Method for Determining the K^0 Spin*†

PHILIPPE EBERHARD
Laboratoire de Physique Atomique et Moleculaire, College De France, Paris, France

AND

M. L. GOOD
University of Wisconsin, Madison, Wisconsin

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A method is presented for determining the K^0 spin, using only angular momentum conservation and the rules of quantum mechanics. The method is based on a proof that, in the reaction $\pi^- + p \rightarrow \Lambda + K^0$, the K_1^0 decay intensity associated with a given direction of the Λ spin cannot be isotropic at any production angle, if the K^0 spin is equal to or greater than two.

Account is taken of the particle-mixture aspect of the K_1^0 and of possible magnetic moment precession.

I. INTRODUCTION

THIS paper presents a method for determining the spin of the K^0 meson. While it is commonly accepted that the K^0 spin is zero, the most compelling argument for spin zero is that of simplicity.

The observed decay into two π^0's rules out odd spin. Even values of the spin other than zero makes difficulty with the observed K^0 decay rate, but this inevitably brings in questionable dynamical assumptions. The situation with respect to the K^+ is not much different. Odd spin values are unlikely on the basis of the τ decay spectrum[1] (dynamical assumptions again are involved) but the τ spectrum provides no evidence against even, nonzero, spin. There is an argument against spin 2 due to Dalitz[2] (i.e., that $K^+ \rightarrow \pi^+ + \gamma$ could occur for spin 2) which, of course, is of a dynamical nature. Finally, the polarization of the μ^+ from K^+ decay[3] is consistent with spin zero, but does not rule out higher spin, since very little is known about the possible polarization of the K^+ mesons, and the measurements were made at one angle.

All this is to be contrasted with the situation for the Λ hyperon, where the beautiful method of Lee and Yang[4] has made possible an unambiguous spin determination[5] using only angular momentum conservation and the rules of quantum mechanics.

We therefore seek here a method of "legal proof" of spin zero for the K^0 in order that the answer to such an important question have a foundation which is as firm as possible.

The method of Adair[6] is applicable to the K^0, but so far has not yielded conclusive results, because of uncertainties in the number of partial waves present in the production process.[7]

We examine the reaction

$$\pi^- + p \rightarrow \Lambda + K^0. \tag{1}$$

The Λ spin is now known to be $\frac{1}{2}$.[5] Thus all spins in reaction (1) are known, except for the K^0. Further, the Λ is known[8] to have a large asymmetry parameter (α). This means that it is possible, on a statistical basis, to distinguish between Λ spin up and Λ spin down, relative to some preferred axis. The way we utilize this is the following:

In reaction (1), at any given production angle, the quantum-mechanical state of an observed K^0 is a statistical mixture of four components, corresponding to spin up or spin down for the struck proton, and, for each of these, to spin up or spin down for the associated Λ. By using the decay asymmetry of the Λ, one can project out the K^0 intensity associated with Λ spin up. This is then only a two-component mixture (corresponding to the two possible spin orientations of the struck proton).

We then prove that for a K^0 spin state which is a statistical mixture of Q pure states, isotropy cannot be achieved in the K_1^0 decay for K_0 spin J such that $J \geqslant Q$. Thus for the two component mixture, $J \geqslant 2$ can be ruled out. Even without observing the Λ, we have $Q=4$ and so $J \geqslant 4$ can be ruled out.

The proof is in the form of an inequality which can be applied to the data; that is to say, we give an inequality, based on the above proof, for each even, nonzero spin J. The inequality must be satisfied if the spin has the assumed value J, and will be violated if the spin is zero.

II. FORMULATION OF THE PROBLEM

Let us choose $\hat{n} = \mathbf{p}_{in} \times \mathbf{p}_\Lambda / |\mathbf{p}_{in} \times \mathbf{p}_\Lambda|$ as the axis of quantization (Fig. 1). (This choice is for definiteness

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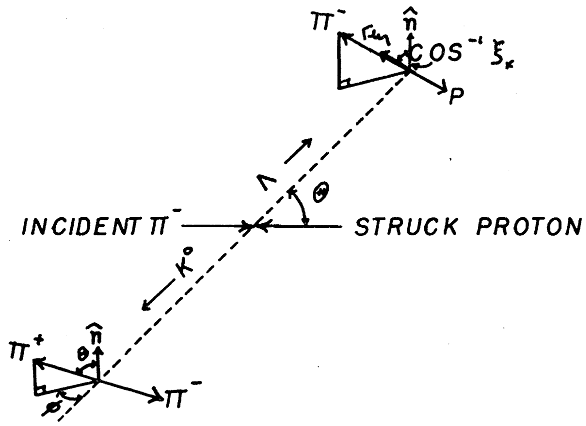

FIG. 1. Schematic diagram of $\pi^- + p \rightarrow \Lambda + K^0$, $\Lambda \rightarrow p + \pi^-$, $K^0 \rightarrow \pi^+ + \pi^-$, as seen in the center-of-mass system of the production process. The decay angles θ, ϕ, $\cos^{-1}\xi_x$ are measured in the particle's rest frame.

only; our results are independent of the axis of quantization.)

The most general wave function for the products of reaction (1) is

$$\psi_+ = \begin{pmatrix}1\\0\end{pmatrix} \sum_m A_m^{(1)} \chi_J^m + \begin{pmatrix}0\\1\end{pmatrix} \sum_m A_m^{(2)} \chi_J^m,$$
$$\psi_- = \begin{pmatrix}1\\0\end{pmatrix} \sum A_m^{(3)} \chi_J^m + \begin{pmatrix}0\\1\end{pmatrix} \sum A_m^{(4)} \chi_J^m, \tag{2}$$

where ψ_+ is for struck proton spin up, ψ_- for down. The $A_m^{(q)}$ are functions of the production angle Θ;

$$\begin{pmatrix}1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\1\end{pmatrix}$$

are Λ spin eigenfunctions; and the χ_J^m are K^0 spin eigenfunctions, for spin J, and spin projection m in the direction $\hat{n}$. For an unpolarized target, the intensity is:

$$|\psi_+|^2 + |\psi_-|^2.$$

Upon K_1^0 decay, angular momentum conservation coupled with zero spin of the pions requires

$$\chi_J^m \rightarrow Y_J^m(\theta,\phi),$$

where

$$Y_J^m(\theta,\phi)$$

are the spherical harmonics.

Upon Λ decay,

$$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \left(\frac{1+\alpha\xi_x}{2}\right) d\xi_x, \quad \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \left(\frac{1-\alpha\xi_x}{2}\right) d\xi_x, \tag{3}$$

where α is the Λ decay asymmetry parameter.

The intensity of the decay products is then

$$I(\xi,\theta,\phi,\Theta) = \left(\frac{1+\alpha\xi_x}{2}\right) \{ |\sum_m A_m^{(1)} Y_J^m|^2 + |\sum_m A_m^{(3)} Y_J^m|^2 \} + \left(\frac{1-\alpha\xi_x}{2}\right) \{ |\sum_m A_m^{(2)} Y_J^m|^2 + |\sum_m A_m^{(4)} Y_J^m|^2 \}, \tag{4}$$

which may be written

$$I = \left(\frac{1+\alpha\xi_x}{2}\right) I_+ + \left(\frac{1-\alpha\xi_x}{2}\right) I_-. \tag{5}$$

We now want to project out the Λ-spin up and Λ-spin down intensities I_+ and I_-. By examining (5) it is readily seen that the functions

$$f_+ = \tfrac{1}{2}(1+3\xi_x/\alpha), \quad f_- = \tfrac{1}{2}(1-3\xi_x/\alpha), \tag{6}$$

have the desired property, namely:

$$\int I(\xi_x,\theta,\phi,\Theta) f_\pm(\xi_x) d\xi_x = I_\pm(\theta,\phi,\Theta). \tag{7}$$

The two-component mixtures I_+, I_- may thus be directly obtained from the data.

We are therefore concerned with the properties of functions of the general form

$$I = \sum_q |\sum_m A_m^{(q)} Y_J^m|^2 = \sum_q \sum_{m,m'} A_m^{(q)} A_{m'}^{(q)*} Y_J^m Y_J^{m'*}. \tag{8}$$

First we note that $Y_J^m Y_J^{m'*} = (-1)^{m+m'} Y_J^{-m'} Y_J^{-m*}$ (i.e., that some of our functions are essentially identical because of a symmetry).

We therefore write

$$I = \tfrac{1}{2} \sum_q \sum_{m,m'} (A_m^{(q)} A_{m'}^{(q)*} + (-1)^{m+m'} A_{-m'}^{(q)} A_{-m}^{(q)*}) Y_J^m Y_J^{m'*}.$$

We define a coefficient $B_{mm'}$:

$$B_{mm'} = \sum_q A_m^{(q)} A_{m'}^{(q)*} + (-1)^{m+m'} A_{-m'}^{(q)} A_{-m}^{(q)*},$$

in terms of which the intensity is then quite simply

$$I = \tfrac{1}{2} \sum_{m,m'} B_{mm'} Y_J^m Y_J^{m'*}. \tag{9a}$$

We can get rid of the complex conjugation symbol, since

$$Y_J^{m*} = (-1)^m Y_J^{-m},$$

and

$$\sum_{m,m'} B_{mm'} Y_J^m Y_J^{m'*} = \sum_{m,m'} B_{mm'} (-1)^{m'} Y_J^m Y_J^{-m'}.$$

Thus, if we define $\tilde{B}_{mm'} = (-1)^{-m'} B_{m,-m'}$, the intensity

becomes

$$I=\tfrac{1}{2}\sum_{m,m'}\tilde{B}_{mm'}Y_J{}^mY_J{}^{m'}. \tag{9b}$$

The $B_{mm'}$ are (except for $m'=m$), the coefficients of typical interference terms in the $K_1{}^0$ decay angle distribution. It is these interference terms which cause the anisotropy.

The $\tilde{B}_{mm'}$ are now all observable, provided the $Y_J{}^mY_J{}^{m'}$ are all linearly independent, except for the above-noted symmetry. (They are linearly independent, as we show in Appendix I, by demonstrating functions which project them out.) The symmetry insures that

$$\tilde{B}_{mm'}=\tilde{B}_{m'm}=(-1)^{m+m'}\tilde{B}_{-m,-m'}{}^* \\ =(-1)^{m+m'}\tilde{B}_{-m',-m}{}^*. \tag{10}$$

The real part of $\tilde{B}_{mm'}$ shows up as the coefficient of $P_J{}^mP_J{}^{m'}\cos((m+m')\phi)$ (where the $P_J{}^m$ are the associated Legendre functions). The imaginary part is the coefficient of $P_J{}^mP_J{}^{m'}\sin((m+m')\phi)$, so both are observable.

III. PROOF OF THE INEQUALITY

Since we suspect not all the interference terms can vanish, we might form the sum of the squares of the $B_{mm'}$ for $m\neq m'$. To increase the symmetry, we include the diagonal terms as well, and form

$$F_J=\sum_{m,m'}|B_{mm'}|^2=\sum_{m,m'}|\tilde{B}_{mm'}|^2, \tag{11}$$

where the sum extends from $m=-J$ to $m=+J$, and the same for m'.

We now show that this function cannot be so small as to correspond to isotropic $K_1{}^0$ decay. Multiplying out Eq. (11), we have

$$\begin{aligned}F_J&=\sum_{mm'}\Big|\sum_q A_m{}^{(q)}A_{m'}{}^{(q)*}+(-1)^{m+m'}A_{-m'}{}^{(q)}A_{-m}{}^{(q)*}\Big|^2\\&=\sum_{m,m',q,q'}\big[(A_m{}^{(q)}A_{m'}{}^{(q)*}A_m{}^{(q')*}A_{m'}{}^{(q')}\\&\quad+A_{-m'}{}^{(q)}A_{-m}{}^{(q)*}A_{-m'}{}^{(q')*}A_{-m}{}^{(q')})\\&\quad+(-1)^{m+m'}(A_m{}^{(q)}A_{m'}{}^{(q)*}A_{-m'}{}^{(q')*}A_{-m}{}^{(q')}\\&\quad+A_m{}^{(q')*}A_{m'}{}^{(q')}A_{-m'}{}^{(q)}A_{-m'}{}^{(q)*})\big].\end{aligned}$$

We may interchange the order of summation and rearrange as follows:

$$\begin{aligned}F_J=\sum_{q,q'}\Big[&\Big|\sum_m A_m{}^{(q)}A_m{}^{(q')*}\Big|^2+\Big|\sum_{-m'}A_{-m'}{}^{(q)}A_{-m'}{}^{(q')*}\Big|^2\\&+\Big|\sum_m(-1)^mA_m{}^{(q)}A_{-m}{}^{(q')}\Big|^2\\&+\Big|\sum_{-m'}(-1)^{m'}A_{-m'}{}^{(q)}A_{m'}{}^{(q')}\Big|^2\Big],\end{aligned} \tag{12}$$

$$F_J=2\sum_{q,q'}\Big[\Big|\sum_m A_m{}^{(q)}A_m{}^{(q')*}\Big|^2+\Big|\sum_m(-1)^mA_m{}^{(q)}A_{-m}{}^{(q')}\Big|^2\Big].$$

The terms on the right are all positive; the terms in the first squared expression, for $q=q'$ only, give

$$|\textstyle\sum_m|A_m{}^{(q)}|^2|^2=w^2(q),$$

where $w(q)$ is the statistical weight of the qth component in the mixture. Thus

$$F_J\geqslant 2\textstyle\sum_q w^2(q). \tag{13}$$

This has a minimum, since

$$\textstyle\sum_q w(q)=W,$$

where W is the total weight of the mixture. The minimum is at

$$w(1)=w(2)=\cdots=w(Q)=W/Q.$$

(Q=number of components in mixture) and the value of the function at the minimum is

$$2Q(W/Q)^2=2W^2/Q.$$

We have then, finally, the basic inequality:

$$F_J\geqslant 2W^2/Q \quad \text{if the } K^0 \text{ spin is } J. \tag{14}$$

[For a wave function normalized to unity ($W=1$) and a two-component mixture ($Q=2$), (14) gives $F_J\geqslant 1$.] [We have tried to strengthen the inequality (14) by invoking the other terms on the right side of Eq. (12), with no appreciable success.]

If, on the other hand, the spin is zero, then the decay distribution is isotropic. For isotropy, the measured $B_{mm'}$ will be zero for $m\neq m'$; for $m=m'$, i.e., the diagonal terms, the B_{mm} will be weighted equally. Thus, in this case $B_{mm}=2W/(2J+1)$, so that

$$(F_J)_{\text{isotropic}}=\sum_{mm'}|B_{mm'}|^2{}_{\text{isotropic}} \\ =(2J+1)\left(\frac{2W}{2J+1}\right)^2=\frac{4W^2}{(2J+1)}.$$

A contradiction is thus encountered if the spin is zero and the trial spin J is such that $2W^2/Q\geqslant 4W^2/(2J+1)$, which amounts to

$$J\geqslant Q. \tag{15}$$

The result is intuitively plausible: one way to fake isotropy, with spin J, is to have a "thermal mixture," i.e., each of the $2J+1$ substates present, all weighted equally, and all incoherent. This in turn is indistinguishable from a mixture containing $m=0$ and each value of $|m|$, the $m=0$ component being given half the weight of each of the others. Such a mixture has $J+1$ components. Equation (14) says that nothing less than this will do; if $Q<J+1$, or in other words $J\geqslant Q$, then isotropy is not possible.

IV. METHOD OF APPLYING THE INEQUALITY TO THE DATA

In Appendix I we show that the functions

$$F_J{}^{mm'}=2\sum_{L\text{ even},M}\frac{C_{JJ}(LM,mm')[4\pi(2L+1)]^{\frac{1}{2}}}{C_{JJ}(L0,00)(2J+1)}Y_L{}^{M*}$$

project out the $\tilde{B}_{mm'}$: i.e.,

$$\int I(\theta,\phi,\Theta)F_J{}^{mm'}d\Omega_{\theta,\phi}=\tilde{B}_{mm'}(\Theta),$$

where the C's are Clebsch-Gordan coefficients, and I denotes a statistical mixture, such as the spin-up intensity I_+ (θ,ϕ,Θ). The integration is over θ, ϕ, at fixed Θ. In the future we denote such an integration by $\langle\ \rangle_{\text{av}}$. If necessary, the $\langle\ \rangle_{\text{av}}$ will also include an average over the Λ decay coordinate, ξ_x.

The inequality (14) then can be put in a form whereby it can be compared with the data. If we project out the Λ spin up (down) intensity, we have

$$F_J=\sum_{mm'}|\langle f_\pm(\xi_x)F_J{}^{mm'}(\theta,\phi)\rangle_{\text{av}}|^2\geqslant|\langle f_\pm(\xi_x)\rangle_{\text{av}}|^2, \quad (15\text{a})$$

where we have taken $W_\pm=\langle f_\pm\rangle_{\text{Av}}$ and $Q=2$, corresponding to the two orientations of struck proton spin. If we do not project out the Λ spin, we have $Q=4$, $W=1$, and

$$F_J=\sum_{mm'}|\langle F_J{}^{mm'}(\theta,\phi)\rangle_{\text{av}}|^2\geqslant\tfrac{1}{2}. \quad (15\text{b})$$

In the form (15a or b) the inequality could be applied to the data; however, the expressions (15a or b) may be simplified greatly, by noting that

$$|\langle f\rangle_{\text{Av}}|^2=\int\psi^*(1)f^*(1)\psi(1)d\tau_1\int\psi^*(2)f(2)\psi(2)d\tau_2, \quad (16)$$

where 1, 2 refer to the coordinates of two *different* events, and f is any function. Thus (15b) becomes

$$4\sum_{mm'}\int\psi^*(1)\left(\sum_{L\text{ even},M}\frac{C_{JJ}(LM,mm')[4\pi(2L+1)]^{\frac{1}{2}}}{C_{JJ}(L0,00)(2J+1)}Y_L{}^M(1)\right)\psi(1)d\tau_1$$
$$\times\int\psi^*(2)\left(\sum_{L'\text{ even},M'}\frac{C_{JJ}(L'M',mm')[4\pi(2L'+1)]^{\frac{1}{2}}}{C_{JJ}(L'0,00)(2J+1)}Y_{L'}{}^{M'*}(2)\right)\psi(2)d\tau_2, \quad (17)$$

which by the orthogonality of the C's is

$$4\sum_{L\text{ even}}\frac{4\pi(2L+1)}{[C_{JJ}(L0,00)]^2(2J+1)^2}\left(\sum_M\langle Y_L{}^M(1)Y_L{}^{M*}(2)\rangle_{\text{av}}\right).$$

By the addition theorem of spherical harmonics, this is

$$\sum_{L\text{ even}}\frac{4(2L+1)^2}{[C_{JJ}(L0,00)]^2(2J+1)^2}\langle P_L(\Delta_{12})\rangle_{\text{Av}}, \quad (18)$$

where Δ_{12} is the spatial angle between the decay π^+ of the events 1 and 2. The $P_L(\Delta)$ are normalized so that

$$\int_{-1}^{+1}(P_L(x))^2dx=\frac{2}{2L+1}.$$

Equation (18) is obviously independent of the choice of coordinate axes. For $J=2$, and projecting out the Λ spin up (down), we have from (7), (15a), and (18);

$$\frac{2}{N_p}\sum_{i>j}\left(1\pm\frac{3\xi_x(i)}{\alpha}\right)\left(1\pm\frac{3\xi_x(j)}{\alpha}\right)$$
$$\times[-1+70P_2(\Delta_{ij})+126P_4(\Delta_{ij})]\geqslant0, \quad (19)$$

where N_p=number of pairs$=N(N-1)$. Finally, by adding the (+) and (−) inequalities and writing the same thing for y and z, we have, for $J=2$:

$$\frac{2}{N_p}\sum_{i>j}\left[1+\frac{3}{\alpha^2}(\xi_i\cdot\xi_j)\right]$$
$$\times[-1+70P_2(\Delta_{ij})+126P_4(\Delta_{ij})]\geqslant0. \quad (20\text{a})$$

This last form is advantageous as it turns out to have a smaller statistical error.

For $J=4$, (7), (15a), and (18) give

$$\frac{2}{N_p}\sum_{i>j}\left[1+\frac{3}{\alpha^2}(\xi_i\cdot\xi_j)\right](-1+15.4P_2$$
$$+44.2P_4+75P_6+67P_8)\geqslant0. \quad (20\text{b})$$

(We do not give the form for $J=4$, Λ spin not projected out, as its statistical error is rather large.)

This (20a and b) is the final form of the inequality. The inequality is to be applied to N events in a "narrow" histogram interval in Θ. The question of how narrow the interval must be depends on the number of partial waves present in the production reaction (1). This in turn affects the statistical error. This point is discussed in Sec. VI. In a similar way, the beam momentum must be sufficiently well defined that the $B_{mm'}$ do not change very much over the width of the beam momentum interval. This point is best investigated experimentally, but should cause no trouble.

V. STATISTICAL ERROR

Whether any of this is useful or not depends on the statistical error associated with the inequalities.

In this section we must distinguish between two kinds of averages: one, an average over the data, including a finite number of events and an attendant statistical error. Two, an average over many experiments. This becomes the quantum mechanical expectation value in the limit of a large number of experiments. Therefore, this average we denote by $\langle\ \rangle$.

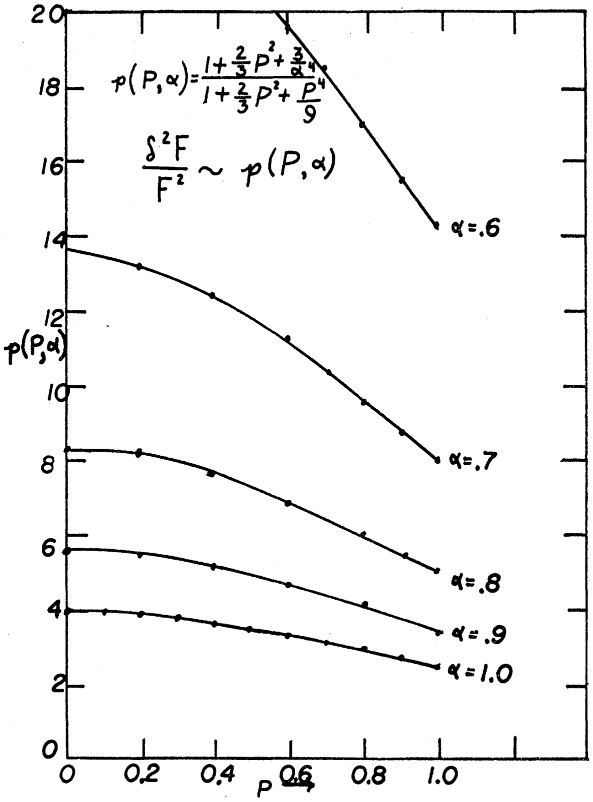

FIG. 2. The function $p(P,\alpha)$.

In the limit of large numbers, both averages are the same. We distinguish between them here, because of statistical error.

We have here to deal with functions F which are averaged over the coordinates of *pairs* of particles, i.e., suppose we have:

$$F=(2/N_p)\sum_{i>j} f_{ij}=(1/N_p)\sum_{i\neq j} f_{ij} \qquad f_{ij}=f(x_i,x_j), \tag{21}$$

then

$$\langle F\rangle=\langle f_{ij}\rangle=\langle f\rangle,$$

and

$$\delta F=F-\langle F\rangle=(1/N_p)\sum_{i\neq j} f_{ij}-\langle f\rangle.$$

The mean-squared fluctuation is

$$\langle\delta^2 F\rangle=(1/N_p^2)\sum_{i\neq j,\, i'\neq j'}[\langle f_{ij}f_{i'j'}\rangle-\langle f\rangle^2]. \tag{22}$$

The terms in $i'=i$, $j'=j$ (and vice versa) give

$$\langle\delta^2 F\rangle=(2/N_p)(\langle f^2\rangle-\langle f\rangle^2). \tag{23}$$

There is also a contribution from $i'=i$, $j'\neq j$; we have evaluated this, and find it (for our case) very small compared to (23).

(23) shows a very curious dependence on the number of events, i.e., the rms fluctuation goes as $1/N$ instead of the usual $1/\sqrt{N}$. This is because we are dealing with pairs of particles, and the number of pairs goes like N^2.

With F given by (20a), we evaluate (23) for a distribution which is actually flat in θ, ϕ, and obtain

$$\langle\delta^2 F\rangle/\langle F\rangle^2{}_{(J=2)}=(2/N_p)2740p(P,\alpha), \tag{24a}$$

where $p(P,\alpha)$ is plotted in Fig. 2; P is the Λ polarization, and α the Λ decay asymmetry parameter. [$p(P,\alpha)\simeq 8$ for 1.12 Bev/c $\pi^-+p\rightarrow\Lambda+K^0$, and for $\alpha=0.8$.]

The statistical error given by (24a) is enormous for small N; this is because we are looking for a large number of different possible deviations from isotropy. The statistical error drops rapidly with increasing N, however. This results in the situation that our method is useless with the present world supply of $\Lambda+K^0$ events, but becomes very good very rapidly with an increased supply of events.

Similarly, for $J=4$

$$\langle\delta^2 F\rangle/\langle F\rangle^2=(2/N_p)960p(P,\alpha), \tag{24b}$$

which is slightly more favorable.

VI. DISCUSSION

We must remember that the K_1^0 is a particle mixture, and see if this in any way affects the results. Also, the experiments are done in bubble chambers with magnetic field, and we must deal with the complication that a K^0 of nonzero spin might possess a magnetic moment which (it would seem) would precess in the field and thus wash out the anisotropies we are looking for.

These questions are examined in Appendix II, where it is proved that they in fact cause no difficulty. The essential feature is that a K_1^0 with nonzero spin and magnetic moment does not precess "as a whole" in a magnetic field, but rather its K^0 and $\bar{K}^0$ components precess *oppositely*, the net result being a small component of K_1^0 decays in the long-lived K_2^0 group.

By looking for this effect, and (presumably!) not seeing it, one sets a limit, that the fraction of K_1^0 decays in the K_2^0 group is less than some number ϵ. We then show that our inequalities are still good, to order ϵ, so that their usefulness is unimpaired. The proof is quite general, being based only on CPT invariance.

In conclusion we have compared this method to the Adair method.[6] We find that the statistical uncertainty is smaller in the Adair method for small numbers of events, and smaller in our method for large numbers of events. The other point of interest is the sensitivity of the method used to the number of orbital angular momentum states participating in the reaction. In this respect we find our method superior; if we define $L_{\max}$ to be that orbital angular momentum against which the method is "safe," we find $L_{\max}\sim N^2$, so that as a practical matter, uncertainty about what angular momentum states are involved soon vanishes. We estimate, that for 3 standard deviations, and safe against S, P, D, and F waves in the production process,

of the order of 5000 events are needed, all at one energy (5000 seen Λ's).

In deriving our inequality we have used only angular momentum conservation, *CPT* invariance, and the rules of quantum mechanics.

It would seem that we might do better statistically, by invoking parity conservation in the production process. We have investigated this, and find that to utilize the restriction of parity conservation to its fullest leads to cumbersome results, without greatly improving the statistics. We do not give the results here.

APPENDIX I

We now show that $\tilde{B}_{mm'}$ (and hence $B_{mm'}$) are observable, by demonstrating functions $F_J^{mm'}$ with the property that

$$\int I(\theta,\phi)F_J^{mm'}d\Omega_{\theta,\phi}=\tilde{B}_{mm'}. \tag{25}$$

The requirement that (25) be satisfied is just that

$$\int F_J^{mm'}Y_J^{m''}Y_J^{m'''}d\Omega_{\theta,\phi} =\delta\binom{m}{m''}\delta\binom{m'}{m'''}+\delta\binom{m}{m'''}\delta\binom{m'}{m''}. \tag{26}$$

First we expand $Y_J^mY_J^{m'}$ in terms of the Y_L^M:

$$Y_J^mY_J^{m'}=\sum_{L,M}C_{JJ}(LM,mm') \times C_{JJ}(L0,00)\frac{2J+1}{[4\pi(2L+1)]^{\frac{1}{2}}}Y_L^M, \tag{27}$$

now

$$C_{JJ}(LM,mm')=(-1)^LC_{JJ}(LM,m'm). \tag{28}$$

Since the left side of (27) is unchanged upon $m\rightleftharpoons m'$, the right side must be also, and so the coefficients of the Y_L^M must vanish for odd L. This is indeed so, since (28) implies that

$$C_{JJ}(L0,00)=(-1)^LC_{JJ}(L0,00), \tag{29}$$

or

$$C_{JJ}(L0,00)=0 \quad \text{for odd } L. \tag{30}$$

Therefore

$$Y_J^mY_J^{m'} =\sum_{L\text{ even},M}\frac{C_{JJ}(LM,mm')C_{JJ}(L0,00)(2J+1)}{[4\pi(2L+1)]^{\frac{1}{2}}}Y_L^M. \tag{31}$$

We try

$$F_J^{mm'}=2\sum_{L\text{ even},M}\frac{C_{JJ}(LM,mm')[4\pi(2L+1)]^{\frac{1}{2}}}{C_{JJ}(L0,00)(2J+1)}Y_L^{M*}, \tag{32}$$

so that

$$\begin{aligned}\int d\Omega_{\theta,\phi}F_J^{mm'}Y_J^{m''}Y_J^{m'''} &=2\sum_{L\text{ even},M}\frac{C_{JJ}(LM,mm')}{C_{JJ}(L0,00)}\sum_{L'\text{ even},M'}\left(\frac{2L+1}{2L'+1}\right)^{\frac{1}{2}} \\ &\quad\times C_{JJ}(L'M',m''m''')C_{JJ}(L'0,00) \\ &\quad\times\int Y_L^{M*}Y_{L'}^{M'}d\Omega_{\theta,\phi} \\ &=2\sum_{L\text{ even},M}C_{JJ}(LM,mm')C_{JJ}(LM,m''m''') \\ &=\sum_{L,M}C_{JJ}(LM,mm')C_{JJ}(LM,m''m''') \\ &\quad+C_{JJ}(LM,m'm)C_{JJ}(LM,m''m''') \\ &=\delta\binom{m}{m''}\delta\binom{m'}{m'''}+\delta\binom{m}{m'''}\delta\binom{m'}{m''},\end{aligned} \tag{33}$$

which is the required property, and shows that (32) is the desired projection operator. We are now able to project out any of the characteristic interference terms in the total intensity or the Λ-spin-up (down) intensity.

APPENDIX II. "PARTICLE MIXTURE" PROBLEM

This note deals with the particle mixture aspect of the K^0 spin determination problem.

It is necessary to look into this because:

(1) The K^0 is in a magnetic field; if it has a large magnetic moment, this might precess and "wash out" the interference terms in $\cos2\phi$, $\cos4\phi$ on which the inequality is based.

(2) The K^0 is in hydrogen rather than in vacuum, so this might have some effect on its spin states.

(With respect to item (1) it is no more difficult to do the most general case than to do any, so we do not assume *CP* invariance, and allow an arbitrary magnetic moment and spin.) The results are then based on:

(i) *CPT* invariance.

(ii) Strangeness of $+1$ for the K^0. (Since from these assumptions alone, the Lee-Yang-Oehme[9] treatment of the particle-mixture scheme follows.) If, for the time being, we neglect nuclear spin-flip processes, then the basic equations of motion are given by Good.[10] We need modify them only to this extent; the strong-interaction term in (n, n' = complex index of refraction for K^0, $\bar{K}^0$) should have added to it the magnetic energy of the K^0, ($\bar{K}^0$) magnetic moment. Thus, for a *given spin substate m* (taken along the magnetic field),

$$n_m=1+\frac{2\pi N}{k^2}A(0)-\frac{m\mu H}{\hbar\beta ck}, \tag{1}$$

$$n_m'=1+\frac{2\pi N}{k^2}A'(0)+\frac{m\mu H}{\hbar\beta ck}, \tag{1'}$$

[9] T. D. Lee, C. N. Yang, and R. Oehme, Phys. Rev. **106**, 340 (1957).

[10] M. L. Good, Phys. Rev. **106**, 591 (1957).

where βc= velocity of particle, μ= magnetic moment of K^0, H= magnetic field strength, $m=0, \pm1, \pm2\cdots$ = component of K^0 spin along magnetic field, k= wave number of K^0, and N= number of atoms per cm^3. The reversal of sign between (1), (1′) is because, for the *same* value of m, the K^0, $\bar{K}^0$ will have opposite signs for $\boldsymbol{\mu}\cdot\mathbf{H}$; i.e., the $\bar{K}^0$ magnetic moment is the negative of that of the K^0. (This follows from *CPT* invariance.)

(1,1′) is correct for a particle moving exactly perpendicular to the magnetic field, since for these the field in the particle's rest frame is $H_{\text{c.m.}}=\gamma H_{\text{lab}}$; in the rest frame the Larmor frequency is

$$\omega_{L,\text{c.m.}}=m\mu H_{\text{c.m.}}/\hbar,$$

and in the lab system it is

$$\omega_{L,\text{lab}}=\omega_{L,\text{c.m.}}/\gamma=m\mu H_{\text{lab}}/\hbar,$$

which is what appears in (1,1′).

For particles with a velocity component *along* the magnetic field, the effect is somewhat reduced, since only the part of H perpendicular to the path gets multiplied by γ, in transforming to the c.m. system. We can think of this effect as reducing the effective value of H for these particles, and consider it no further.

We neglect spin-flip processes in the hydrogen, for the time being. [This is why there is no subscript m on $A(0)$ in (1,1′).]

The general solution is (we temporarily suppress the index m)

$$\begin{pmatrix}\alpha_1(t)\\ \alpha_2(t)\end{pmatrix}=\frac{1-iR}{2^{\frac{1}{2}}(1-R^2)}\begin{pmatrix}1\\ R\end{pmatrix}e^{-\lambda_1 t}+\frac{i(1+iR)}{2^{\frac{1}{2}}(1-R^2)}\begin{pmatrix}R\\ 1\end{pmatrix}e^{-\lambda_2 t}, \quad (2)$$

where

$$R=\frac{\beta ck(n-n')}{-i(\omega_2{}^0-\omega_1{}^0)/\gamma-(1/2\gamma)(1/\tau_2-1/\tau_1)+2\Delta},$$

$$\lambda_1=\omega+\Delta;\quad \lambda_2=\omega-\Delta,$$

$$\omega=\tfrac{1}{2}[i(\omega_1+\omega_2)+(1/2\gamma)(1/\tau_1+1/\tau_2)-i\beta ck(n+n')],$$

$$\Delta=\tfrac{1}{2}\{[i(\omega_1{}^0-\omega_2{}^0)/\gamma+(1/2\gamma)(1/\tau_1-1/\tau_2)]^2+[i\beta ck(n-n')]^2\}^{\frac{1}{2}},$$

where α_1, α_2 are the amplitudes for K_1, K_2 if *CP* is conserved; and are the amplitudes for the (in vacuo) short-lived and long-lived particles, if *CP* is not conserved.[10]

Without invoking *CP*, we now appeal to experiment: in vacuo, the long-lived particle does not decay into $\pi^++\pi^-$ [rate $\leqslant(10^{-2}\times\tau_1/\tau_2\simeq10^{-2}\times2\times10^{-3}\simeq2\times10^{-5})$ of $K_1{}^0$ rate].[11] So what we are interested in is the short-lived particle amplitude, α_1. (We can refer to it as the K_1 amplitude henceforth.) We note first that for small R, there is a *long-lived* K_1 component, of intensity $|R|^2$ compared to the short K_1 component. [See Eq. (2).]

Let us assume that this is *looked for and not seen*. This sets a limit, $|R|^2\leqslant\epsilon$, $\epsilon\ll1$. We can then expand (2) for small R, and believe the result.

Small R corresponds to

$$\beta ck(n-n')\ll(1/\gamma)[i(\omega_2{}^0-\omega_1{}^0)+1/(2\tau_1)];$$

in this limit:

$$R\simeq\frac{\beta\gamma ck(n-n')}{2[-i(\omega_2{}^0-\omega_1{}^0)+1/(2\tau_1)]},$$

$$\Delta\simeq\tfrac{1}{2}\{-i(\omega_2{}^0-\omega_1{}^0)/\gamma+1/(2\gamma\tau_1)\}, \quad (3)$$

$$\lambda_{1,2}\simeq i\omega_{1,2}+1/(2\gamma\tau_{1,2})-i\beta ck(n+n')/2.$$

We have the key result that the eigenvalues (λ_1,λ_2) do not depend on the magnetic field. This is because in $n+n'$, the magnetic term just cancels. There will therefore be no precession of the spin at the Larmor frequency. Rather, to order R, initial $K_1{}^0$ turns into $K_2{}^0$ and vice versa.

Expanding the solution to first order in R gives

$$\alpha_1\simeq2^{-\frac{1}{2}}((1-iR)e^{-\lambda_1 t}+iRe^{-\lambda_2 t})$$
$$\simeq2^{-\frac{1}{2}}[e^{-\lambda_1 t}+iR(e^{-\lambda_2 t}-e^{-\lambda_1 t})]. \quad (4)$$

We are now ready to see what effect the magnetic field has on the inequality proved in the text.

Let us take first those events in which the plane of production is flat in the chamber, and choose the direction of quantization along the magnetic field. For these, quantities like $(A_m{}^*A_{m'})$ appearing in the text [Eq. (8), say] are to be replaced by

$$\int_0^\infty A_m{}^*(t)A_{m'}(t)\frac{dt}{(\gamma\tau_1)},$$

which in turn is

$$2A_m{}^*(0)A_{m'}(0)\int\alpha_{1m}{}^*(t)\alpha_{1m'}(t)dt/(\gamma\tau_1),$$

where α_{1m}, $\alpha_{1m'}$ are the $K_1{}^0$ amplitude, as a function of time, in the two magnetic substates corresponding to m, m'. So what we need is

$$\int\alpha_{1m}{}^*\alpha_{1m'}dt/(\gamma\tau_1);$$

from (4) this is

$$\int[e^{-\lambda_1 t}+iR_m(e^{-\lambda_2 t}-e^{-\lambda_1 t})]^*\times[e^{-\lambda_1 t}+iR_{m'}(e^{-\lambda_2 t}-e^{-\lambda_1 t})]dt/(\gamma\tau_1),$$

[11] M. Lederman, *1958 Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958), p. 275.

which may be integrated readily for $R \ll 1$:

$$\simeq\left[1-iR_m^*\left(\frac{1}{\gamma\tau_1(\lambda_1+\lambda_2^*)}-1\right)+iR_{m'}\left(\frac{1}{\gamma\tau_1(\lambda_1^*+\lambda_2)}-1\right)\right]$$

$$\simeq[1-R_m^*\eta+iR_{m'}\eta],$$

where

$$\eta\cong\frac{1-2i\tau_1(\omega_2^0-\omega_1^0)}{1+2i\tau_1(\omega_2^0-\omega_1^0)}.$$

(η is 1 if the mass difference $\omega_2^0-\omega_1^0$ is zero, and -1 if it is infinite. So η is of order 1.) Also

$$\int_0^\infty A_m^*A_{m'}dt/\gamma\tau_1 \simeq A_m^*(0)A_{m'}(0)(1+i\eta R_m)^*(1+i\eta R_{m'}).$$

Now this is just what we would have had for $\int A_m^*(t)A_{m'}(t)dt/\gamma\tau_1$ had the magnetic field been turned off and had $A_m(0)$, $A_{m'}(0)$ been replaced by

$$A_m''(0)=A_m(0)(1+i\eta R_m),$$
$$A_{m'}''(0)=A_{m'}(0)(1+i\eta R_{m'}),$$

and so on. But, since the inequality was proved for arbitrary A_m, $A_{m'}$, $\cdots$ *it is still true to order* R.

Any corrections to a given $B_{mm'}$ are of order R^2, i.e., of the order of the intensity of the long-lived tail of K_1^0 decays. The fractional correction to $\sum_{mm'}|B_{mm'}|^2$ is therefore of this same order, i.e., completely negligible.

All the above argument applied to the quantization direction chosen along the magnetic field.

This restriction can now be lifted. Suppose the quantization axis makes an angle Φ to the field. Call the amplitudes of the various substates quantized along this axis a_i, and call the amplitudes obtained by quantizing instead along the field, ξ_i. Then

$$a_i=\textstyle\sum_j C_{ij}(\Phi)\xi_j,$$

where the C_{ij} are the spin transformation matrix, and do not depend on the time. Now form

$$\int a_i^*a_jdt=\sum_{m,m'}C_{im}^*C_{jm'}\int\xi_m^*\xi_{m'}dt.$$

The integrals on the right are the ones we have studied. They are the same as if the field were turned off and the ξ_m replaced by

$$\xi_m''=\xi_m(0)(1+iR_m\eta),\quad \text{etc.},$$

so

$$\int a_i^*a_jdt=\sum_{mm'}C_{im}^*C_{jm'}\xi_m''^*\xi_{m'}''.$$

But, this is just what we would have had for $\int a_i^*a_jdt$ had the field been turned off and the original population of ξ_m replaced by the ξ_m''. But since the inequality was proved for arbitrary ξ_m, it is still true, to order R, and any corrections are still of the order of the intensity of the long-lived tail.

We can now go back and see what inclusion of a spin-dependent cross section would do. Arguing by analogy to the case of turning K_1 into K_2 by differential absorption of K^0, $\bar{K}^0$, the biggest effect we can get on the *amplitude* of any one spin substate is of the order of the ratio of the K_1 decay length to the mean free path for the particular spin-flip process concerned [i.e., $(m=+2)\rightarrow(m=-1)$, say]. On the intensity, the effect is of this order at worst, and may be of this order squared. Now this order is necessarily smaller than the ratio of K_1^0 decay length to total absorption length.

This latter ratio is observed experimentally, and is $\sim\frac{1}{2}\times10^{-2}$. We are therefore safe in neglecting nuclear spin-flip processes.

All the above arguments are good for arbitrary spin, so we should not have to consider the particle-mixture aspect any further.

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