simple case that the observed process can be considered as a fluctuation in an isolated system, the theory of Van Hove can be related to the experimental situation in a natural way.

5. Ordinary probabalistic reasoning shows that in experimental situations the anti-Boltzmann is never useful for prediction of the future behavior of a non-equilibrium distribution function, even in cases for which the distribution function is changing according to that equation.

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Temperature Dependence of the Low-Momentum Excitations in a Bose Gas of Hard Spheres*

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The pseudopotential method is used to calculate the temperature dependence of the low-momentum excitations in a Bose gas of hard spheres to an order beyond that previously known. The correction term is shown to be small compared to the leading term for all temperatures less than the critical temperature T_c. The excitations become poorly defined for temperatures slightly below T_c, however, since it is shown that in this region the excitation lifetimes are quite small.

INTRODUCTION

At any temperature T less than the critical temperature T_c, the low-lying excitation energies in a system of bosons are associated with unique wave numbers and, therefore, they exhibit particle-like properties. The (temperature-dependent) energy-momentum relation for these quasi-particles has been measured for liquid He II by neutron scattering experiments.[1] On the other hand, a theoretical calculation of quasi-particle energies has been made only for a Bose gas of hard spheres in the low-density limit. For this idealized model the leading term in the quasi-particle energy, as determined by Lee and Yang,[2] is

$$\omega_0(k)=(2M)^{-1}\hbar^2k(k^2+2k_0^2)^{\frac{1}{2}},$$

where $k_0^2=8\pi\rho aX$ (ρ=density, a=diameter of a hard sphere, and $X(T)$=fraction of bosons in zero-momentum state). The expression is valid for all $T<T_c$. Several calculations have been made of the leading correction to $\omega_0(k)$ at $T=0$ (where $X=1$).[3] On the other hand, the leading correction to $\omega_0(k)$ for $T\neq 0$ has not previously been published.

The problem of calculating the quasi-particle energies for the ground state ($T=0$) of a Bose gas of hard spheres possesses the simplifying feature that only two lengths characterize the system; namely, $l\equiv\rho^{-\frac{1}{3}}$ and a. The only dimensionless parameter which is encountered in the low-density case is found to be the small parameter $(\rho a^3)^{\frac{1}{2}}$.

When $T\neq 0$ another length occurs in the problem; namely, the thermal wavelength,

$$\lambda_T=(2\pi\hbar^2/kTM)^{\frac{1}{2}}.$$

The introduction of the thermal wavelength increases the number of dimensionless parameters which can be used to describe the Bose gas of hard spheres. Now, whereas the low-density Bose gas is certainly not the same system as real liquid He II, it is of value for orientation purposes to write down the lengths a, l, and λ_c ($=\lambda_T$ at $T_c=2.18$°K) of real He II,[4] together with various dimensionless combinations which are encountered in this paper. These are:

$$a=2.30\text{ A},\qquad a/\lambda_c=0.39,$$
$$l=3.58\text{ A},\qquad (\rho a^3)^{\frac{1}{2}}=0.515,$$
$$\lambda_c=5.91\text{ A},\qquad \rho\lambda_c^3=4.5,$$
$$\rho a\lambda_c^2=1.75.$$

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[1] H. Palevsky, K. Otnes, and E. Larsson, Phys. Rev. **112**, 11 (1958); Y. L. Yarnell, G. P. Arnold, P. J. Bendt, and E. C. Kerr, Phys. Rev. **113**, 1379 (1959); D. G. Henshaw, Phys. Rev. Letters **1**, 127 (1958).

[2] T. D. Lee and C. N. Yang, Phys. Rev. **112**, 1419 (1958).

[3] S. T. Beliaev, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 433 (1958) [translation: Soviet Phys.—JETP **7**, 299 (1958)]; F. Mohling and A. Sirlin, Phys. Rev. **118**, 370 (1960), hereafter referred to as MS.

[4] See, e.g., F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1954), Vol. II.

It is seen that although liquid He II is not a low-density system, it is also not a high-density system.

For the low-temperature Bose gas of hard spheres, the most important dimensionless parameter which involves λ_T seems to be the parameter

$$t \equiv 2X\rho a\lambda_T^2.$$

In this paper we investigate the momentum dependence of the corrections to the quasi-particle energy $\omega_0(k)$ for $k \ll k_0$ and for various regions of the parameter t. In particular, we consider both the regions $t \gg 1$ (very low temperatures) and $t \ll 1$ ("high" temperatures), even though the above numbers suggest that the region $t \ll 1$ may have little to do with the real He II system. The interest in the calculation is twofold. In the first place, there is the question as to whether or not the corrections to $\omega_0(k)$ for low densities are *small* corrections for all $T<T_c$. Secondly, one is also interested in knowing the quasi-particle lifetime as a function of both T and k, which is related to the imaginary part of the correction to $\omega_0(k)$.

Our calculation, which uses the pseudopotential method,[5] is outlined in Secs. I and II, and is patterned after the zero-temperature calculation of MS. It is shown in Sec. IV that the real part of the correction to $\omega_0(k)$ is small for all $T<T_c$. Then, in Sec. V, it is shown for $t \ll 1$ that the quasi-particle lifetime τ decreases with increasing temperature and that $(\tau\omega_0)$ approaches zero in the low-momentum limit.

I. HAMILTONIAN OF THE SYSTEM

A Bose gas of free particles at a temperature T less than the critical temperature T_c can be characterized by a parameter $X(T)$, which is the fraction of bosons in the zero-momentum state. That is, if we define

$$m_{\mathbf{k}}(T) \equiv \text{number of bosons in the (unperturbed) plane wave state } \mathbf{k}\ (\mathbf{k}\neq 0) \text{ at a given } T, \quad (1)$$

then

$$\sum_{\mathbf{k}\neq 0} m_{\mathbf{k}} \equiv \sum_{\mathbf{k}}{}' m_{\mathbf{k}} = (1-X)N, \quad (2)$$

where N is the total number of particles in the system. The free-particle eigenfunctions characterized by the wave number $\mathbf{k}$ are assumed to be normalized in a large cubic box of volume Ω.

For a dilute Bose gas of hard spheres we can use ordinary perturbation theory to calculate energy levels. In this case, the unperturbed state vectors are characterized by the occupation numbers $m_{\mathbf{k}}$ of (1). We also introduce the number operator $N_{\mathbf{k}} = a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}$ for particles of momentum $\mathbf{k}$, and define the operator

$$\delta_{\mathbf{k}} \equiv N_{\mathbf{k}} - m_{\mathbf{k}}, \quad \mathbf{k}\neq 0, \quad (3)$$

which has a nonvanishing expectation value for the interacting system. We treat the zero-momentum state in a special manner by using the parameter $X(T)$ to characterize the unperturbed eigenstates, because for $T<T_c$, $X(T)$ is an appreciable fraction, whereas $m_{\mathbf{k}}/N \sim 1/N$ for $\mathbf{k}\neq 0$. Thus, Lee and Yang[2] have shown in the limit $(N,\Omega)\to\infty$ and $\rho = N/\Omega = \text{constant}$, that $X(T)$ decreases monotonically with increasing temperature from the value $X(0)=1$ to the value $X(T_c)=0$. We observe that although $X(0)=1$, this does not imply that at $T=0$ all of the bosons in the interacting system are in the zero-momentum state. In fact, it has been shown[6] that for a dilute Bose gas of hard spheres,

$$\lim_{T\to 0}\left(\frac{1}{N}\sum_{\mathbf{k}}{}'\langle N_{\mathbf{k}}\rangle\right) = \frac{8}{3\sqrt{\pi}}(\rho a^3)^{\frac{1}{2}} + O(\rho a^3), \quad (4)$$

where a is the diameter of a single hard sphere.

In this paper our objective is to calculate the excitation energy $\omega(k)$ for a dilute Bose gas of hard spheres as a function of the temperature of the gas. The leading term in $\omega(k)$, namely,

$$\omega_0(k) = k(k^2+2k_0^2)^{\frac{1}{2}}, \quad (5)$$

where we use units such that $\hbar = 2M = 1$ (M = mass of boson), and

$$k_0^2 = 8\pi a X\rho \quad (6)$$

has already been calculated by Lee and Yang using the pseudopotential method.[2] Using the methods of statistical mechanics, these authors have also determined the dominant behavior of the quantities $m_{\mathbf{k}}$ and X as a function of temperature. We shall take their expressions for $m_{\mathbf{k}}$ and X to be given quantities and shall use them in a perturbation calculation of the leading correction to $\omega_0(k)$.

We also use the pseudopotential method in which the approximate many-body Hamiltonian, to be used in conjunction with ordinary perturbation theory, can be written as[7]

$$H \cong \sum_{\mathbf{k}} k^2 N_{\mathbf{k}} + \frac{4\pi a}{\Omega}\sum_{\mathbf{k}1,\mathbf{k}2,\mathbf{k}3,\mathbf{k}4} a_1^{\dagger}a_2^{\dagger}a_3a_4 \times\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\cos\tfrac{1}{2}\epsilon|\mathbf{k}_3-\mathbf{k}_4|, \quad (7)$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$ are, respectively, the annihilation and creation operators of the free-particle state vectors $|\mathbf{k}\rangle$. The sum over $\mathbf{k}1$, etc., means the sum over $\mathbf{k}_1$, etc. Using expressions from Appendix I of MS,[3] the Hamiltonian H can be written as follows:

$$H = H_0 + H_1 + H_2 + H_3 + \text{(higher order terms)}, \quad (8)$$

[5] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).

[6] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

[7] T. T. Wu, Phys. Rev. **115**, 1390 (1959).

where

$$H_0=4\pi a\rho N[1+(1-X)^2]+\sum_{\mathbf{k}}{}'(k^2+k_0^2)N_{\mathbf{k}}$$
$$+\tfrac{1}{2}k_0^2\sum_{\mathbf{k}}{}'[a_{\mathbf{k}}{}^\dagger a_{-\mathbf{k}}{}^\dagger+a_{\mathbf{k}}a_{-\mathbf{k}}\cos k\epsilon],$$
$$H_1=-\frac{4\pi a}{\Omega}\Big(\sum_{\mathbf{k}}{}'\,\delta_{\mathbf{k}}\Big)^2$$
$$-\frac{4\pi a}{\Omega}\Big(\sum_{\mathbf{k}}{}'\,\delta_{\mathbf{k}}\Big)\sum_{\mathbf{l}}{}'[a_{\mathbf{l}}{}^\dagger a_{-\mathbf{l}}{}^\dagger+a_{\mathbf{l}}a_{-\mathbf{l}}\cos l\epsilon], \tag{9}$$
$$H_2=\frac{8\pi a(XN)^{\frac{1}{2}}}{\Omega}\sum_{\mathbf{k},\mathbf{k}'}{}'\,[a_{\mathbf{k}}{}^\dagger a_{\mathbf{k}'}a^\dagger{}_{\mathbf{k}+\mathbf{k}'}\cos\tfrac{1}{2}\epsilon|\mathbf{k}+\mathbf{k}'|$$
$$+a_{\mathbf{k}+\mathbf{k}'}{}^\dagger a_{\mathbf{k}}a_{\mathbf{k}'}\cos\tfrac{1}{2}\epsilon|\mathbf{k}-\mathbf{k}'|\,],$$
$$H_3=\frac{4\pi a}{\Omega}\sum_{\mathbf{k}\neq\pm\mathbf{k}'}{}'\,a_{\mathbf{k}}{}^\dagger a_{-\mathbf{k}}{}^\dagger a_{\mathbf{k}'}a_{-\mathbf{k}'}\cos k'\epsilon.$$

In the zero-temperature limit, i.e., in the limit $X=1$, the approximate Hamiltonian (8) and (9) reduces to Eqs. (I.4) and (I.5) of MS. As in MS *we must take the limit* $\epsilon\to 0+$ *at the end of any calculation*, since the role of the $\cos k\epsilon$ term in Eq. (7) is to eliminate spurious divergences which would arise in perturbation theory calculations should we set $\epsilon=0$ before performing momentum-space sums.[7]

The various parts of the approximate Hamiltonian (8) have a significance similar to that discussed in MS for the corresponding zero-temperature parts. Thus, H_0 gives the leading contribution to the energy levels of a dilute Bose gas of hard spheres at a given temperature, i.e., for a given X, as first calculated by Lee and Yang. In particular, the dominant part $\omega_0(k)$ of the quasi-particle energy comes from H_0. The terms H_1, H_2, and H_3 then give the leading correction to the energy levels in the low-density limit. The quantity H_1 arises because of the changes in the free-particle occupation numbers from $m_{\mathbf{k}}$ to $\langle N_{\mathbf{k}}\rangle$ which occur when we switch (adiabatically) from the unperturbed free-particle states to corresponding states of the interacting system. We may say that H_1 corrects the error of using the number X in H_0 instead of the operator $(1-N^{-1}\sum_{\mathbf{k}}{}'N_{\mathbf{k}})$. Now, H_0 includes the leading effect of the hard core interaction in converting pairs of zero momentum particles into pairs of particles in the states $|\mathbf{k},-\mathbf{k}\rangle$. Similarly, H_2 includes the leading terms of the interaction which change the occupation number of the zero-momentum state by one. Finally, H_3 gives the effect of the interaction in converting pairs of particles in the states $|\mathbf{k}',-\mathbf{k}'\rangle$ into other states $|\mathbf{k},-\mathbf{k}\rangle$.

II. OUTLINE OF THE CALCULATION

It is convenient to imagine the denumerably infinite set of plane wave states to be ordered in a sequence $(\mathbf{k}_1,\mathbf{k}_{-1},\mathbf{k}_2,\mathbf{k}_{-2},\cdots)$, where $\mathbf{k}_{-n}=-\mathbf{k}_n$. Then the occupation numbers $m_{\mathbf{k}}$ of the unperturbed states can be written in the corresponding ordered sequence $(m_1,m_{-1},m_2,m_{-2},\cdots)$. Now the numbers $m_{\mathbf{k}}$ are also the quantum numbers which characterize the eigenstate vectors of the dominant part H_0 of the many-body Hamiltonian (8), i.e., they are also the (approximate) occupation numbers of the quasi-particle states. Thus the "right" eigenstates of H_0 are[8]

$$|m_1m_{-1}m_2\cdots m_i\cdots\rangle$$
$$=\prod_{i>0}K_{\mathbf{k}i}{}^{(m_i,m_{-i})}(\xi_i{}^\dagger)^{m_i}(\xi_{-i}{}^\dagger)^{m_{-i}}|0_{\mathbf{k}i}\rangle, \tag{10}$$

where the $K_{\mathbf{k}i}{}^{(m_i,m_{-i})}$ are normalization constants, and

$$\xi_{\mathbf{k}}{}^\dagger\equiv(1-\alpha_k{}^2)^{-\frac{1}{2}}(a_{\mathbf{k}}{}^\dagger+\alpha_k a_{-\mathbf{k}}),$$
$$\alpha_k=(k_0{}^2\cos k\epsilon)^{-1}\{k^2+k_0{}^2-[k^4+2k_0{}^2k^2$$
$$+k_0{}^4(1-\cos k\epsilon)]^{\frac{1}{2}}\}\underset{k\to\infty}{\longrightarrow}\frac{1}{2}\left(\frac{k_0}{k}\right)^2\left[1+0\left(\frac{k_0}{k}\right)^2\right]. \tag{11}$$

Similarly, the "left" eigenstates of H_0 are

$$\langle m_1m_{-1}m_2\cdots m_i\cdots|$$
$$=\prod_{i>0}\bar{K}_{\mathbf{k}i}{}^{(m_i,m_{-i})}\langle 0_{\mathbf{k}i}|\,(\bar{\xi}_i)^{m_i}(\bar{\xi}_{-i})^{m_{-i}}, \tag{12}$$

where

$$\bar{\xi}_k\equiv(1-\bar{\alpha}_k{}^2)^{-\frac{1}{2}}(a_{\mathbf{k}}+\bar{\alpha}_k a_{-\mathbf{k}}{}^\dagger),\quad \bar{\alpha}=\alpha_k\cos k\epsilon. \tag{13}$$

We shall now describe the calculation of the correction to the quasi-particle energy $\omega_0(k)$, Eq. (5), in terms of the above right and left eigenstates. We begin by recalling the parameter argument of the zero-temperature calculation (Sec. III of MS). At $T=0$, the only parameter which enters into the calculation of the energy of a dilute Bose gas of hard spheres is the small parameter $(\rho a^3)^{\frac{1}{2}}$ of Eq. (4). In terms of this parameter, it was argued in MS that the perturbation-theory contributions of H_1, H_2, and H_3 to the zero-temperature quasi-particle energies were small compared to $\omega_0(k)$ and large compared to any other contributions. Although there are other, temperature-dependent parameters when $T\neq 0$, we assume for the present that the zero-temperature argument still suffices to pick out the leading corrections to $\omega_0(k)$. In analogy with Eqs. (III.1) and (III.2) of MS we therefore write the quasi-particle energy as

$$\omega(k)=\omega_0(k)+\omega_1(k)+\omega_{2,+1}(k)+\omega_{2,-1}(k)+\omega_{2,+3}(k)$$
$$+\omega_{2,-3}(k)+\omega_3(k)+\text{(higher order terms)}, \tag{14}$$

[8] The distinction between "right" and "left" eigenstates is explained in Sec. II of MS and in reference 7.

where

$$\omega_1(k)=\lim_{\epsilon\to0+}\lim_{\Omega\to\infty}\{\langle m_{\mathbf{k}}+1|H_1|m_{\mathbf{k}}+1\rangle-\langle m_{\mathbf{k}}|H_1|m_{\mathbf{k}}\rangle\},$$

$$\omega_{2,\pm1}(k)=\pm\lim_{\epsilon\to0+}\lim_{\Omega\to\infty}\sum_{\substack{11\leqq12\\(13=11+1_2)}}{}'\left[\frac{1}{\omega_0(l_3)-\omega_0(l_2)-\omega_0(l_1)\pm i\delta}\right]$$
$$\times\{\langle m_{\mathbf{k}}+1|H_2|m_{11}\pm1,\,m_{12}\pm1,\,m_{13}\mp1,\,m_{\mathbf{k}}+1\rangle\langle m_{11}\pm1,\,m_{12}\pm1,\,m_{13}\mp1,\,m_{\mathbf{k}}+1|H_2|m_{\mathbf{k}}+1\rangle$$
$$-\langle m_{\mathbf{k}}|H_2|m_{11}\pm1,\,m_{12}\pm1,\,m_{13}\mp1,\,m_{\mathbf{k}}\rangle\langle m_{11}\pm1,\,m_{12}\pm1,\,m_{13}\mp1,\,m_{\mathbf{k}}|H_2|m_{\mathbf{k}}\rangle\},\tag{15}$$

$$\omega_{2,\pm3}(k)=\mp\lim_{\epsilon\to0+}\lim_{\Omega\to\infty}\sum_{\substack{11\leqq12\leqq13\\(11+12+13=0)}}{}'\left[\frac{1}{\omega_0(l_1)+\omega_0(l_2)+\omega_0(l_3)}\right]$$
$$\times\{\langle m_{\mathbf{k}}+1|H_2|m_{11}\pm1,\,m_{12}\pm1,\,m_{13}\pm1,\,m_{\mathbf{k}}+1\rangle\langle m_{11}\pm1,\,m_{12}\pm1,\,m_{13}\pm1,\,m_{\mathbf{k}}+1|H_2|m_{\mathbf{k}}+1\rangle$$
$$-\langle m_{\mathbf{k}}|H_2|m_{11}\pm1,\,m_{12}\pm1,\,m_{13}\pm1,\,m_{\mathbf{k}}\rangle\langle m_{11}\pm1,\,m_{12}\pm1,\,m_{13}\pm1,\,m_{\mathbf{k}}|H_2|m_{\mathbf{k}}\rangle\},$$

$$\omega_3(k)=\lim_{\epsilon\to0+}\lim_{\Omega\to\infty}\{\langle m_{\mathbf{k}}+1|H_3|m_{\mathbf{k}}+1\rangle-\langle m_{\mathbf{k}}|H_3|m_{\mathbf{k}}\rangle\}.$$

In Eqs. (15) we have suppressed all quantum numbers $m_{\mathbf{l}}$ which are the same for each of the matrix elements of a given term. Furthermore, the subscripts $\mathbf{l}_1$, $\mathbf{l}_2$, and $\mathbf{l}_3$ of m are conventionally printed as 11, 12, and 13, respectively.

Normalization of the State Vectors

In order to evaluate the matrix elements of Eqs. (15), we must first determine the normalization constants $K_{\mathbf{k}}^{(m_{\mathbf{k}},m_{-\mathbf{k}})}$ and $\bar{K}_{\mathbf{k}}^{(m_{\mathbf{k}},m_{-\mathbf{k}})}$ of Eqs. (10) and (12). Thus, the normalization condition

$$\begin{aligned}1&=\langle m_{\mathbf{k}},n_{-\mathbf{k}}|m_{\mathbf{k}},n_{-\mathbf{k}}\rangle\\&=\bar{K}_{\mathbf{k}}^{(m,n)}K_{\mathbf{k}}^{(m,n)}\\&\quad\times\langle0_{\mathbf{k}}|(\bar{\xi}_{\mathbf{k}})^{m_{\mathbf{k}}}(\bar{\xi}_{-\mathbf{k}})^{n_{-\mathbf{k}}}(\xi_{-\mathbf{k}}{}^\dagger)^{n_{-\mathbf{k}}}(\xi_{\mathbf{k}}{}^\dagger)^{m_{\mathbf{k}}}|0_{\mathbf{k}}\rangle,\end{aligned}\tag{16}$$

gives an expression for each of the products $\bar{K}_{\mathbf{k}}^{(m,n)}K_{\mathbf{k}}^{(m,n)}$.

In order to evaluate the matrix element on the right-hand side of Eq. (16), we need two identities. We define two operators $b_{\mathbf{k}}$ and $\bar{b}_{\mathbf{k}}$ as follows:

$$\begin{aligned}b_{\mathbf{k}}&\equiv(1-\alpha_k{}^2)^{\frac12}\xi_{\mathbf{k}}=a_{\mathbf{k}}+\alpha_k a_{-\mathbf{k}}{}^\dagger,\\\bar{b}_{\mathbf{k}}&\equiv(1-\bar{\alpha}_k{}^2)^{\frac12}\bar{\xi}_{\mathbf{k}}=a_{\mathbf{k}}+\bar{\alpha}_k a_{-\mathbf{k}}{}^\dagger.\end{aligned}\tag{17}$$

Then the first identity, for $m\leqq n$, is:

$$(\bar{b}_{\mathbf{k}})^n(b_{\mathbf{k}}{}^\dagger)_m=\sum_{p=0}^{m}\frac{m!n!}{(m-p)!(n-p)!p!}\times(1-\alpha_k\bar{\alpha}_k)^p(b_{\mathbf{k}}{}^\dagger)^{m-p}(\bar{b}_{\mathbf{k}})^{n-p},\tag{18}$$

as can be verified with the aid of the usual commutation relations for the plane wave operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}{}^\dagger$. The second identity,

$$\langle0_{\mathbf{k}}|(a_{\mathbf{k}})^n(a_{-\mathbf{k}})^n|0_{\mathbf{k}}\rangle=\frac{(n!)(-\alpha_k)^n}{(1-\alpha_k\bar{\alpha}_k)^n},\tag{19}$$

can be proved with the aid of the relations $b_{\mathbf{k}}|0_{\mathbf{k}}\rangle=\langle0_{\mathbf{k}}|\bar{b}_{\mathbf{k}}{}^\dagger=0$. After substituting the identities (18) and (19) into Eq. (16), we obtain the following expression:

$$1=\bar{K}_{\mathbf{k}}^{(m,n)}K_{\mathbf{k}}^{(m,n)}\frac{(1-\bar{\alpha}_k{}^2)^{\frac12(m-n)}(1-\alpha_k{}^2)^{\frac12(m-n)}}{(1-\alpha_k\bar{\alpha}_k)^{m-n}}\times\sum_{p=0}^{n}\left(\frac{n!}{p!}\right)^2\frac{(m+p)!}{(n-p)!}(-1)^p[\phi(k,\epsilon)]^{2p},\tag{20}$$

where

$$\begin{aligned}[\phi(k,\epsilon)]&\equiv\left(\frac{\alpha_k-\bar{\alpha}_k}{1-\alpha_k\bar{\alpha}_k}\right)\\&=\frac{(1-\cos k\epsilon)}{2[(k/k_0)^4+2(k/k_0)^2+(1-\cos k\epsilon)]^{\frac12}}\\&\underset{k\to\infty}{\longrightarrow}\frac{k_0{}^2}{2k^2}(1-\cos k\epsilon).\end{aligned}\tag{21}$$

We now observe that in the high-momentum region, $k\gg k_0$, both $\phi(k,\epsilon)$ and α_k are of $\sim k_0{}^2k^{-2}$ and are therefore very small quantities. We conclude that the high-momentum behavior of any integrals is not changed if we set $\epsilon=0$ wherever the factors $(1-\alpha_k{}^2)$, $(1-\bar{\alpha}_k{}^2)$, $(1-\alpha_k\bar{\alpha}_k)$, and $\phi(k,\epsilon)$ appear. In particular, we may set $\epsilon=0$ in Eq. (20) to obtain $1=[\bar{K}_{\mathbf{k}}^{(m,n)}K_{\mathbf{k}}^{(m,n)}]m!n!$, or

$$K_{\mathbf{k}}^{(m_{\mathbf{k}},m_{-\mathbf{k}})}=\bar{K}_{\mathbf{k}}^{(m_{\mathbf{k}},m_{-\mathbf{k}})}=K^{(m_{\mathbf{k}})}K^{(m_{-\mathbf{k}})},\tag{22a}$$

with

$$K^{(m)}=(m!)^{-\frac12}.\tag{22b}$$

In a similar manner, we may show that the various state vectors (10) and (12) are orthogonal. We conclude that these state vectors may be considered to form an orthonormal set, although this is rigorously true only in the limit $\epsilon=0$.

Evaluation of the Perturbation Matrix Elements

The evaluation of the matrix elements of H_1 and H_3 which appear in Eqs. (15) is readily performed with

the aid of the identities:

$$\langle m_{\mathbf{k}},m_{-\mathbf{k}}|a_{\mathbf{k}}a_{-\mathbf{k}}|m_{\mathbf{k}},m_{-\mathbf{k}}\rangle = -(m_{\mathbf{k}}+m_{-\mathbf{k}}+1)\left(\frac{\alpha_k}{1-\alpha_k\bar{\alpha}_k}\right),$$

$$\langle m_{\mathbf{k}},m_{-\mathbf{k}}|a_{\mathbf{k}}^{\dagger}a_{-\mathbf{k}}^{\dagger}|m_{\mathbf{k}},m_{-\mathbf{k}}\rangle = -(m_{\mathbf{k}}+m_{-\mathbf{k}}+1)\left(\frac{\bar{\alpha}_k}{1-\alpha_k\bar{\alpha}_k}\right), \quad (23)$$

$$\langle m_{\mathbf{k}},m_{-\mathbf{k}}|\delta_{\mathbf{k}}|m_{\mathbf{k}},m_{-\mathbf{k}}\rangle = (m_{\mathbf{k}}+m_{-\mathbf{k}}+1)\left(\frac{\alpha_k\bar{\alpha}_k}{1-\alpha_k\bar{\alpha}_k}\right).$$

In each of these identities, terms proportional to $[\phi(k,\epsilon)]^p$ with $p>0$ have been dropped, since such terms do not affect the high-momentum behavior of the integrals which occur in $\omega_1(k)$ and $\omega_3(k)$. To evaluate the matrix elements of H_2, it is simpler to write H_2 of Eqs. (9) in terms of $\xi_{\mathbf{k}}$ and $\xi_{\mathbf{k}}^{\dagger}$ [see Eq. (11)], of in terms of $\bar{\xi}_{\mathbf{k}}$ and $\bar{\xi}_{\mathbf{k}}^{\dagger}$.

We now write down the expressions which are obtained for the various terms of Eqs. (15) in the limit of infinite volume. Introducing the dimensionless parameter

$$y \equiv k/k_0, \quad (24)$$

we derive the following integrals:

$$\omega_1 = \lim_{\epsilon\to 0+}\left(\frac{8\pi k_0^3 a}{1-\alpha_y^2}\right)\int\frac{d^3y_1}{(2\pi)^3}[1+m(\mathbf{y}_1)+m(-\mathbf{y}_1)]\left(\frac{\alpha_1\cos\epsilon y_1}{1-\alpha_1^2}\right)[(1+\alpha_y^2)-(1-\alpha_y)^2\alpha_1], \quad (25a)$$

$$\omega_{2,+1}(k)+\omega_{2,-1}(k) = \omega_R^{(a)}(k)+\omega_R^{(b)}(k)-i[\omega_I^{(a)}(k)+\omega_I^{(b)}(k)], \quad (25b)$$

$$\omega_R^{(a)} = \lim_{\epsilon\to 0+}\frac{16\pi k_0^3 a}{(1-\alpha_y^2)}\mathrm{P}\int\frac{d^3y_1}{(2\pi)^3}\left[\frac{1}{\omega_0(y)-\omega_0(y_1)-\omega_0(y_2)}\right][1+m(\mathbf{y}_1)+m(\mathbf{y}_2)]\frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)}\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}\},$$

$$\omega_R^{(b)} = \lim_{\epsilon\to 0+}\frac{32\pi k_0^3 a}{(1-\alpha_y^2)}\mathrm{P}\int\frac{d^3y_1}{(2\pi)^3}\left[\frac{1}{\omega_0(y_2)-\omega_0(y_1)-\omega_0(y)}\right][m(\mathbf{y}_2)-m(\mathbf{y}_1)]\frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)}\{\mathbf{y},-\mathbf{y}_1,\mathbf{y}_2\},$$

$$\omega_I^{(a)} = \lim_{\epsilon\to 0+}\frac{16\pi^2 k_0^3 a}{(1-\alpha_y^2)}\int\frac{d^3y_1}{(2\pi)^3}\delta[\omega_0(y)-\omega_0(y_1)-\omega_0(y_2)] \times[1+m(\mathbf{y}_1)+m(\mathbf{y}_2)+2m(\mathbf{y}_1)m(\mathbf{y}_2)]\frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)}\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}\},$$

$$\omega_I^{(b)} = \lim_{\epsilon\to 0+}\frac{32\pi^2 k_0^3 a}{(1-\alpha_y^2)}\int\frac{d^3y_1}{(2\pi)^3}\delta[\omega_0(y_2)-\omega_0(y_1)-\omega_0(y)] \times[m(\mathbf{y}_1)+m(\mathbf{y}_2)+2m(\mathbf{y}_1)m(\mathbf{y}_2)]\frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)}\{\mathbf{y},-\mathbf{y}_1,\mathbf{y}_2\},$$

$$\begin{aligned}\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}\} = {}&[\cos\tfrac12\epsilon y+\alpha_1\alpha_y\cos\tfrac12\epsilon y_1+\alpha_2\alpha_y\cos\tfrac12\epsilon y_2-\alpha_1\cos\tfrac12\epsilon|\mathbf{y}_1+\mathbf{y}|-\alpha_2\cos\tfrac12\epsilon|\mathbf{y}_2+\mathbf{y}|\\ &-\alpha_1\alpha_2\alpha_y\cos\tfrac12\epsilon|\mathbf{y}_1-\mathbf{y}_2|][\cos\tfrac12\epsilon|\mathbf{y}_1-\mathbf{y}_2|+\alpha_1\alpha_y\tfrac12\epsilon|\mathbf{y}_2+\mathbf{y}|+\alpha_2\alpha_y\cos\tfrac12\epsilon|\mathbf{y}_1+\mathbf{y}|\\ &-\alpha_1\cos\tfrac12\epsilon y_2-\alpha_2\cos\tfrac12\epsilon y_1-\alpha_1\alpha_2\alpha_y\cos\tfrac12\epsilon y],\end{aligned}$$

$$\begin{aligned}\omega_{2,3}(k) \equiv {}&\omega_{2,+3}(k)+\omega_{2,-3}(k)\\ = {}&-\lim_{\epsilon\to 0+}\frac{16\pi k_0^3 a}{(1-\alpha_y^2)}\int\frac{d^3y_1}{(2\pi)^3}\left[\frac{1}{\omega_0(y)+\omega_0(y_1)+\omega_0(y_2)}\right][1+m(\mathbf{y}_1)+m(\mathbf{y}_2)]\frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)}\\ &\times[\alpha_1\alpha_2\cos\tfrac12\epsilon|\mathbf{y}_1-\mathbf{y}_2|+\alpha_1\alpha_y\cos\tfrac12\epsilon|\mathbf{y}_1+\mathbf{y}|+\alpha_2\alpha_y\cos\tfrac12\epsilon|\mathbf{y}_2+\mathbf{y}|-\alpha_y\cos\tfrac12\epsilon y\\ &-\alpha_1\cos\tfrac12\epsilon y_1-\alpha_2\cos\tfrac12\epsilon y_2][\alpha_1\alpha_2\cos\tfrac12\epsilon y+\alpha_1\alpha_y\cos\tfrac12\epsilon y_2+\alpha_2\alpha_y\cos\tfrac12\epsilon y_1\\ &-\alpha_y\cos\tfrac12\epsilon|\mathbf{y}_1-\mathbf{y}_2|-\alpha_1\cos\tfrac12\epsilon|\mathbf{y}_2+\mathbf{y}|-\alpha_2\cos\tfrac12\epsilon|\mathbf{y}_1+\mathbf{y}|],\end{aligned} \quad (25c)$$

$$\omega_3 = \lim_{\epsilon\to 0+}16\pi k_0^3 a\left(\frac{\alpha_y}{1-\alpha_y^2}\right)\int\frac{d^3y_1}{(2\pi)^3}[1+m(\mathbf{y}_1)+m(-\mathbf{y}_1)]\left(\frac{\alpha_1\cos\epsilon y_1}{1-\alpha_1^2}\right), \quad (25d)$$

where

$$\mathbf{y}_2 = \mathbf{y}-\mathbf{y}_1,$$
$$\alpha_y = 1+y^2-y(2+y^2)^{\frac12}, \quad \alpha_1 \equiv \alpha_{y_1}, \quad (26)$$
$$\omega_0(y) = y(2+y^2)^{\frac12}.$$

In the zero temperature limit we have $m(y)=0$, and the above expressions reduce to Eqs. (III.4) of MS.

We conclude this section with an order of magnitude examination of the $m(y_i)$-dependent terms in $\omega(k)$. At very low temperatures the quantity $m(y_i)$ is extremely

small and therefore the zero-temperature parts of $\omega(k)$ are much larger than the $m(y_i)$-dependent parts. At higher temperatures, however, the reverse situation holds. It will be shown in the following section that the relevant parameter for investigating the temperature dependence of these expressions is

$$t=k_0\lambda_T/(4\pi)^{\frac{1}{2}}=(2X\rho a\lambda_T^2)^{\frac{1}{2}}, \tag{27}$$

where

$$\lambda_T=(4\pi\beta)^{\frac{1}{2}}=(4\pi/kT)^{\frac{1}{2}}=\text{thermal wavelength.} \tag{28}$$

At very low temperatures, $t\gg1$, whereas at a temperature just below the critical temperature $t\ll1$. It is the latter region in which we shall be most interested.

In the region $t\ll1$, each factor of $m(y_i)$ *in the real part of* $\omega(k)$ behaves as t^{-1} to first approximation, and hence the $m(y_i)$ dependent terms dominate in each of the integrals of Eqs. (25). We shall show, in fact, that for $y=k/k_0\ll1$

$$\mathrm{Re}\left[\frac{\omega(k)-\omega_0(k)}{\omega_0(k)}\right]\sim k_0at^{-1}\sim\frac{a}{\lambda_T}\ll1. \tag{29}$$

Thus, the real part of the correction to the quasiparticle energy $\omega_0(k)$ is a small correction for all T.

A similar situation holds for $\omega_I^{(a)}(k)$ of Eq. (25b). On the other hand $\omega_I^{(b)}\sim k_0\lambda_T^{-1}$ in the temperature region $t\ll1$. This term is therefore large compared to $\omega_0(k)$. It will be discussed in greater detail in Sec. V.

III. CALCULATION OF $m(y)$ AND X

Explicit expressions for the quantities $m(y)$ and X of Eqs. (1) and (2) have been derived by Lee and Yang, by substituting the energy eigenvalues of H_0, Eq. (9), into the partition function of statistical mechanics. The expressions which they have derived are therefore only first approximations to these quantities, but this, however, is all we require for the calculation of the leading correction to $\omega_0(k)$. From their paper on the equilibrium properties of a low-density Bose gas of hard spheres[2] we obtain

$$m(y)=\frac{\zeta e^{-t\omega_0(y)}}{1-\zeta e^{-t\omega_0(y)}}, \tag{30}$$

and

$$(1-X)\rho=\frac{k_0^3}{(2\pi)^3}\int d^3y\, m(y), \tag{31}$$

where

$$\epsilon\equiv-\ln\zeta=\frac{k_0at}{\pi^2}\int d^3y\left[1-\frac{y}{(2+y^2)^{\frac{1}{2}}}\right]m(y). \tag{32}$$

The quantity t is defined by Eq. (27). We also define a quantity δ,

$$\delta\equiv\epsilon/t, \tag{33}$$

and shall show below that $\delta\ll1$ for all $T<T_c$.

There are two temperature-density regions in which we shall be interested; namely, the regions $t\ll1$ and $t\gg1$. Before proceeding to evaluate the various integrals of Eqs. (25), it is well to have a feeling for the orders of magnitude of the quantities ϵ and X for the different regions of t. Moreover, the approximation techniques for performing the integrals of Eqs. (25) are somewhat involved and are best illustrated by the simpler calculations of X and ϵ. We shall therefore begin by rederiving the approximate expressions which Lee and Yang have obtained for these latter quantities.[2]

We first consider the "high" temperature region $t\ll1$ and make the change of variable in Eq. (31)

$$z=\omega_0(y)=y(2+y^2)^{\frac{1}{2}} \tag{34}$$

to obtain

$$\begin{aligned}(1-X)\rho&=\frac{k_0^3}{(2\pi)^2}\int_0^\infty zdz\left[\frac{(1+z^2)^{\frac{1}{2}}-1}{1+z^2}\right]^{\frac{1}{2}}\left[\frac{e^{-t(\delta+z)}}{1-e^{-t(\delta+z)}}\right]\\&=\frac{k_0^3}{(2\pi)^2}\int_0^\infty zdz\left\{z^{-\frac{1}{2}}-\frac{z^{-\frac{3}{2}}}{2}+\left(\left[\frac{(1+z^2)^{\frac{1}{2}}-1}{1+z^2}\right]^{\frac{1}{2}}-z^{-\frac{1}{2}}+\frac{z^{-\frac{3}{2}}}{2}\right)\right\}\left[\frac{e^{-t(\delta+z)}}{1-e^{-t(\delta+z)}}\right]\\&=\frac{k_0^3}{(4\pi t)^{\frac{3}{2}}}[g_{\frac{3}{2}}(\epsilon)-tg_{\frac{1}{2}}(\epsilon)]+k_0^3\Delta(t,\delta)+\frac{k_0^3}{(2\pi)^2t}\int_0^\infty\frac{zdz}{(z+\delta)}\left\{\left[\frac{(1+z^2)^{\frac{1}{2}}-1}{1+z^2}\right]^{\frac{1}{2}}-z^{-\frac{1}{2}}+\frac{z^{-\frac{3}{2}}}{2}\right\},\end{aligned} \tag{35}$$

where

$$g_p(\epsilon)\equiv\sum_{n=1}^\infty n^{-p}e^{-n\epsilon}, \tag{36}$$

and

$$\Delta(t,\delta)\equiv\frac{1}{(2\pi)^2}\int_0^\infty zdz\left\{\left[\frac{(1+z^2)^{\frac{1}{2}}-1}{1+z^2}\right]^{\frac{1}{2}}-z^{-\frac{1}{2}}+\frac{z^{-\frac{3}{2}}}{2}\right\}\left[\frac{e^{-t(\delta+z)}}{1-e^{-t(\delta+z)}}-\frac{1}{t(\delta+z)}\right]\to0\quad\text{as}\quad t\to0. \tag{37}$$

We see that by asymptotically expanding in powers of t, the most difficult part of the integral for $(1-X)\rho$ does not need to be done in the region $t\ll1$. The reason for this simplification is that the last term in the third

line of Eq. (35) is a convergent integral. Thus, one finds for the quantity $(1-X)\rho$:

$$(1-X)\rho=\frac{k_0^3}{(4\pi t)^{\frac{3}{2}}}\left\{g_{\frac{3}{2}}(\epsilon)-t\left[g_{\frac{1}{2}}(\epsilon)-\left(\frac{\pi}{\epsilon}\right)^{\frac{1}{2}}\right]+A(\delta)t^{\frac{1}{2}}\right\}+k_0^3\Delta(k), \quad (38)$$

where

$$A(\delta)=\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{zdz}{(z+\delta)}\left\{\left[\frac{(1+z^2)^{\frac{1}{2}}-1}{1+z^2}\right]^{\frac{1}{2}}-z^{-\frac{1}{2}}\right\}$$
$$=-4(2/\pi)^{\frac{1}{2}}+2(\pi\delta)^{\frac{1}{2}}+O(\delta). \quad (39)$$

The complete integrated expression for $A(\delta)$ is given by Eq. (A16) of reference 2. With the aid of the expressions[9]

$$\begin{aligned}g_{\frac{1}{2}}(\epsilon)-(\pi/\epsilon)^{\frac{1}{2}}&=-1.460+O(\epsilon),\\ g_{\frac{3}{2}}(\epsilon)&=2.612-2(\pi\epsilon)^{\frac{1}{2}}+1.460\epsilon+O(\epsilon^2).\end{aligned} \quad (40)$$

We finally obtain for $(1-X)$:

$$(1-X)=(\rho\lambda_T^3)^{-1}[2.612-4(2/\pi)^{\frac{1}{2}}t^{\frac{1}{2}}+1.460t+O(\epsilon^{\frac{3}{2}})]. \quad (41)$$

In a similar manner one may show that for $t\ll1$

$$\epsilon=2\pi^{-\frac{1}{2}}(k_0a)[(\pi-2)(2/\pi)^{\frac{1}{2}}-1.460t^{\frac{1}{2}}+O(\delta^{\frac{1}{2}})]. \quad (42)$$

From Eq. (42) it is seen that in the region $t\ll1$, the quantity δ of Eq. (33) is $\sim a/\lambda_T$. We also note that $(\delta/t)^2\sim(a/\lambda_T)(1-X)/X\ll1$. Now at very low temperatures, i.e., when $t\gtrsim1$ or $t\gg1$, δ is a completely negligible quantity and the quantities $(1-X)$ and ϵ can be written as follows:

$$(1-X)=\frac{k_0^3}{(2\pi)^2\rho}\int_0^\infty\frac{zdz}{(1+z^2)}[(1+z^2)^{\frac{1}{2}}-1]^{\frac{1}{2}}\left[\frac{e^{-tz}}{1-e^{-tz}}\right]$$
$$=2(2/\pi)^{\frac{1}{2}}(a/\lambda_T)X[2g_3(1)t^{-2}-15g_5(1)t^{-4}+O(t^{-6})], \quad (43)$$

and

$$\epsilon=\sqrt{2}\pi^{-1}(k_0a)[2g_3(1)t^{-2}-3g_4(1)t^{-3}+O(t^{-4})]. \quad (44)$$

Thus, the quantities $(1-X)$, ϵ, and δ are all extremely small in the region $t\gg1$, and for all practical purposes we may assume that they are zero.

We now return to the calculation of the correction to the quasi-particle energy $\omega_0(k)$, and use Eq. (30) for $m(y)$, where both X and ϵ are the known quantities of Eqs. (31) and (32).

IV. LOW MOMENTUM EXCITATIONS, $k\ll k_0$

The leading correction to the quasi-particle energy $\omega_0(k)$, Eq. (14), is determined by performing the integrals of Eqs. (25). We have only evaluated the leading terms of these integrals in the low-momentum region. $y=k/k_0\ll1$.

We consider first the very low temperature-density region $t\sim k_0\lambda_T\gg1$. In this region we obtain

$$\omega(k)=[\omega(k)]_{T=0}[1+O(k_0at^{-2}\ln t)] \quad t\gg1, \quad (45)$$

where $[\omega(k)]_{T=0}$ is given by Eq. (IV.2) of MS.

$$[\omega(k)]_{T=0}=\sqrt{2}k_0k\left[1+8\pi^{-\frac{1}{2}}(\rho a^3)^{\frac{1}{2}}+O\left(\frac{k}{k_0}\right)^2+O(\rho a^3)\right]. \quad (46)$$

Thus, the zero temperature term is the dominant term in $\omega(k)$ for $t\gg1$. The coefficient of the leading temperature-dependent term of (45) is rather complex to calculate, and comes from the integral $\omega_R^{(b)}$ of Eq. (25b).

In the "high" temperature region $t\sim k_0\lambda_T\ll1$, we find that the $m(y_i)$-dependent parts of the integrals of Eqs. (25) give the leading corrections to the quasi-particle energy $\omega_0(k)$. To evaluate these integrals we must use the technique demonstrated in Sec. III in the evaluation of $(1-X)$. The final result for the *real* part of the correction to the energy $\omega_0(k)$ is

$$\begin{aligned}\mathrm{Re}\,\omega(k)&=\sqrt{2}k_0k+2\sqrt{2}(k_0^2ka/\pi t)\times[-1.48+3.88t^{\frac{1}{2}}]+O(k_0^2ka)\\&=\sqrt{2}k_0k\{1+4(2/\pi)^{\frac{1}{2}}(a/\lambda_T)\times[-1.48+3.88t^{\frac{1}{2}}]+O(k_0a)\}.\end{aligned} \quad (47)$$

We see that in the region $t\ll1$, the real part of the correction to $\omega_0(k)$ is a small term which is of $\sim(a/\lambda_T)\omega_0(k)$ where we always assume that $a/\lambda_T\ll1$.

V. QUASI-PARTICLE LIFETIME FOR $k\ll k_0$ AND $t\ll1$

The most important "correction" term to the energy $\omega_0(k)$ is the region $t\ll1$ is the imaginary part of $\omega(k)$, which is given by $\omega_I^{(a)}$ and $\omega_I^{(b)}$ of Eqs. (25b). These two terms may be simplified to the expressions

$$\begin{aligned}\omega_I^{(a)}&=\frac{k_0^3a}{8yz\alpha_z}\int_0^z dz_1\frac{(1+2m_1+2m_1m_2)}{\alpha_1\alpha_2(1+z_1^2)^{\frac{1}{2}}(1+z_2^2)^{\frac{1}{2}}}[1-\alpha_1-\alpha_2+\alpha_1\alpha_z+\alpha_2\alpha_z-\alpha_1\alpha_2\alpha_z]^2, \quad (z_2=z-z_1)\\ \omega_I^{(b)}&=\frac{k_0^3a}{4yz\alpha_z}\int_0^\infty dz_1\frac{(m_1+m_2+2m_1m_2)}{\alpha_1\alpha_2(1+z_1^2)^{\frac{1}{2}}(1+z_2^2)^{\frac{1}{2}}}[1-\alpha_1-\alpha_z+\alpha_z\alpha_2+\alpha_1\alpha_2-\alpha_1\alpha_2\alpha_z]^2, \quad (z_2=z+z_1)\end{aligned} \quad (48)$$

[9] See, e.g., the Appendix of reference 4.

$$
\begin{aligned}
z&=y(2+y^2)^{\frac{1}{2}},\\
\alpha_z&=(1+z^2)^{\frac{1}{2}}-z, \quad \alpha_1\equiv\alpha_{z1},\\
m_1&=\frac{e^{-t(\delta+z_1)}}{1-e^{-t(\delta+z_1)}}.
\end{aligned} \tag{49}
$$

In the very low-momentum region $k\ll k_0$ and for $t\ll 1$, one can show that $\omega_I{}^{(a)}$ is very much smaller than $\omega_I{}^{(b)}$. Therefore, we only consider the latter term, which can be written as follows:

$$
\omega_I{}^{(b)}=\frac{k_0{}^3a}{\sqrt{2}}\int_0^\infty dz_1\left[\frac{z_1{}^2}{(1+z_1{}^2)}-(1+z_1{}^2)^{-\frac{1}{2}}+1\right]^2 \times m_1(1+m_1)[1+O(y,t)] \underset{y,t\to 0}{\longrightarrow}\left(\frac{2\pi^{\frac{3}{2}}}{\pi-2}\right)\left(\frac{k_0}{\lambda_T}\right). \tag{50}
$$

We see that in the region $t\ll 1$, $\omega_I{}^{(b)}/\omega_0\sim(k\lambda_T)^{-1}$ becomes very large for $k\ll k_0$. This means that the half-width of a low-momentum excitation is large compared to the excitation energy itself. The lifetime $\tau=[-2\,\mathrm{Im}\omega(k)]^{-1}$ of the quasi-particles is extremely short for $t\ll 1$, i.e., for temperatures just below the critical temperature, and the excitations are therefore not very well defined. *In a dilute Bose gas of hard spheres, low-momentum quasi-particles can only be well defined in the low-temperature region* $t\gtrsim 1$. The result of Eq. (47) for the real part of $\omega(k)$ becomes less and less physically meaningful as the temperature $T<T_c$ is increased.

It is of interest to compare the qualitative result of Eq. (50) with the experimental results for the inelastic scattering of neutrons in liquid He II. In Fig. 2 of a note by Henshaw[1] are plotted the spectra of inelastically scattered neutrons at a fixed angle and for four different temperatures: $T=1.27$°K, 1.57°K, 2.08°K and 4.21°K. Now, the critical temperature of liquid helium is 2.18°K, and yet one sees from Henshaw's curves that already at 2.08°K the unique energy-momentum relation, which defines the quasi-particles in He II at lower temperatures, has started to wash out. *The experimentally determined quasi-particles of He II are also not very well defined at temperatures just below the critical temperature*, and the result of Eq. (50) is qualitatively compatible with experiment.

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Thermal Conduction in Rotating Liquid Helium II

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When a heat current flows in a wide channel filled with liquid helium II, the resulting temperature gradient is approximately proportional to the cube of the heat current density. The establishment of this gradient requires a time τ which is a function of heat current, temperature, and past history of the helium. The present experiments concern the effect of uniform slow rotation, about an axis normal to the direction of heat flow, upon grad$T$ and τ. Grad$T$ was measured at 0 and 1.3 rad/sec, the highest angular velocity at which this measurement could conveniently be carried out. No effect of rotation could be observed; however, an approximate calculation suggests that grad$T$ might increase detectably at somewhat higher rates of rotation. τ was measured at a number of angular velocities between 0 and 4 rad/sec; it was found that τ was appreciably reduced by rotation, the effect being greatest at small heat currents and high angular velocities. These results can be explained on the assumption that mutual friction results from turbulence in the superfluid component, taking the form of a tangled mass of vortex line. The delay time τ characterizes the rate of growth of this turbulence when a heat current is switched on; rotation reduces τ by introducing an initial length of vortex line which accelerates this growth.

I. INTRODUCTION

When a heat current flows in liquid helium II, the resulting temperature gradient is approximately proportional to the cube of the heat current density. This fact, together with the results of experiments on fountain pressures, led Gorter and Mellink[1] to postulate the existence of a mutual friction force between the normal and superfluid components given by

$$
F_{sn}=A\rho_s\rho_n(|v_s-v_n|-v_0)^3, \tag{1}
$$

where v_s and v_n are the superfluid and normal fluid velocities, respectively. A is approximately independent of channel width and is a slowly varying function of temperature; v_0 is of the order of 1 cm/sec, varies somewhat with temperature, and decreases with increasing channel width.

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[1] C. J. Gorter and J. H. Mellink, Physica 15, 285 (1949).