetic energy is conveniently symmetric. This can be contrasted with its form in the common center-of-mass coordinates, obtained by setting $d=1$ in Eq. (1), where a different reduced mass μ^α appears for each value of α in the kinetic energy. To gain this symmetry, it is worth paying the small price that ordinary physical distances are measured not by $[\sum_i(\xi_i^\alpha)^2]^{\frac{1}{2}}$, but by $d[\sum_i(\xi_i^1)^2]^{\frac{1}{2}}$ and $d^{-1}[\sum_i(\xi_i^2)^2]^{\frac{1}{2}}$, respectively. With this caveat, it is still possible to say that $\boldsymbol{\xi}^\alpha$ describes a physical vector representing the relative positions of certain particles.[7]

The transformation of Eq. (1) is not the only one with the desired properties. This one has the additional feature that the vector $\boldsymbol{\xi}^1$ describes the relative motion of particles B and C, and $\boldsymbol{\xi}^2$ describes the relative motion of A and the center of mass of BC. There are two other such privileged coordinate systems, related to it by orthogonal transformations. The system $\{\boldsymbol{\xi}'\}$, where $\boldsymbol{\xi}'^1$ represents the relative motion A and C, and $\boldsymbol{\xi}'^2$ the motion of B with respect to AC, is related to $\{\boldsymbol{\xi}\}$ by

$$\bar{\xi}_i' = O''\bar{\xi}_i, \quad O'' = \begin{Bmatrix} \cos\beta'' & \sin\beta'' \\ -\sin\beta'' & \cos\beta'' \end{Bmatrix}, \tag{5}$$

where β'' is an obtuse angle such that

$$\tan^2\beta'' = \frac{m_C}{m_A} + \frac{m_C}{m_B} + \frac{m_C^2}{m_A m_B}. \tag{6}$$

This transformation may be called a *kinematic rotation*,[8] to distinguish it from ordinary rotations of the vectors in Cartesian space. A similar kinematic rotation leads to the coordinate system $\{\boldsymbol{\xi}''\}$. Figure 1 illustrates these coordinates, and the trajectories (or their projections), in the plane of ξ_1^1 and ξ_2^2. The momenta obviously transform in the same way as the coordinates.

When the collision of 4 or more particles is in question, it is again possible to set up a normalized center-of-mass coordinate system. In the general case, with N particles, the kinetic energy is a form like (4), with a reduced mass

$$\mu = \Big(\prod_{\alpha=1}^{N} m^\alpha \Big/ \sum_{\alpha=1}^{N} m^\alpha\Big)^{1/(N-1)}. \tag{7}$$

C. Collision on a Line

Let us now consider the collision of 3 particles on a line. This can be described by the normalized center-of-mass coordinates (ξ_1^1, ξ_1^2), and illustrated by a trajectory like (c) or (e) in Fig. 1. At first, the particles are far apart and approaching each other on a trajectory described by

$$\xi_1^\alpha(t) = \xi_1^\alpha(t_0) + (t-t_0)\mu^{-1}\pi_1^\alpha, \tag{8}$$

or

$$\bar{\xi}_1(t) = \bar{\xi}_1(t_0) + (t-t_0)\mu^{-1}\bar{\pi}_1. \tag{9}$$

[If we had not used the normalized coordinate system, a different reduced mass μ^α would have been associated with each value of α in Eq. (8), and it would have been impossible to write the simple vector equivalent, Eq. (9).] As expressed here, the problem is formally identical with the center-of-mass description of a 2-body collision in a plane. An initial trajectory aimed at the origin in Fig. 1 would lead to the simultaneous collision of 3 mass points (in the absence of a potential). The extent to which a trajectory misses being such a simultaneous collision course can be measured by a generalized

[7] Similar coordinates, often chosen so that $\mu=1$, have been used before. See, for instance, D. W. Jepsen and J. O. Hirschfelder, Proc. Natl. Acad. Sci. U. S. 45, 249 (1959).

[8] F. T. Smith, J. Chem. Phys. 31, 1352 (1959).

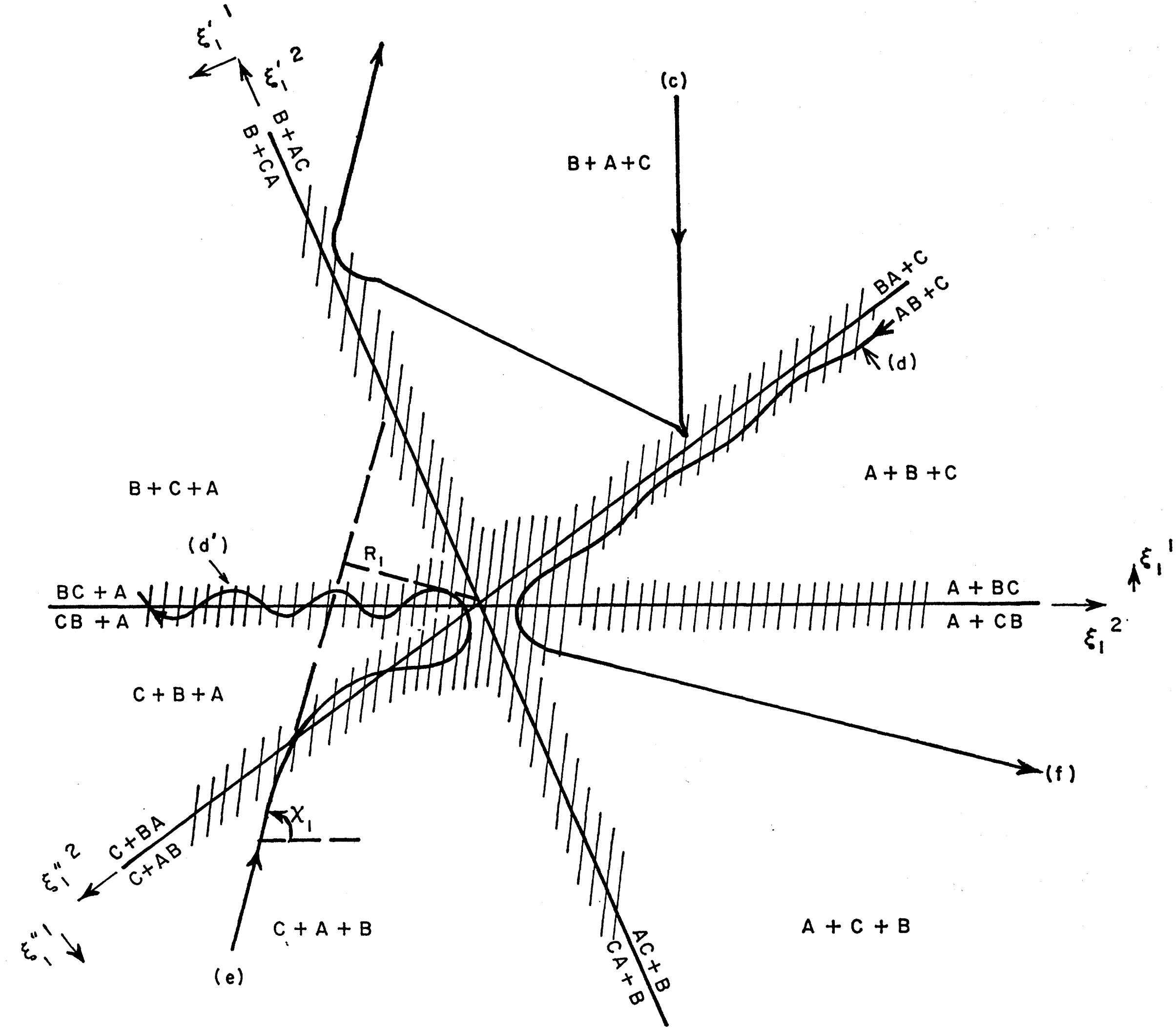

FIG. 1. Some possible 3-body trajectories. The trajectories lie in, or are projected onto, the plane ξ_1^1, ξ_1^2, where ξ_1^1 is the x component of the normalized vector between particles B and C, and ξ_1^2 is the x component of the normalized vector between their center of mass and A (the coordinates are described further in the text). The hatched regions are regions of interaction, where the potential is nonzero. (c) represents two successive 2-body collisions, (d) represents a vibrating molecule AB approaching collision with C, (f) represents 3 particles separating, starting from a region of pure 3-body interaction, (e) shows A and B colliding and revolving about each other long enough to collide with C, (d') shows a bound molecule BC rotating as it departs from A.

angular momentum,

$$\begin{aligned}\Lambda_{11}^{12} &= \xi_1^1(t)\pi_1^2 - \xi_1^2(t)\pi_1^1 \\ &= \xi_1^1\pi_1^2 - \xi_1^2\pi_1^1,\end{aligned} \tag{10}$$

which is independent of the time t, by Eq. (8), and also invariant under an orthogonal transformation of the coordinates. Alternatively, one can use the generalized impact parameter R_1; it is related to Λ_{11}^{12} by

$$(\Lambda_{11}^{12})^2 = 2\mu E R_1^2. \tag{11}$$

Clearly, if the potential has a finite range, a trajectory with R_1 greater than some value R_0 cannot lead to a pure 3-body collision.

It is worth noting that the angle χ_1, which gives the orientation of the trajectory (e) in Fig. 1, has a physical interpretation in the distribution of kinetic energy among the 3 colliding particles. In fact,

$$\begin{aligned}&E_A = E(1-m_A/M)\cos^2\chi_1, \\ &E_B = E(1-m_B/M)\cos^2\chi_1', \\ &\qquad (\chi_1' = \chi_1 - \beta'' = \chi_1 + \beta + \beta'), \\ &E_C = E(1-m_C/M)\cos^2\chi_1'', \quad (\chi_1'' = \chi_1 + \beta').\end{aligned} \tag{12}$$

The energies, of course, are all in the center-of-mass system.

D. Collision in Space

The collision of three particles in space can be dealt with by an immediate generalization of the notions of

the preceding section. If the initial straight-line trajectory in the 6-dimensional space $\{\xi_i\}$ is to lead to an exact 3-body collision (that is, pass through the origin of the coordinate system), all of the quantities

$$\Lambda_{ij}=\xi_i\pi_j-\xi_j\pi_i, \quad (i, j=1, \cdots, 6) \tag{13}$$

must vanish. This condition is equivalent to the vanishing of the single, positive definite, quantity

$$\Lambda^2=\tfrac{1}{2}\sum_{i,j}(\Lambda_{ij})^2. \tag{14}$$

Λ^2 is a suitable measure of the closeness with which the trajectory approaches a 3-body collision course.

The array of the Λ_{ij} forms an antisymmetric 6×6 tensor, which will be denoted $\mathbf{\Lambda}$ and called the *grand angular momentum tensor*. Λ^2, the total squared grand angular momentum, is an invariant of the straight-line trajectory, independent of the coordinate system. Λ^2 is generally *not* invariant in a collision with interaction, and its initial and final values may differ (this is true even when only two of the particles interact and the third passes by without interaction).

In the six-dimensional coordinate space, the distance from any point to the origin will be denoted ρ:

$$\begin{aligned}\rho^2&=\sum_{i=1}^{6}(\xi_i)^2\\&=\mu\{m_1^{-1}|\mathbf{x}^2-\mathbf{x}^3|^2+m_2^{-1}|\mathbf{x}^3-\mathbf{x}^1|^2+m_3^{-1}|\mathbf{x}^1-\mathbf{x}^2|^2\}\\&=(2\mu M)^{-1}\sum_{i,j=1}^{3}m^im^j|\mathbf{x}^i-\mathbf{x}^j|^2.\end{aligned} \tag{15}$$

(The first and last forms can be extended directly to define a generalized distance coordinate for the problem of N particles.) The associated momentum is

$$p_\rho=\mu d\rho/dt=\rho^{-1}\sum_i\xi_i\pi_i. \tag{16}$$

These identities follow immediately:

$$\sum_i\xi_i\Lambda_{ij}=\rho^2\pi_j-\rho p_\rho\xi_j, \tag{17}$$

$$\sum_j\Lambda_{ij}\pi_j=2\mu T\xi_i-\rho p_\rho\pi_i, \tag{18}$$

$$\Lambda^2=\rho^2(2\mu T-p_\rho^2). \tag{19}$$

The minimum value of ρ on a straight-line trajectory, say R, is the analog of the impact parameter of a 2-body collision, and will be called the *3-body impact distance* of the trajectory. It is related to the invariant Λ^2 by

$$\Lambda^2=2\mu TR^2. \tag{20}$$

$\mathbf{\Lambda}$ obviously includes ordinary angular momenta among its elements. We may write, as an alternative form of (17),

$$\Lambda_{ij}{}^{\alpha\beta}=\xi_i{}^\alpha\pi_j{}^\beta-\xi_j{}^\beta\pi_i{}^\alpha, \quad (i, j=1, 2, 3; \alpha, \beta=1, 2), \tag{21}$$

and

$$\mathbf{\Lambda}=\begin{Bmatrix}\mathbf{\Lambda}^{11} & \mathbf{\Lambda}^{12}\\ \mathbf{\Lambda}^{21} & \mathbf{\Lambda}^{22}\end{Bmatrix}. \tag{22}$$

$\mathbf{\Lambda}^{11}$ is just the usual angular momentum of relative motion of B and C expressed as a tensor, and $\mathbf{\Lambda}^{22}$ is the same for A and BC. The total angular momentum is

$$\mathbf{L}=\mathbf{\Lambda}^{11}+\mathbf{\Lambda}^{22}. \tag{23}$$

The other terms, $\mathbf{\Lambda}^{12}$ and $\mathbf{\Lambda}^{21}$ are more complicated; one is the negative transpose of the other, and they can be analyzed in terms of a symmetric tensor $\mathbf{\Sigma}$ and an antisymmetric tensor $\mathbf{A}$:

$$\begin{gathered}\mathbf{\Sigma}=\mathbf{\Lambda}^{12}-\mathbf{\Lambda}^{21}, \quad \mathbf{A}=\mathbf{\Lambda}^{12}+\mathbf{\Lambda}^{21},\\ \text{so}\\ 2\mathbf{\Lambda}^{12}=\mathbf{A}+\mathbf{\Sigma}, \quad 2\mathbf{\Lambda}^{21}=\mathbf{A}-\mathbf{\Sigma}.\end{gathered} \tag{24}$$

As will be shown below, $\mathbf{A}$ can also be identified with a combination of ordinary angular momenta. It is closely associated with the quantity $\mathbf{Y}$ defined by

$$\mathbf{Y}=\mathbf{\Lambda}^{11}-\mathbf{\Lambda}^{22}. \tag{25}$$

The full tensor $\mathbf{\Lambda}$ can now be written in the form

$$2\mathbf{\Lambda}=\begin{Bmatrix}\mathbf{L} & \mathbf{\Sigma}\\ -\mathbf{\Sigma} & \mathbf{L}\end{Bmatrix}+\begin{Bmatrix}\mathbf{Y} & \mathbf{A}\\ \mathbf{A} & -\mathbf{Y}\end{Bmatrix}, \tag{26}$$

which is the natural form for displaying the effect of the kinematic rotation, Eq. (5), to the new coordinate system $\{\xi_i'\}$. $\mathbf{L}$ and $\mathbf{\Sigma}$ are invariant under such a transformation, and we have

$$2\mathbf{\Lambda}'=2O''\mathbf{\Lambda}\tilde{O}''=\begin{Bmatrix}\mathbf{L} & \mathbf{\Sigma}\\ -\mathbf{\Sigma} & \mathbf{L}\end{Bmatrix}+\begin{Bmatrix}\mathbf{Y}' & \mathbf{A}'\\ \mathbf{A}' & -\mathbf{Y}'\end{Bmatrix}. \tag{27}$$

$\mathbf{Y}$ and $\mathbf{A}$ transform together like the components of a vector, rotated through the angle $2\beta''$:

$$\begin{aligned}\mathbf{Y}'&=\mathbf{Y}\cos2\beta''+\mathbf{A}\sin2\beta'',\\ \mathbf{A}'&=-\mathbf{Y}\sin2\beta''+\mathbf{A}\cos2\beta''.\end{aligned} \tag{28}$$

E. Tensor in 3-Particle Coordinates

Let us now define the grand angular momentum tensor in the initial 3-body system, with the coordinates taken from the center of mass. The coordinates and momenta are subject to the constraints:

$$\sum_{\alpha=1}^{3}m^\alpha x_i{}^\alpha=0, \quad \sum_\alpha p_i{}^\alpha=0. \tag{29}$$

The grand angular momentum tensor L can now be defined by

$$L_{ij}{}^{\alpha\beta}=(m^\alpha/m^\beta)^{\frac{1}{2}}x_i{}^\alpha p_j{}^\beta-(m^\beta/m^\alpha)^{\frac{1}{2}}x_j{}^\beta p_i{}^\alpha, \tag{30}$$

or

$$L_{ij}=(m_i/m_j)^{\frac{1}{2}}x_ip_j-(m_j/m_i)^{\frac{1}{2}}x_jp_i.$$

[The mass coefficients enter here because the $L_{ii}^{\alpha\beta}$ are related to the symmetry of the kinetic energy in the form $T=\frac{1}{2}\sum_\alpha(P_i^\alpha)^2$ that results when the coordinates and momenta are transformed to $X_i^\alpha=(m^\alpha)^{\frac{1}{2}}x_i^\alpha$ and $P_i^\alpha=(m^\alpha)^{-\frac{1}{2}}p_i^\alpha$.] In view of the conditions (29), it is easy to show that

$$\sum_\alpha(m^\alpha)^{\frac{1}{2}}\mathbf{L}^{\alpha\gamma}=0. \tag{31}$$

The 3×3 tensors $\mathbf{L}^{\alpha\alpha}$, lying along the diagonal of L are obviously just the angular momenta of the particles about the common center of mass. The others can be written as a sum of symmetric and antisymmetric parts,

$$2\mathbf{L}^{\alpha\beta}=\mathbf{A}^{\alpha\beta}+\mathbf{S}^{\alpha\beta}, \quad (A_{ij}^{\alpha\beta}=-A_{ji}^{\alpha\beta}=A_{ij}^{\beta\alpha},\; S_{ij}^{\alpha\beta}=S_{ji}^{\alpha\beta}=-S_{ij}^{\beta\alpha}), \tag{32}$$

and the antisymmetric part, by (31), can be expressed in terms of the ordinary angular momenta:

$$(m^\alpha m^\beta)^{\frac{1}{2}}\mathbf{A}^{\alpha\beta}=m^\gamma\mathbf{L}^{\gamma\gamma}-m^\alpha\mathbf{L}^{\alpha\alpha}-m^\beta\mathbf{L}^{\beta\beta}. \tag{33}$$

The symmetric parts, $\mathbf{S}^{\alpha\beta}$, by (31), are all related to a single symmetric matrix, $\boldsymbol{\sigma}$:

$$(m^\alpha m^\beta)^{\frac{1}{2}}\mathbf{S}^{\alpha\beta}=-(m^\alpha m^\beta)^{\frac{1}{2}}\mathbf{S}^{\beta\alpha}=\boldsymbol{\sigma}, \quad (\alpha\beta=12,\,23,\,31). \tag{34}$$

The 9×9 tensor L can thus be expressed in terms of the three angular momenta $\mathbf{L}^{\alpha\alpha}$ and the symmetric 3×3 tensor $\boldsymbol{\sigma}$.

The elements of the symmetric tensor $\boldsymbol{\sigma}$ are related to the quantity Λ_{11}^{12} of Eq. (10), which appeared in the discussion of the 3-body collision on a line where the true angular momentum was necessarily zero. They thus refer to the relative simultaneity of the 2-body collisions implicit in the 3-body trajectory. In general, $\boldsymbol{\sigma}$ may not vanish even when all the angular momenta $\mathbf{L}^{\alpha\alpha}$ are zero.

Applying the transformation (1), one can relate the components of $\boldsymbol{\Lambda}$ to those of L as follows:

$$\mathbf{L}=\mathbf{L}^{11}+\mathbf{L}^{22}+\mathbf{L}^{33}, \tag{35}$$

$$\mu\boldsymbol{\Sigma}=\boldsymbol{\sigma}, \tag{36}$$

$$\mu\mathbf{A}=\frac{m_3-m_2}{m_2+m_3}m_1\mathbf{L}^{11}+m_2\mathbf{L}^{22}-m_3\mathbf{L}^{33}, \tag{37}$$

$$\mathbf{Y}=\left(1+\frac{2m_1}{m_2+m_3}\right)\mathbf{L}^{11}-\mathbf{L}^{22}-\mathbf{L}^{33}. \tag{38}$$

Finally, it can be shown that the invariant Λ^2 can be computed directly from the $L_{ij}^{\alpha\beta}$:

$$\Lambda^2=\tfrac{1}{2}\sum_{ij\alpha\beta}(L_{ij}^{\alpha\beta})^2=\tfrac{1}{2}\sum_{ij=1}^{9}(L_{ij})^2. \tag{39}$$

To prove this, note that the transformation (1) implies that

$$\mu\rho^2=\mu\sum_i(\xi_i)^2=\sum_i m^i(x_i)^2, \tag{40}$$

and

$$\rho p_\rho=\sum_i\xi_i\pi_i=\sum_i x_ip_i. \tag{41}$$

Hence,

$$\tfrac{1}{2}\sum_{ij}(L_{ij})^2=2T\sum_i m^i(x_i)^2-(\sum_i x_ip_i)^2=\Lambda^2. \tag{42}$$

It is interesting to note that a simple case of the grand angular momentum tensor can be constructed in the case of a 2-particle collision, if the coordinates of the separate particles are measured from the center of mass. In this case, $\mathbf{L}=\boldsymbol{\Lambda}$ is the total angular momentum of the pair, and L is the 6×6 tensor

$$L=(m_1+m_2)^{-1}\begin{Bmatrix} m_2\mathbf{L} & -(m_1m_2)^{\frac{1}{2}}\mathbf{L} \\ -(m_1m_2)^{\frac{1}{2}}\mathbf{L} & m_1\mathbf{L}\end{Bmatrix}. \tag{43}$$

The preceding theorem, Eq. (42), is obviously obeyed.

F. Properties of the Grand Angular Momentum Tensor

In the 6-dimensional coordinate space, the grand angular momentum tensor $\boldsymbol{\Lambda}$ defines a magnitude $|\Lambda|$ and the orientation of a 2-dimensional plane containing the coordinate origin and the straight trajectory from which $\boldsymbol{\Lambda}$ was derived. If the coordinate system is rotated so that the axis of the new coordinate ξ_1' is parallel to the 6-vector $\boldsymbol{\pi}$, and the axis of ξ_2' is parallel to the 6-vector $\boldsymbol{\xi}(R)$ running from the origin perpendicularly to $\boldsymbol{\pi}$, $\boldsymbol{\Lambda}'$ has only two nonzero elements,

$$\Lambda_{12}'=-|\Lambda|, \quad \Lambda_{21}'=|\Lambda|. \tag{44}$$

This may be thought of as the normal form of $\boldsymbol{\Lambda}$; it shows that $\boldsymbol{\Lambda}$ has four zero roots, and two that are conjugate pure imaginaries, $\pm i|\Lambda|$. Generally if $\boldsymbol{\Lambda}$ is known, an orthogonal coordinate transformation can be found that will put $\boldsymbol{\Lambda}$ into its normal form and identify the plane in which the trajectory lies.

Of the 15 elements of $\boldsymbol{\Lambda}$, how many are algebraically independent? Certainly not more than the 12 independent coordinates and momenta. Clearly, the total kinetic energy T is independent of $\boldsymbol{\Lambda}$; when T is known, the trajectory is limited to a family of straight lines tangent to a circle of radius R in the plane defined by $\boldsymbol{\Lambda}$. One additional parameter suffices to determine the particular straight line (for instance, the angle χ_1 in Fig. 1 or, by Eq. (12), an additional energy such as E_A). The velocity with which this trajectory is traced out is known from T, but the initial position at time $t=0$ requires one further independent parameter. Three parameters in addition to $\boldsymbol{\Lambda}$ are thus generally needed to specify the straight-line motion completely (but in the singular case $|\Lambda|^2=0$, these 3 do not suffice). This suggests that $\boldsymbol{\Lambda}$ implicitly contains in general $(12-3=9)$ independent quantities.

Since $\boldsymbol{\Lambda}$ has only two nonzero roots, a set of implicit relations among its elements can be obtained by constructing third-order determinants from its elements and setting them equal to zero. This leads to a set of identities (which can be verified directly by expanding

in terms of x's and p's):

$$(ij,kl)\equiv\Lambda_{ij}\Lambda_{kl}+\Lambda_{jk}\Lambda_{il}+\Lambda_{ki}\Lambda_{jl}=0, \qquad (45)$$

where i, j, k, l are all unequal; because of the antisymmetry of $\mathbf{\Lambda}$, the cyclic permutation in the sum may be taken over any 3 of the 4 indices. Only 6 of these equations are independent, for instance the set

$$(12,kl)=0, \quad (k>2, l>k), \qquad (46)$$

and the rest can be expressed as combinations of them by using the identity (which depends only on the form of the definition (45) and the antisymmetry of $\mathbf{\Lambda}$):

$$\Lambda_{im}(ij,kl)=\Lambda_{ij}(im,kl)+\Lambda_{ik}(ij,ml)+\Lambda_{il}(ij,km). \qquad (47)$$

Another set of relations can be found in the Poisson brackets containing the Λ_{ij}; these are in all respects parallel to the commutation rules to be derived in the next chapter.

G. Many-Body Collisions

Although the almost simultaneous collision of 4 or more free particles is very rarely a matter of concern, the opposite process, N-body breakup after the collision of 2 or 3 particles, is often of physical importance. All the development of the preceding sections can be extended immediately to the description of such N-body events.

In the $3N-3$ dimensions of a normalized center-of-mass coordinate system, the N-body grand angular momentum tensor $\mathbf{\Lambda}_N$ is constructed as in Eq. (13); it is related to the tensor $\boldsymbol{L}_N$ defined in the coordinate system of the N particles by an equation like (30). Either of these can be used as in Eqs. (14) and (39) to construct the quantity Λ_N^2 which is an invariant of the straight-line trajectory. $\mathbf{\Lambda}_N$ has two roots, and can be transformed by an orthogonal coordinate transformation to the normal form of Eq. (44), which identifies (except when $\Lambda_N^2=0$) a 2-dimensional plane through the origin in the hyperspace. The $\frac{1}{2}(3N-3)(3N-4)$ elements Λ_{ij} are connected by $\frac{1}{2}(3N-5)(3N-6)$ independent identities of the form of Eq. (46), leaving $(6N-9)$ independent parameters to fix this plane. The trajectory's closeness to an N-body simultaneous collision course is characterized by Λ_N^2 or by the N-body impact distance R_N that is related to it by Eq. (20).

The tensor $\mathbf{\Lambda}_N$ incorporates the elements of the tensors $\mathbf{\Lambda}_{N-1}$, etc., of lower order which characterize the collision trajectories of the various possible subgroupings of the N particles. To exhibit these various possible groupings, different sets of center-of-mass coordinates must be used; these are related to one another by orthogonal transformations like that of Eq. (5), but with, in general, $N-1$ rows and columns (for examples of these kinematic rotations, and an indication of how they can be decomposed into a sequence of simple rotations, see reference 8). Under these transformations, $\mathbf{\Lambda}_N$ behaves as in the first part of Eq. (27).

II. QUANTUM MECHANICS

Introduction

In quantum as well as classical mechanics, a 3-body collision can be considered as an event causing a transition from an approach trajectory to a retreat trajectory. The uncertainty principle limits the specification of the initial and final trajectories, but it leaves us with a compensating freedom to choose from a number of possible representations one that comes close to representing an experimental situation or seems particularly convenient for calculation.

In the theory of 2-body collisions, it is common to begin by describing an idealized experiment in terms of the scattering of a plane wave, with a well-defined momentum vector. In dealing with low-energy collisions, at least, one quickly analyzes the plane wave into a set of spherical waves with well-defined angular momenta about the center of mass, and from then on carries out the analysis in this more convenient representation. A parting glance may finally be given to the plane wave, in order to compute the interference effects that may be observed in experiment. The prime reason for the use of spherical waves is that they are concentrated on the region of interaction, which may indeed, at low energies, not extend far enough to affect any but the first one or two partial waves. A related reason is that the outgoing scattered wave is in any case conveniently described in spherical terms, having lost the directional character of the incident wave; when the experimental observation is a function of angle made at a large distance from a small region of interaction, the spherical description is particularly appropriate.

Similar considerations apply to the specification of 3-body collisions. Analogy suggests that 3-body collisions can usefully be analyzed in terms of some sort of spherical or hyperspherical waves, classified in terms of a generalization of the angular momentum. The grand angular momentum introduced in Sec. I proves to have the desired properties. By Eq. (20) of Sec. I it is related to R, the 3-body impact distance that measures the closeness of the 3-body impact, in just the way the ordinary angular momentum is to the 2-body impact parameter.

A. Commutation Rules and Commuting Observables

The coordinate transformations of the previous section are linear with constant coefficients and unit Jacobian, and they and their consequences for the momenta and the quantities derived from them (kinetic energy, angular momenta) carry over to the quantum operators without change. It will suffice from here on to use a center-of-mass coordinate system, and I shall return to the use of Latin instead of Greek letters for the coordinates and momenta. Also, in deference to the usual terminology, ordinary angular momenta will be denoted by the letter L, but Λ will still be used for the

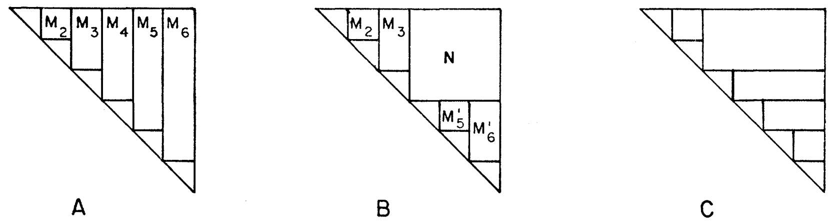

FIG. 2. Representations of the grand angular momentum for 3-body collisions.

full grand angular momentum tensor. We may redefine

$$\mathbf{L}^1=\mathbf{\Lambda}^{11},\quad \mathbf{L}^2=\mathbf{\Lambda}^{22},$$

$$\mathbf{\Lambda}=\begin{Bmatrix} \mathbf{L}^1 & \tfrac{1}{2}(\mathbf{\Lambda}+\mathbf{\Sigma}) \\ \tfrac{1}{2}(\mathbf{\Lambda}-\mathbf{\Sigma}) & \mathbf{L}^2 \end{Bmatrix}, \tag{1}$$

and

$$\Lambda_{ij}=\frac{\hbar}{i}\left(x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i}\right)=x_ip_j-x_jp_i, \qquad (i,\,j=1,\,\cdots,\,6). \tag{2}$$

The following commutation rules are obeyed:

$$[\Lambda_{ij},x_i]=-[\Lambda_{ji},x_i]=i\hbar x_j, \tag{3}$$

$$[\Lambda_{ij},p_i]=-[\Lambda_{ji},p_i]=i\hbar p_j, \tag{4}$$

$$[\Lambda_{ij},\Lambda_{ik}]=i\hbar\Lambda_{jk}, \tag{5}$$

$$[\Lambda_{ij},\Lambda_{mk}]=0, \quad (i,\ j,\ k,\ m \text{ all unequal}),$$

$$[\Lambda_{ij},\ (\Lambda_{ik}^2+\Lambda_{jk}^2)]=0, \quad (i\neq k,\ j\neq k). \tag{6}$$

The kinetic energy matrix, $T=(1/2\mu)\sum_{i=1}^{6}p_i^2$, commutes with each element Λ_{ij},

$$[\Lambda_{ij},T]=0. \tag{7}$$

Quantum mechanically, the approach stage of a 3-body collision, in the region where the potential is negligible, is completely described in terms of the eigenvalues of a commuting set of operators. Ordinarily, the total kinetic energy, T, is one of the set. In view of Eq. (7) any commuting set of operators derived from the Λ_{ij} will also commute with T. Let us see what can be done with combinations of the Λ_{ij}. From Eqs. (5) and (6), $\mathbf{\Lambda}_{12}$ commutes with each of the operators

$$M_i=\sum_{j=1}^{i}\Lambda_{ji}^2. \tag{8}$$

The M_i also commute among themselves,

$$[M_i,M_j]=0; \tag{9}$$

for instance,

$$[M_3,M_4]=[\Lambda_{13}^2,\ (\Lambda_{14}^2+\Lambda_{34}^2)]+[\Lambda_{23}^2,\ (\Lambda_{24}^2+\Lambda_{34}^2)]=0.$$

Thus we have the set of commuting operators,

$$\text{(A)} \qquad \{\Lambda_{12},M_3,M_4,M_5,M_6,T\}. \tag{10}$$

Since $\Lambda^2=\sum_{i<j}\Lambda_{ij}^2=L_{12}^2+\sum_{i=3}^{i=6}M_i$, it is possible to substitute Λ^2 instead of M_6 in the set (A), or to use any other complete set derived from Eq. (10) by linear combination.

The five angular momentum operators of (A) can be represented graphically as shown in Fig. 2(A), since the M_i represent sums of squares of the matrix elements in successive columns of the upper triangle of the matrix $\mathbf{\Lambda}$. Some alternative ways of partitioning this triangle are also shown; each represents a set of commuting operators.

Of the representations shown in Fig. 2, (A) and (B) are the most important. (B) in particular can be related to familiar quantities, since only N is unfamiliar:

$$\begin{aligned} &M_2=\Lambda_{12}^2=(L_z^1)^2, && M_2+M_3=(L^1)^2,\\ &M_5'=(L_z^2)^2, && M_5'+M_6'=(L^2)^2,\\ &N=\sum_{i,j=1}^{3}(\Lambda_{ij}^{12})^2, \end{aligned} \tag{11}$$

where the usual symbols for angular momenta and their z components are used. Instead of N, Λ^2 may be taken as the fifth operator of the set. Then we have

$$\text{(B}'\text{)} \qquad \{E,\Lambda^2,(L^1)^2,L_z^1,(L^2)^2,L_{z'}^2\}, \tag{12}$$

as a complete set of operators. (L_z^1 and $L_{z'}^2$ need not be referred to the same direction in space.) Alternatively, we can use the total angular momentum, $\mathbf{L}$:

$$\text{(B}''\text{)} \qquad \{E,\Lambda^2,L^2,L_z,(L^1)^2,(L^2)^2\}, \tag{13}$$

since

$$L^2=(L^1)^2+(L^2)^2+2\mathbf{L}^1\cdot\mathbf{L}^2, \quad L_z=L_z^1+L_z^2. \tag{14}$$

The initial and final states of a 3-body collision can now be specified in terms of the quantum numbers associated with the eigenvalues of a set of operators like (A), (B′), or (B″). The sets (B′) or (B″) involve the familiar eigenvalues of the ordinary angular momenta,

$$\begin{aligned} &(L^\alpha)^2=l^\alpha(l^\alpha+1)\hbar^2,\\ &L_z^\alpha=\hbar m_l^\alpha \quad (|m_l^\alpha|\leqslant l^\alpha), \end{aligned} \tag{15}$$

and an additional eigenvalue, Λ^2. This is of the form

$$\Lambda^2=\lambda(\lambda+4)\hbar^2,$$

with

$$\lambda-l^1-l^2=2q\geqslant 0, \quad (q \text{ integral}). \tag{16}$$

The initial and final states may then be labeled with the quantum numbers

$$\text{(B}'\text{)} \qquad \{E,\lambda,l^1,m_l^1,l^2,m_l^2\},$$

or

$$(B'') \qquad \{E,\lambda,l,m_l,l^1,l^2\}, \tag{17}$$

where l and m_l refer to the total angular momentum,

$$(L)^2=l(l+1)\hbar^2, \quad l\geqslant l^1,$$
$$L_z=m_l\hbar, \quad |m_l|\leqslant l. \tag{18}$$

These quantum numbers are convenient for labelling the elements of the scattering matrix for 3-body collisions.

The representation (B″) is properly adapted to expressing the conservation of total angular momentum, but it has a disadvantage for some purposes in that the 3 particles enter asymmetrically in the definitions of $\mathbf{L}^1$ and $\mathbf{L}^2$. In Sec. I, Eqs. (26) and (27), we saw that the symmetric tensor $\mathbf{\Sigma}$, as well as the total angular momentum $\mathbf{L}$, is invariant under the kinematic rotation which represents a change to a different pairing of the particles. If a set of 5 independent commuting operators can be derived from Λ^2, $\mathbf{L}$, and $\mathbf{\Sigma}$, we shall have a representation that treats the particles symmetrically. Such a representation will apparently involve three angular momentum operators, connected with the Euler angles of the plane of the three bodies, and the trace of $\mathbf{\Sigma}$,

$$\Sigma_{\mathrm{Tr}}=\Sigma_{11}+\Sigma_{22}+\Sigma_{33}. \tag{19}$$

This representation may be advantageous for some forms of the 3-body problem.[9]

B. *N*-Body Operators

Similar principles apply when energy is available to produce four or more particles, and a representation in terms of a discrete spectrum continues to be available. The treatment can be sketched briefly. Let $\mathbf{x}^1$ be the normalized vector between two particles, $\mathbf{x}^2$ run from their center of mass to a third, and $\mathbf{x}^3$ run from the center of mass of all three to the fourth. Then, letting i run from 1 to 9, the grand angular momentum can be defined as before. A complete set of commuting operators now includes 8 composed from the Λ_{ij}, plus T, the kinetic energy. A possible representation is shown in Fig. 3, where

$$\Lambda_2{}^2=\sum_{j=1}^{6}\sum_{i<j}\Lambda_{ij}{}^2,$$

and

$$\Lambda_3{}^2=\sum_{j=1}^{9}\sum_{i<j}\Lambda_{ij}{}^2. \tag{20}$$

The eigenvalue of $\Lambda_3{}^2$ has the form

$$\Lambda_3{}^2=\lambda_3(\lambda_3+7)\hbar^2,$$

where

$$\lambda_3-\lambda_2-l^3=2q'\geqslant 0, \quad (q' \text{ integral}). \tag{21}$$

When five or more particles are involved in the collision, the hierarchy of grand angular momenta can be extended as far as needed. It is convenient to use the convention that $\mathbf{\Lambda}_n$ refers to a system of $(n+1)$ particles, so that $\mathbf{\Lambda}_1$ is an ordinary angular momentum. The eigenvalues of $\Lambda_n{}^2$ are then

$$\Lambda_n{}^2=\lambda_n(\lambda_n+3n-2)\hbar^2. \tag{22}$$

For $(n+1)$ particles, a complete set of observables in the center-of-mass system would be:

$$\{T,\Lambda_n{}^2,\Lambda_{n-1}{}^2,\cdots\Lambda_2{}^2,(L^n)^2,\cdots(L^1)^2,L_z{}^n,\cdots L_z{}^1\}. \tag{23}$$

As the number of particles goes up, so does the variety of possible choices for a set of commuting operators. With five particles, for instance, the hierarchy $\mathbf{\Lambda}_4$, $\mathbf{\Lambda}_3$, $\mathbf{\Lambda}_2$, can be replaced by $\mathbf{\Lambda}_4$, $\mathbf{\Lambda}_2$, $\mathbf{\Lambda}_2'$, where $\mathbf{\Lambda}_2$ involves coordinate indices running from 1 to 6, and $\mathbf{\Lambda}_2'$ involves coordinate indices from 7 to 12.

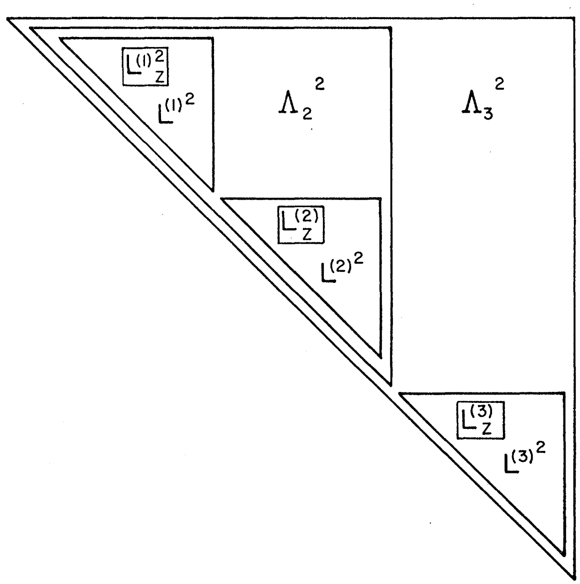

FIG. 3. A representation of the hierarchy of angular momenta for a 4-body collision.

[9] I hope to discuss this representation more fully elsewhere.

C. Coordinates and Eigenfunctions

Just as spherical polar coordinates are associated with problems involving ordinary angular momenta, so hyperspherical polar coordinates are naturally used for problems involving the grand angular momentum operators. Corresponding to the different possible sets of commuting angular momentum operators are different choices of the $(3n-1)$ angular coordinates in the problem of $(n+1)$ bodies, but the hyperradial coordinate r is independent of the choice of angular coordinates:

$$r^2=\sum_{i=1}^{3n}(x_i)^2. \tag{24}$$

With the operator

$$p_r^2=-\hbar^2 r^{-n+1}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right), \tag{25}$$

the kinetic energy operator is

$$T=\frac{1}{2\mu}p_r^2+\frac{1}{2\mu r^2}\Lambda_n^2. \tag{26}$$

For the collision problem with short-range forces there is a region at large r where the Schrödinger equation can be separated into an angular and a hyperradial part. The hyperradial equation can be solved by a Bessel function; for instance,

$$R_{n,\lambda n}(r)=r^{(1-n/2)}J_{(\lambda_n-1+n/2)}(kr); \tag{27}$$

incoming or outgoing waves can be expressed similarly in terms of Hankel functions.[2]

The solutions of the equation in the angular coordinates ω,

$$[\Lambda_n^2-\hbar^2\lambda_n(\lambda_n+3n-2)]\Omega(\omega)=0, \tag{28}$$

are hyperspherical harmonics. These have different forms depending on the angular coordinates chosen; the ones of interest here can be written as simple products of functions of a single angle each. Let us look briefly at a couple of these representations for the 3-body problem.

If we introduce what may be called the regular hyperspherical polar coordinate system (A), defined by

(A)
$$\begin{aligned} x_6&=r\cos\zeta_5,\\ x_5&=r\sin\zeta_5\cos\zeta_4,\\ &\vdots\\ x_1&=r\sin\zeta_5\sin\zeta_4\sin\zeta_3\sin\zeta_2\sin\zeta_1, \end{aligned} \tag{29}$$

we find that the operator Λ^2 is naturally subdivided in a way related to the partitioning of Eq. (10) and Fig. 2(A). Separation of variables leads to angular functions related to the Gegenbauer polynomials.[10]

The coordinate system (A) is not very useful, however, because it is not adapted to singling out the ordinary angular momentum of the system. For this purpose, Λ^2 is more appropriately subdivided as in Fig. 2(B) and Eq. (17). The associated coordinates $(r,\chi,\theta_1,\phi_1,\theta_2,\phi_2)$ are derived from the ordinary spherical polar coordinates $(r_1,\theta_1,\phi_1;r_2,\theta_2,\phi_2)$ by

$$r_1=r\cos\chi,\quad r_2=r\sin\chi. \tag{30}$$

In these coordinates, Λ^2 becomes

$$\Lambda^2=\frac{-1}{\sin^2 2\chi}\frac{\partial}{\partial\chi}\left(\sin^2 2\chi\frac{\partial}{\partial\chi}\right)+\frac{(L^1)^2}{\cos^2\chi}+\frac{(L^2)^2}{\sin^2\chi}, \tag{31}$$

and generates a hyperspherical harmonic which is a product of ordinary spherical harmonics and a function $X(\chi)$ which can be expressed[2,3] in terms of the Jacobi polynomial ${}_2F_1$:

$$X(\chi)=\cos^{l_1}\chi\sin^{l_2}\chi\times{}_2F_1\left(\frac{l_1+l_2-\lambda}{2},\frac{l_1+l_2+\lambda+4}{2},l_2+\tfrac{3}{2},\sin^2\chi\right), \tag{32}$$

where $\lambda\geq l_1+l_2$. [Note that Delves calls "λ" what appears as $\frac{1}{2}(\lambda-l_1-l_2)$ here.] Delves shows how to transform between this representation and the momentum (plane-wave) representation.

Similar expressions can be found for the hyperspherical harmonics needed in the angular momentum analysis of a system of four or more particles. If the m-dimensional grand angular momentum tensor $\mathbf{\Lambda}$ is partitioned in a way involving the m_a-dimensional tensor $\mathbf{\Lambda}_a$ and the m_b-dimensional tensor $\mathbf{\Lambda}_b$ (where $m_a+m_b=m$) the angular eigenfunction can be written:

$$\Omega(\omega)=X_m(\chi)\Omega_a(\omega_a)\Omega_b(\omega_b). \tag{33}$$

$X_m(\chi)$ satisfies the equation:

$$\left[\cos^{-m_a+1}\chi\sin^{-m_b+1}\chi\frac{d}{d\chi}\left(\cos^{m_a-1}\chi\sin^{m_b-1}\chi\frac{d}{d\chi}\right)-\frac{\lambda_a(\lambda_a+m_a-2)}{\cos^2\chi}-\frac{\lambda_b(\lambda_b+m_b-2)}{\sin^2\chi}+\lambda(\lambda+m-2)\right]X_m(\chi)=0. \tag{34}$$

The solution is

$$\begin{aligned} X_m(\chi)&=X(m,m_a,m_b;\lambda,\lambda_a,\lambda_b;\chi)\\ &=N\cos^{\lambda_a}\chi\sin^{\lambda_b}\chi F\left(\frac{\lambda_a+\lambda_b-\lambda}{2},\frac{\lambda+\lambda_a+\lambda_b+m-2}{2},\lambda_b+\frac{m_b}{2};\sin^2\chi\right), \end{aligned} \tag{35}$$

where $(\lambda-\lambda_a-\lambda_b)$ is an even non-negative integer.

The normalizing constant can be fixed by

$$N^2=\frac{[\frac{1}{2}(\lambda-\lambda_a-\lambda_b)]![\Gamma(\lambda_b+\frac{1}{2}m_b)]^2\Gamma[\frac{1}{2}(\lambda+\lambda_a-\lambda_b+m_a-2)]}{(2\lambda+m-2)\Gamma[\frac{1}{2}(\lambda+\lambda_a+\lambda_b+m-2)]\Gamma[\frac{1}{2}(\lambda-\lambda_a+\lambda_b+m_b)]}, \tag{36}$$

[10] L. Infeld and T. E. Hull, Revs. Modern Phys. **23**, 21 (1951) where these functions are called "generalized spherical harmonics." A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Chap. XI. Similar representations hav been used by G. A. Gallup, J. Mol. Spectroscopy **3**, 673 (1959), and by J. D. Louck and W. H. Schaffer, J. Mol. Spectroscopy **4**, 285, 298 (1960) in connection with the n-dimensional isotropic harmonic oscillator: they construct the generalized angular moementum tensor and associated raising and lowering operators.

so that

$$\int_0^{\pi/2} X(m,m_a,m_b;\lambda,\lambda_a,\lambda_b;\chi)X(m,m_a,m_b;\lambda',\lambda_a,\lambda_b;\chi) \times\cos^{m_a-1}\chi\,\sin^{m_b-1}\chi d\chi=\delta_{\lambda,\lambda'}, \quad (37)$$

which corresponds to unit normalization over a hypersphere.

D. Description of Collisions

The eigenfunctions in the hyperspherical coordinate system make it easy to describe the asymptotic form of a collision involving three or more particles. The wave functions can be normalized to unit total inward or outward flux through a hypersphere. When a 3-body collision leads to a bound state (of a molecule BC, for instance) the quantum number λ is replaced by v^1, the vibrational quantum number of the molecule. Conversely, a collision of the type $A+BC\rightarrow A+B+C$ can be described by a term of the scattering matrix leading from the incoming state (E,v^1,l,m_l,l^1,l^2) to the outgoing state $(E,\lambda,l,m_l,l'^1,l'^2)$, where E, l, and m_l are conserved and no other labels are needed if A, B, and C are structureless, spinless particles. When three or more bodies are produced from a 2-body collision, the convenience of a representation in which all the quantum numbers except the energy remain in the discrete spectrum instead of passing into the continuum is obvious. In such a representation, problems of normalization also disappear; for a bound state BC the condition of unit flux through a hypersphere reduces to unit flux through a sphere in the coordinates of $\mathbf{x}^2$ describing the relative motion of A and BC and so the same prescription suffices for 3-body trajectories and for the bound states $A+BC$, $AC+B$, and $AB+C$. At sufficient distances, the free 3-body states labeled by λ and the bound states labeled by v^1, v'^1, and v''^1 are asymptotically orthogonal, so the description in these terms is unique.

How do metastable 2-body states like $A+BC^*$, where BC^* has enough energy to dissociate, fit into this picture? In principle, these contribute ultimately to outgoing waves with large values of Λ^2, which appear to originate near the potential trough along the axis of the coordinates $\mathbf{x}^2$ ($\xi_1{}^2$ in Fig. 1). Strictly, they should be described in terms of a superposition of these 3-body states with various values of λ (and certain phase relations). This will describe their effect everywhere except in the potential trough—and asymptotically, at large enough r, this state will have decayed away from the trough and will not be noticeable there. Practically, observations will be made at a finite distance, and it may be more realistic to treat the metastable state like a true bound state and ignore its contribution to waves with large λ. In doing so, it must be recognized that the procedure is not completely self-consistent, and becomes especially fuzzy for metastable states with short half-lives which may contribute a good fraction of their flux to states of low λ.

A simple criterion for such metastable processes can be found as follows: If the velocity in the trough is

$$v_2=(2E_2/\mu)^{\frac{1}{2}}, \quad (38)$$

(assuming the ordinary angular momenta are negligible, $l_1=l_2=0$), and if the average metastable lifetime is

$$\tau_{\rm m}=\hbar/2\Gamma_{\rm m}, \quad (39)$$

then the average normalized distance traveled in the trough is

$$\langle x_2\rangle=v_2\tau_{\rm m}=(\hbar/\Gamma_{\rm m})(E_2/2\mu)^{\frac{1}{2}}. \quad (40)$$

If the characteristic range of the 3-body interactions is σ, the processes going by way of metastables are effectively distinguishable from 3-body events when $\langle x_2\rangle\gg\sigma$. Defining the energy level of the metastable BC^* with respect to the dissociated fragments $B+C$ as $E_{\rm m}$, it is interesting to note that the average value of $|\Lambda|$ becomes

$$|\bar{\Lambda}_{\rm m}|=|p_1|\langle x_2\rangle=\hbar(E_{\rm m}E_2)^{\frac{1}{2}}/\Gamma_{\rm m}=\hbar\bar{\lambda}_{\rm m}. \quad (41)$$

The metastable state will produce waves with different values of λ with relative probabilities given by

$$P_{\rm m}(\lambda)=\bar{\lambda}_{\rm m}{}^{-1}\exp(-\lambda/\bar{\lambda}_{\rm m}). \quad (42)$$

The largest value of λ that can be involved in a pure 3-body interaction with range σ, when the total energy is $E=E_{\rm m}+E_2$, is

$$\lambda_\sigma=\hbar^{-1}(2\mu E)^{\frac{1}{2}}\sigma. \quad (43)$$

The criterion for processes involving metastables then becomes $\lambda_{\rm m}\gg\lambda_\sigma$, or

$$E_{\rm m}(E-E_{\rm m})\gg\Gamma_{\rm m}{}^2\lambda_\sigma{}^2. \quad (44)$$

It has been implicitly assumed in this argument that the half-width $\Gamma_{\rm m}$ is much smaller than $E_{\rm m}$ and $(E-E_{\rm m})$; when this is not the case, more detailed discussion is required.[11]

E. Applications to Experiments

It must be admitted that the sets of observables derived from the grand angular momentum tensor are not always the most convenient ones to compare with experiment. But in this, after all, they do not differ greatly from ordinary angular momentum, which is not usually observed directly in scattering experiments. Instead, one deduces angular momentum effects from the angular dependence of scattering cross sections. Likewise the contribution of states of different values of λ to 3-body processes can be deduced from experimental measurement of the distribution of energy between the three particles. Delves[2] discusses threshold phenomena from this point of view; states with $\lambda=0$ (or $\lambda=l_1+l_2$) should predominate here. At higher energies contributions from states of larger λ will appear and produce more structure in an energy distribution plot. Metastable

[11] See also F. T. Smith, Phys. Rev. **118**, 349 (1960).

bound states will of course give a peak of half-width Γ_m at a critical value of the energy $E_{BC}=E_m$. Further study of the hyperspherical harmonics should disclose other ways in which experimental observations can be used in this sort of analysis—various combinations of data including energies, angular correlations, and temporal coincidences or delays, could be examined for 3-body effects.[12]

[12] Compare the study of 3-body events for effects of 2-body forces: G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959). See also L. Fonda and R. G. Newton, Phys. Rev. **119**, 1394 (1960).

In some problems of chemical kinetics, 3-body reactions occur in a statistical assemblage of colliding particles. Similar events, governed by short-range forces, occur in the 3-body attachment of electrons to atoms or molecules. In cases like these, it should be possible to introduce the angular momentum description of 3-body collisions into a statistical argument. In such a description it is important to look carefully into the relative contributions of pure 3-body processes and events involving a 2-body metastable.

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Quenching of Magnetic Moments in Nuclei*

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Starting from the premise that with modern dispersion-theoretical techniques one has a reliable method for calculating the anomalous magnetic moment of a nucleon, we have calculated the modification or "quenching" of this moment for a nucleon in nuclear matter. The effect we consider here is due to the fact that nucleons are not allowed by the exclusion principle to recoil into states already occupied by other nucleons in the nucleus. The actual technique we have used in our calculation is to sum all the Feynman diagrams that are included in the dispersion-theory calculation of the single-nucleon moment. We then write the nucleon propagator as a sum over states and remove those states in which the nucleon is inside the Fermi sea. Our result is that the anomalous moment is reduced by $\approx 7\%$.

I. INTRODUCTION

In this paper we wish to re-examine the question of the quenching of the intrinsic magnetic moments of nucleons in nuclear matter. The idea of quenching the spin-g factor of a nucleon (g_s) in nuclear matter was proposed in 1951, independently, by Bloch,[1] Candler,[2] Miyazawa,[3] and de-Shalit.[4] Their arguments were based on the observation that in almost every case the observed magnetic moments of odd-A nuclei could be explained by a single-particle calculation with the intrinsic nucleon moment lying somewhere between the free-nucleon moment

$$\mu_p=(1+1.79)\text{ nm},\quad \mu_n=-1.91\text{ nm},\tag{1.1}$$

and a completely quenched moment

$$\mu_p=1\text{ nm},\quad \mu_n=0\text{ nm}.\tag{1.2}$$

If one plots the magnetic moments of the odd-A nuclei vs the nuclear spin one obtains from the single-particle model two Schmidt lines[5,6] for $l=I\pm\frac{1}{2}$, where l is the orbital angular momentum and I the total angular momentum, or spin, of the odd nucleon considered to be moving in the spherically symmetric potential provided by the even-even core. The experimental moments are found to cluster near these lines but the fit is greatly improved if the value of the intrinsic moment for a nucleon in nuclear matter is taken to lie between values (1.1) and (1.2).

The physical assumption underlying use of unquenched values (1.1) in nuclear matter is that the currents in the meson cloud about a nucleon are not altered by the presence of other nucleons in the nuclear matter. The values (1.2) would apply in the case that the presence of other nucleons at the density of normal nuclear matter completely discouraged a nucleon from developing its normal meson currents. Thereby the moment would be quenched all the way down to the Dirac value which obtains in the absence of all meson-current effects. There have been several attempts to implement this idea with an accurate calculation.[3,7] However, several major obstacles have barred the way:

1. It has not been possible to calculate from meson theory the magnetic moments of free nucleons with any accuracy. Indeed until the dispersion-theory methods of the past two years there has not even existed a systematic approach to a nucleon magnetic

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