

# Nuclear Deformation in the Spheroidal Shell Model\*

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In the usual shell-model procedure, the effective Hamiltonian contains only half the sum of the shell-model potentials of nucleons in order to avoid counting average pairwise interactions twice. Because of the factor one-half, the nondiagonal elements of this Hamiltonian in the harmonic oscillator representation do not vanish, but they have been neglected in previous calculations of nuclear deformations by Nilsson and others, in which one minimizes total shell-model energy at constant volume. It is here shown in typical cases (without taking spin-orbit coupling into account) that the equilibrium deformation is unaltered in second and third order and that the fourth-order modification arising from the nondiagonal elements is very small. The relation of these nondiagonal elements to those of the pairwise interactions is also discussed.

## INTRODUCTION

THE shell-model potential is to be thought of as a function of the position coordinate of one nucleon giving the average dependence of the potential energy of the entire system on that coordinate. It thus consists of a sum over the  $(A-1)$  other nucleons:

$$V_i = \sum_j v_{ij}, \quad j \neq i. \quad (1)$$

The sum  $\sum_i V_i$  includes each interaction  $v_{ij}$  twice so the average value of the potential energy of the system is one-half of this sum. The mean value of the effective Hamiltonian  $\mathcal{H}_{\text{eff}}$  contains this factor  $\frac{1}{2}$ ;

$$\langle a | \mathcal{H}_{\text{eff}} | a \rangle = \langle a | \sum_i (T_i + \frac{1}{2} V_i) | a \rangle, \quad (2)$$

and is thus not approximately equal to the mean value of the zeroth-order Hamiltonian

$$\langle a | \mathcal{H}^0 | a \rangle = \langle a | \sum_i (T_i + V_i) | a \rangle, \quad (3)$$

the difference being made up by the first-order contribution to the energy

$$E_a - E_a^{(0)} \approx E_a^{(1)} = \langle a | \mathcal{H}^1 | a \rangle = \langle a | \mathcal{H} - \mathcal{H}^0 | a \rangle = -\langle a | \frac{1}{2} \sum_i V_i | a \rangle. \quad (4)$$

If for  $V_i$  we take an oscillator potential, the virial theorem for the harmonic oscillator,  $\langle a_i | T_i | a_i \rangle = \langle a_i | V_i | a_i \rangle$ , simplifies the evaluation of (3) and the corresponding relation with a minus sign for nondiagonal elements,

$$\langle a_i | T_i | b_i \rangle = -\langle a_i | V_i | b_i \rangle \quad \text{for } b_i \neq a_i, \quad (5)$$

assures that the nondiagonal elements of the zeroth-order Hamiltonian vanish in the oscillator representation:

$$\langle a | \mathcal{H}^0 | b \rangle = 0 \quad \text{for } a \neq b. \quad (6)$$

Because of the factor  $\frac{1}{2}$  in Eq. (2), the corresponding nondiagonal elements of the Hamiltonian,  $\langle a | \mathcal{H}_{\text{eff}} | b \rangle$ , do not thus vanish in this approximation and it is our purpose here to investigate their importance in a case

of interest. More explicitly, the nondiagonal element of the Hamiltonian may be written

$$\begin{aligned} \langle a | \mathcal{H}_{\text{eff}} | b \rangle &= \langle a | \sum_i T_i + \frac{1}{2} \sum_i v_{ij} | b \rangle \\ &= \langle a_1 \cdots a_A | \frac{1}{2} \sum_i v_{ij} - \sum_i V_i | b_1 \cdots b_A \rangle, \end{aligned} \quad (7)$$

in view of Eq. (5). On the assumption that there is an approximate simple proportionality between the diagonal and nondiagonal elements;

$$\langle a | \sum_j v_{ij} | b \rangle / \langle a | \sum_j v_{ij} | a \rangle \approx \langle a | V_i | b \rangle / \langle a | V_i | a \rangle \quad (8)$$

there would be a systematic tendency for the nondiagonal element similar to (7) to vanish if there were no factor  $\frac{1}{2}$  in the Hamiltonian, but with the actual factor  $\frac{1}{2}$  there is no such tendency.

As a basis for assumption (8), we note that the large matrix elements, between states differing in only one nucleon function so that  $\langle a | V_i | b \rangle$  does not vanish, may be written

$$\begin{aligned} \langle a | \sum_j v_{ij} | b \rangle &= \langle a_1 \cdots a_i \cdots a_A | \sum_j v_{ij} | b_1 \cdots b_i \cdots b_A \rangle \\ &= \langle a_1 \cdots a_i \cdots a_A | \sum_j v_{ij} | a_1 \cdots b_i \cdots a_A \rangle \\ &= \int dx_i a_i^* b_i \sum_j \int dx_j v_{ij} a_j^* a_j \\ &\quad - \sum_j \int dx_i a_i^* a_j \int dx_j v_{ij} a_j^* b_i. \end{aligned} \quad (9)$$

Here the symbol  $a_i^*$  is used alternatively as the wave function  $a_i^*(x_i)$ . The main term is in the third line and contains the expression

$$\sum_j \int dx_j v_{ij} a_j^* a_j \approx V_i, \quad (10)$$

which defines  $V_i$  in a no-exchange approximation. This term if alone in (9) would lead to the relation (8) as an equality. Equation (9) remains a plausible approximation because the exchange term in the last line of (9) is expected to be considerably smaller, in view of the presence of an additional factor  $a_j^* b_i(x_j)$  having the nature of a wave with positive values at some places and negative at others, and also because of the can-

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cellation of positive and negative terms in the sum over  $j$ . We have assumed that  $b$  differs from  $a$  in only one factor  $b_i \neq a_i$ . If it differs also in a second factor  $b_j \neq a_j$  the matrix element is much smaller because it contains only one term of the sum over  $j$ , thus avoiding the cooperative buildup that makes  $V_i$ . There are many such small terms but they appear quadratically in applications and one may hope they are thus relatively unimportant.

This neglect of the sum of squares of single terms compared to the square of a sum is questionable in the light of the saturation nature of nuclear forces. If we want to use realistic interactions  $v_{ij}$ , with exchange properties not sufficiently pronounced to effect saturation,  $v_{ij}$  must have a repulsive core and the perturbation procedure here used is not valid. With this procedure we must mock up the saturation properties by using appropriate exchange operations rather than a repulsive core. Then one nucleon interacts, in effect, with only a few others, the sum over many others in the second line of (9) is not so much greater than some single terms, and the neglect mentioned may not be justified. However, the assumption (8) permits a simple discussion of deviations expected from the simple Nilsson treatment of the nuclear shape, and we wish first to examine its consequences.

#### EQUIVALENT HAMILTONIAN

With the approximation (8) the shell-model potential  $V_i$  may be used in place of  $\sum_j v_{ij}$  in calculating non-diagonal elements as well as diagonal elements. The replacement may then be made in the Hamiltonian itself, and we may write an equivalent Hamiltonian in the convenient form

$$\mathcal{H} = \sum T_i + \frac{1}{2} \sum v_{ij} \approx \mathcal{H}_{\text{eff}} = \sum T_i + \frac{1}{2} \sum V_i \quad (11.1)$$

$$= \frac{3}{4} \sum (T_i + V_i) + \frac{1}{4} \sum (T_i - V_i) = \frac{3}{4} (\mathcal{H}^0 + \sum h_i), \quad (11.2)$$

with

$$h_i = \frac{1}{3} (T_i - V_i). \quad (11.3)$$

With oscillator functions the diagonal elements of  $h_i$  vanish, from the virial theorem, and the expectation value of  $\mathcal{H}$  is  $E = \frac{3}{4} E_0$ , the factor  $\frac{3}{4}$  arising from avoiding counting the pairwise interaction twice. The same relation (5) which makes the nondiagonal elements of  $\mathcal{H}^0$  vanish, as stated in Eq. (6), also makes the nondiagonal elements of  $h_i$  systematically large. The factor  $\frac{1}{3}$  mitigates their effect, however, and we shall see that the higher-order contributions apparently cause no serious trouble in the example to which we now proceed.

#### THE MAGNITUDE OF THE COLLECTIVE DISTORTION

Our knowledge of nuclear forces and computation methods has not yet proved sufficient to calculate well the magnitude of the binding energy of a finite nucleus, this being a small difference between large potential and kinetic energies. In attacking the problem of calculating the equilibrium deformation of a non-

spherical nucleus, one requires not the absolute magnitude of the energy but the change of the energy when the shape is altered, so as to be able to minimize it. It has seemed reasonable to avoid the problem of the saturation properties of nuclear forces, which determine primarily the density of nuclear matter, by assuming the volume to be kept constant as the shape is varied, and then to calculate the energy as the sum of the deformed shell-model energies of the occupied single-nucleon states. It is the change of total energy with change of shape that is then significant. With oscillator functions the nucleus has no definite surface and it is the volume within equipotential surfaces that is kept constant. Since the problem is separable in the Cartesian coordinates  $x, y, z$ , the competition between potential and kinetic energy is resolved in the same way in each direction, and there remains the competition between the energies associated with the different directions to determine the shape, depending upon how the nucleon quantum numbers are distributed between the directions.

This approximation has been used to obtain nuclear equilibrium deformations in connection with the "cranked model" calculation of the moments of inertia of rotating spheroidal nuclei<sup>1</sup> and in connection with determining the parity and angular momentum of low nuclear states with spin-orbit coupled nucleon wave functions in spheroidal nuclei.<sup>2</sup> In the latter problem, Nilsson employed the equivalent Hamiltonian explicitly in the form (11.2), with neglect of the nondiagonal elements of  $h_i$ . Because of the great interest of his results, we wish here to investigate the importance of this neglect, and shall find it gratifying that the inaccuracy is apparently not serious. The same problem arises in very nearly the same way in the simpler problem of calculating the deformation for a nucleus made of nucleons with no spin-orbit coupling, suitably remote from the condition of closed shells in that model (in which the magic numbers are 40 and 70, etc., rather than 50 and 82), and we confine our attention to this model.

One assumes that the shell-model potential is a three-dimensional potential having spheroidal, rather than spherical, equipotential surfaces:

$$V_i = \frac{1}{2} \hbar [\omega_2 (\xi^2 + \eta^2) + \omega_3 \zeta^2], \quad (12)$$

with  $\xi = (M\omega_2/\hbar)^{1/2} x$ , etc. We carry out a perturbation theory in terms of shell-model wave functions  $\phi_a = \prod u_{ai}(i) = \prod a_i$  (but antisymmetrized) satisfying

$$(\mathcal{H}^0 - E_a^0) \phi_a = 0, \quad (\mathcal{H}_i - \mathcal{E}_{ai}) u_{ai}(i) = 0, \\ \mathcal{H}^0 = \sum \mathcal{H}_i = \sum (T_i + V_i). \quad E_a^0 = \sum_i \mathcal{E}_{ai} = \sum \mathcal{E}_i. \quad (13)$$

<sup>1</sup> A. Bohr, *Rotational States of Atomic Nuclei* (E. Munksgaards Forlag, Copenhagen, 1954); D. R. Inglis, *Phys. Rev.* **96**, 1059 (1954); **103**, 1786 (1956); *Am. J. Phys.* **26**, 82 (1958); *Nuclear Phys.* **8**, 125 (1958). A. Bohr and B. M. Mottelson, *Kgl. Danske Videnskab Selskab, Mat.-fys. Medd.* **30**, 1 (1955).

<sup>2</sup> S. G. Nilsson, *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.* **29**, 1 (1955); B. R. Mottelson and S. G. Nilsson, *Phys. Rev.* **99**, 1615 (1955).

In the oscillator representation the shell-model Hamiltonian  $\mathcal{H}_i$  contains a sum of three terms of the form  $\frac{1}{2}\hbar\omega(\xi^2 - \partial^2/\partial\xi^2)$  and  $\phi_a$  contains three factors, each consisting of a Hermite polynomial and an exponential of the form  $\exp(-\xi^2/2)$  where  $\xi^2 = \sigma\alpha x^2$  and  $\sigma\alpha = M\omega_2/\hbar$ , which is related to  $\mathcal{E}_i = \hbar\omega_2(l_i + \frac{1}{2}) + \dots$ . The  $V_i$  appearing here is considered to be the average potential of one nucleon in the field of all the others and therefore the potential which determines the size of the wave functions. This size is related to the observed size of nuclei by an empirical determination of the parameter  $\sigma\alpha$  and thus of the  $\mathcal{E}_i$  by means of these equations.

In calculating expectation values of the equivalent Hamiltonian (11.2)

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_{\text{eff}}^0 + \mathcal{H}_{\text{eff}}', \quad \text{with} \quad \mathcal{H}_{\text{eff}}^0 = \frac{3}{4}\mathcal{H}^0$$

and

$$\mathcal{H}_{\text{eff}}' = \frac{3}{4} \sum h_i,$$

we denote energies on the contracted scale by small  $e$  rather than large  $E$ :

$$e_a = \frac{3}{4}E_a = e_a^{(0)} + e_a^{(1)} + e_a^{(2)} + \dots$$

For the first-order correction to the energy we obtain

$$e_a^{(1)} = \langle a | \mathcal{H}' | a \rangle = \sum_i \langle a_i | \frac{3}{4}h_i | a_i \rangle \\ = \frac{1}{4} \sum_{lmn} \langle lmn | T_i - V_i | lmn \rangle = 0,$$

vanishing for harmonic oscillator functions. Here  $l$ ,  $m$  and  $n$  are the oscillator quantum numbers in the three dimensions,  $\xi$ ,  $\eta$ , and  $\zeta$ . The second-order contribution to the energy of the ground state  $a$  with the Hamiltonian (11.2) is

$$e_a^{(2)} = \sum_{\alpha} |\langle \alpha | \frac{3}{4} \sum h_i | a \rangle|^2 / (e_{\alpha} - e_a) \\ = \sum_{\alpha i} \langle \alpha_i | \frac{3}{4}h_i | a_i \rangle^2 / \frac{3}{4}(\mathcal{E}_{\alpha} - \mathcal{E}_a) \\ = \frac{3}{4} \sum_{lmn} \sum_{\lambda\mu\nu} |\langle \lambda\mu\nu | h_i | lmn \rangle|^2 / (\mathcal{E}_{\lambda\mu\nu} - \mathcal{E}_{lmn}). \quad (14)$$

The  $e_a^{(0)}$  appear directly in the energy denominators, but we express results in terms of the  $\mathcal{E}_i$  because these are empirically determined. The  $\sum_{lmn}$  extends over occupied nucleon states. The other summation is limited only by the selection rules. In this statement we conveniently ignore the Pauli antisymmetry as we may for a single-particle perturbation, since we thereby spuriously include only pairs of terms that cancel one another, having matrix elements between the same pair of states and denominators of opposite sign.

The evaluation of the matrix elements involves  $T_i = -(\hbar^2/2M)\partial^2/\partial x^2 + \dots = -\frac{1}{2}\hbar\omega_2\partial^2/\partial\xi^2 + \dots$  and  $V_i$  from Eq. (12) with the matrix elements implied by the

TABLE I. Coefficients for specific nuclei.

$A$	$b_1$	$b_2$	$c_1$	$c_2$
28	11	17/2	-0.0280	-0.0479
180	135	175/2	-1.2579	-1.0961

following equation relating the  $u_i = u_l(\xi)u_m(\eta)u_n(\zeta)$  (which contain normalized Hermite polynomials):

$$\xi^2 u_l, \quad (\partial^2/\partial\xi^2)u_l = \frac{1}{2}\{[(l+1)(l+2)]^{\frac{1}{2}}u_{l+2} \\ \pm (2l+1)u_l + [l(l-1)]^{\frac{1}{2}}u_{l-2}\}. \quad (15)$$

The choice indicated by the comma refers to the choice of the  $\pm$  sign. The selection rules are clearly  $\lambda = l$ ,  $l \pm 2$ , etc., with only one of the three quantum numbers differing from  $l$ ,  $m$ ,  $n$ , because  $h_i$  is a sum of terms in  $\xi$ ,  $\eta$ , and  $\zeta$ :

$$h_i = -(\hbar/6) \sum_{\xi} \omega_{\xi}(\xi^2 + \partial^2/\partial\xi^2). \quad (16)$$

When the substitution is made in the last member of (14), one obtains the same type of nondiagonal contributions from each of the three dimensions and from each occupied state  $l$ ,  $m$ ,  $n$ :

$$\frac{3}{4}\{l+1)(l+2) - l(l+1)\} (\hbar\omega_2/6)^2 / 2\hbar\omega_2 \\ = \frac{3}{4}(l + \frac{1}{2})\hbar\omega_2/18, \quad (17)$$

which is just one-eighteenth of the zeroth order contribution from the same coordinate. Thus we have the simple result in second order:

$$e = \sum_{lmn} (e_{lmn}^{(0)} + e_{lmn}^{(2)}) = (1 - 1/18)e^0. \quad (18)$$

The equilibrium shape is found by minimizing the total energy,  $de/d\omega_2 = 0$  subject to the volume-conserving condition  $\omega_2^2\omega_3 = \text{constant}$ . This amounts to balancing the energy contribution from the  $\zeta$  direction against those from the  $\xi$  and  $\eta$  directions, and the result is

$$\omega_3/\omega_2 = \sum (l+m+1) / \sum (2n+1) \quad (19)$$

independent of any factor common to all directions. The equilibrium shape is thus the same in second order as in zeroth order though the total energy is altered.

There is no third-order contribution,  $e^{(3)} = 0$ , because the third-order perturbation involves products of three nondiagonal matrix elements of  $h_i$ , two of them containing the ground state and each of them involving a jump of a quantum number by 2: one cannot arrive back at zero with three jumps of magnitude 2.

The fourth-order contribution to the energy may be written

$$e^{(4)} = -\frac{3}{4} \left[ \sum \sum \sum \frac{\langle lmn | h_i | \lambda\mu\nu \rangle \langle \lambda\mu\nu | h_i | \lambda'\mu'\nu' \rangle \langle \lambda'\mu'\nu' | h_i | \lambda''\mu''\nu'' \rangle \langle \lambda''\mu''\nu'' | h_i | lmn \rangle}{(\mathcal{E}_{lmn} - \mathcal{E}_{\lambda\mu\nu})(\mathcal{E}_{lmn} - \mathcal{E}_{\lambda'\mu'\nu'}) (\mathcal{E}_{lmn} - \mathcal{E}_{\lambda''\mu''\nu''})} \right. \\ \left. - \frac{1}{2} \sum \sum \frac{|\langle lmn | h_i | \lambda\mu\nu \rangle|^2 |\langle lmn | h_i | \lambda'\mu'\nu' \rangle|^2}{(\mathcal{E}_{lmn} - \mathcal{E}_{\lambda\mu\nu})^2 (\mathcal{E}_{lmn} - \mathcal{E}_{\lambda'\mu'\nu'})} \right], \quad (20)$$

TABLE II. Energies and equilibrium deformation to fourth order.<sup>a</sup>

$A$	$(\omega_3/\omega_2)_0$	$E^0$	$E^{(2)}$	$E^{(4)}$	$ E^{(4)}/E^{(2)} $	$(\omega_3/\omega_2)_4$
28	0.65	99.2	5.5	-0.328	0.060	1.0033 $(\omega_3/\omega_2)_0$
180	0.77	404.9	22.5	-12.24	0.544	1.0034 $(\omega_3/\omega_2)_0$

<sup>a</sup> Energies are all in the unit of  $\frac{1}{2}\hbar\omega_2$ .

with the sums extending over the sets of Greek indices. It is desired to express this in terms of the occupied-state quantum numbers  $l, m, n$  and the deformation factor  $D=\omega_3/\omega_2$ . The result is

$$e^{(4)} = -\left(\frac{1}{2}\hbar\omega_2/3^4 4^3\right)\{6(l^3+m^3)+10(l^2+m^2)+28(l+m)+24+2lm(l+m+2)-2(l+m+1)\times(n^2+n+1)+D[6n^3+9n^2+25n+11-(l^2+m^2+l+m+2)(2n+1)-\langle-8(l^2+m^2)nD+8(l+m)n^2-4(l^2+m^2)D+8(l+m)(1-D/2)+8n^2+8n(1-2B)\rangle/(1-D^2)+8(ln+mn+1)/(1+D)]\}. \quad (21)$$

The corresponding equation in zeroth order is

$$e^0 = \frac{3}{4}\hbar\omega_2\{(l+m+1)+(n+\frac{1}{2})D\}. \quad (22)$$

When the summation over occupied levels is carried out for specific nuclei, these equations take the form

$$e^0 = \frac{3}{4}\hbar\omega_2(b_1+b_2D), \\ e^{(4)} = \frac{3}{4}\hbar\omega_2(c_1+c_2D+\dots). \quad (23)$$

As examples of moderately light and fairly heavy nuclei we evaluate these sums for the cases  $A=28$ , in which the nucleon configuration is  $(1s)^4(1p)^{12}(1d,2s)^{12}$ , and  $A=180$  in which beyond the "no-spin-orbit-coupling magic number" 70 we have 40 like nucleons  $(1h,2f,3p)^{40}$ , whereas it takes 84 nucleons to fill this "shell." In each case the last major shell is about half filled, so as to provide an example of large distortion. For these examples the coefficients in the last two equations are given in Table I. After carrying out the minimization with the condition  $\omega_2^2\omega_3=\text{constant}$ , in zeroth order [as in Eq. (19)] and in fourth order, we obtain the results shown in Table II. One finds that the fourth-order correction to the equilibrium shape parameter  $\omega_3/\omega_2$  is remarkably small, only 0.3% in each case.

Thus the Hamiltonian (11.2), which is as far as it goes the same as used by Nilsson in his interesting determination of nuclear shapes, has nondiagonal elements, heretofore neglected in that determination, which seem to exert practically no influence on the shape calculated. As we have seen, there are reasons for doubting the adequacy of this simplified Hamiltonian for calculation of such details as the non-diagonal elements even though its use may be justified for the diagonal elements encountered in the original calculation.

In order to illustrate the nature of the inadequacy with as simple an extension of the calculation as possible, we here wish to display the roughness of the approximation of Eq. (8) as it stands, without introducing exchange, for a few specific states of  $A=28$ . Explicit evaluations of several forms of  $v_{ij}$  for harmonic oscillator states have been made.<sup>3</sup> The form

$$v_{ij} = \exp(-\alpha r_{ij}^2) \quad (24)$$

is perhaps the simplest to use, partly because its integrals have been listed explicitly for some cases of interest.<sup>4</sup> In the notation there used, the wave functions are

$$\psi = H \exp\{-\frac{1}{2}\alpha[\sigma(x^2+y^2)+\sigma'z^2]\}. \quad (25)$$

Thus  $\sigma$  and  $\sigma'$  are the parameters expressing nuclear size and shape, and the matrix elements are expressed in terms of  $\tau=\sigma+2$  and  $\tau'=\sigma'+2$ . Minimization without regard to the nondiagonal elements determines a shape corresponding to  $\sigma'/\sigma=0.65$ . In Table III are listed for comparison the values of

$$L = \langle a | \sum_j v_{ij} | b \rangle / \langle a | \sum_j v_{ij} | a \rangle$$

and

$$R = \langle a | v_i | b \rangle / \langle a | v_i | a \rangle \quad (26)$$

for this shape. In order to display the relative insensitivity to the shape, values are listed for three values of  $\sigma'/\sigma$ . The values of  $L$  calculated with a non-exchange interaction  $v_{ij}$  are listed in the first of the two columns for each  $\sigma'/\sigma$ . In the second of the two columns are listed in parentheses the values calculated using a simple form of exchange operator,  $v_{ij}(0.8P^a - 0.2P^a)$ , mainly space-exchange  $P^a$  with a little spin-exchange  $P^s$ . For the nonexchange case, we see that

 TABLE III. Comparison of nondiagonal elements in the  $\sum_j v_{ij}$  and  $V_i$  approximations for  $A=28$ .

												$L$			$R$
State												First values are calculated without exchange, values in parenthesis with exchange.			
$l$	$m$	$n$	$l'$	$m'$	$n'$	$l''$	$m''$	$n''$	$l'''$	$m'''$	$n'''$	$\sigma'/\sigma=0.5$	0.65	0.8	
0	0	0	0	0	1	0	0	0	0	0	3	0.266(0.081)	0.263(0.068)	0.260(0.058)	0.419
1	0	0	0	0	2	1	0	0	0	2	2	0.191(0.093)	0.193(0.101)	0.194(0.109)	0.269
0	1	1	0	0	2	0	1	1	0	0	4	0.394(0.273)	0.385(0.261)	0.375(0.249)	0.429

<sup>3</sup> I. Talmi, *Helv. Phys. Acta* **25**, 185 (1952).

<sup>4</sup> D. R. Inglis, *Phys. Rev.* **51**, 531 (1937); W. J. Kroeger, *Phys. Rev.* **54**, 1048 (1938).

there is some degree of correspondence between the approximations  $L$  and  $R$ , at least enough so that the smallest remains smallest and the largest remains largest. The same may not be said for the exchange case. However, in both cases, the values of  $L$  are uniformly smaller than of  $R$ ; that is, the nondiagonal elements calculated with pair interactions are relatively smaller than those calculated with  $V_i$ . The very small influence of the nondiagonal elements of  $V_i$  on the shape, a third of a percent as indicated in Table II, seems to come about largely by cancellations of effects of various nondiagonal elements but is still some rough measure of the magnitude of the nondiagonal elements. A similar calculation based on the pair interactions rather than of  $V_i$  would apparently involve somewhat smaller matrix elements, among which there would also be

cancellations of some sort. It thus seems plausible to take the smallness of the influence calculated with  $V_i$  as an indication that the result calculated with pair interactions would likewise be quite small.

The determination of nuclear shape by minimizing the oscillator energies at constant volume is a schematic approach which probably owes most of its success to the fact that it takes kinetic energies into account to a fairly good approximation, at the same time making a rough estimate of the change in potential energy which is in reality much too complicated to have been treated adequately. Our remarks on the effect of nondiagonal elements of  $V_i$  display the effective consistency of the approximate treatment. The remarks on pairwise interactions give an inadequate hint of how this might be related to the more fundamental nuclear problem.

## Fine Structure in the Energy Spectra of Photoprotons from $\text{He}^4$

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The energy spectra of the photoprotons from  $\text{He}^4$  irradiated with 31 and 32-Mev bremsstrahlung were studied by means of nuclear photoemulsions in  $\Delta E_\gamma$  steps of 0.133 Mev. In the region of the giant resonance several peaks have been distinguished. The more evident of these have widths less than 0.5 Mev and are located at  $\text{He}^4$  excitation energies of  $24.7 \pm 0.2$  and  $26.1 \pm 0.1$  Mev. Previous experiments on the photoproton spectra from  $\text{He}^4$  could not give evidence of these narrow resonances because the spectra were examined in energy steps much larger than the width of these resonances.

Other types of experiments which have been performed in order to obtain information on the existence of excited states in  $\text{He}^4$  are discussed. The fact that narrow resonances are distinguished only in the present experiment is attributable to the selective nature of the  $(\gamma, p)$  process, the most important contribution in the  $\text{He}^4(\gamma, p)$  reaction coming only from the states of  $\text{He}^4$  having  $J=1^-$  and  $T=1$ .

### I. INTRODUCTION

THE experiments on the  $\text{He}^4(\gamma, p)$  reaction performed up to now give the energy spectra of the photoprotons from helium in large energy steps: Gaertner and Yeater,<sup>1</sup> with a bremsstrahlung beam of  $E_{\gamma\text{max}}=100$  Mev, studied the  $\text{He}(\gamma, p)$  reaction with a cloud chamber. The photoproton spectrum is given up to  $E_\gamma=30$  Mev in  $\Delta E_\gamma$  energy steps of several Mev. Fuller,<sup>2</sup> with bremsstrahlung spectra having  $E_{\gamma\text{max}}=25, 29, 32,$  and  $40$  Mev, studied the same reaction with nuclear emulsions. The photoproton spectra are given in steps of  $\Delta E_\gamma=1$  Mev. Gorbunov and Spiridonov,<sup>3</sup> with a bremsstrahlung beam of 170 Mev, studied the  $\text{He}(\gamma, p)$  reaction with a cloud chamber located in a magnetic field. The energy spectrum of the photoprotons is given in  $\Delta E_p$  steps of several Mev.

The aim of the present experiment is to study the

$\text{He}(\gamma, p)$  reaction in order to obtain photoproton spectra with better energy resolution in the giant resonance region. The peak of the giant resonance is found by Fuller<sup>2</sup> at  $E_\gamma \approx 26$  Mev. According to Reid *et al.*<sup>4</sup> the excitation function appeared to have a maximum in the region of 26 Mev and to decrease more rapidly with increasing energy than indicated by Fuller. According to Gorbunov *et al.*<sup>3</sup> the cross section reaches the maximum value at 27–28 Mev and decreases with increasing energy less rapidly than indicated by Fuller. No evidence for resonances having widths less than several Mev has been found.

### II. EXPERIMENTAL PROCEDURE

A helium gas target has been irradiated with a collimated bremsstrahlung beam of  $E_{\gamma\text{max}}=32$  Mev from the betatron of Turin University. The photoprotons have been recorded by means of Ilford C2 nuclear emulsions 200  $\mu$  thick placed inside a chamber filled with helium gas. Four nuclear emulsions were placed parallel to the  $\gamma$ -ray beam as shown in Fig. 1.

<sup>4</sup> J. M. Reid, P. Swinbank, and J. R. Atkinson, *Physica* **22**, 1142 (1956).

<sup>1</sup> E. R. Gaertner and M. L. Yeater, *Phys. Rev.* **83**, 146 (1951).

<sup>2</sup> E. G. Fuller, *Phys. Rev.* **96**, 1306 (1954).

<sup>3</sup> A. N. Gorbunov and V. M. Spiridonov, *J. Exptl. Theoret. Phys. (U.S.S.R.)* **33**, 21 (1957) [translation: *Soviet Phys.-JETP* **6**, 16 (1958)]; *Comptes Rendue du Congres International de Physique Nucleaire Interactions Nucleaires aux Basses Energies et Structure des Noyaux*, Paris, July, 1958, edited by P. Guggenberger (Dunod, Paris, 1959), p. 682.