Decay of Bound Muons

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The decay rate of bound muons from the K shell is calculated for elements up to lead. The muon is represented by a relativistic wave function for a point nucleus with parameters adjusted for finite-nuclear-size effects. The outcoming electron is represented by a Sommerfeld-Maue wave function. The decay rate is found to be a monotonically decreasing function of the atomic number. An estimate of the error is obtained by comparing with an exact calculation for the lowest electron angular state. The spectrum and angular distribution of the electrons are also presented.

1. INTRODUCTION

THE decay of negative muons in matter is complicated by the formation of μ atoms. Since muons from, say, a 100-Mev beam descend to the ground state in less than 10^{-10} sec.[1] almost all the decays are from the ground state.

In order to calculate the decay it is essential to have good wave functions for the initial muon state and the final electron state. Since we shall be interested mainly in the decay in materials of intermediate atomic number it is clear that the Coulomb interaction between the electron and the nucleus will be important. For large nuclei the finite nuclear size will also be significant.

In this calculation we shall neglect the interaction between the initial muon and the final electron. That is, we shall use wave functions which closely approximate solutions to the Dirac equation with a Coulomb potential.

The electron will be treated as a massless particle and the residual nucleus will be permitted to take up momentum but no energy.

The main results are presented in Table I. The rate is shown to he a monotonically decreasing function of atomic number.[2] This is in conflict with experiments[3] but in agreement with results obtained by Überall[4] who treated the problem in a different way.

2. OUTLINE OF THE CALCULATION

A. General Result

In this section we shall outline the calculation of the spectrum and angular distribution of electrons.

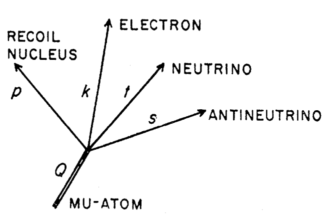

FIG. 1. Bound muon decay.

[1] E. Fermi, E. Teller, and V. Weisskopf, Phys. Rev. **71**, 314 (1947); E. Fermi and E. Teller, Phys, Rev. **72**, 399 (1947).

[2] These results were presented by J. Mathews at the December, 1959, meeting of the American Physical Society at the California Institute of Technology.

The matrix element T for muon decay is given by

$$T=2\sqrt{2}G\int d^3\mathbf{x}\,\bar{\psi}_e(\mathbf{x})\gamma_\sigma a\psi_\mu(\mathbf{x})\bar{\psi}_\nu(\mathbf{x})\times\gamma_\sigma a\psi_{\bar{\nu}}(\mathbf{x})2\pi\delta(Q_4-p_4-k-s-t),\quad(1)$$

where the momenta are defined in Fig. 1.[5]

Define M by

$$T=2\sqrt{2}GM2\pi\delta(Q_4-p_4-k-s-t).\quad(2)$$

Then the total rate for $\mu_{\text{bound}}\rightarrow e+\nu+\bar{\nu}$ is

$$R=8G^2\int\frac{d^3\mathbf{p}}{(2\pi)^3}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{s}}{(2\pi)^3}\int\frac{d^3\mathbf{t}}{(2\pi)^3}\times\sum_{\text{electron spin}}|M|^2(2\pi)^4\delta^4(Q-p-k-s-t).\quad(3)$$

So far we are dealing with oriented muons. If we insert suitable projection operators we can take traces and obtain a simpler expression.

The neutrinos are unobserved so it is convenient to eliminate them from the calculation at this stage. The neutrino wave functions are

$$\psi_\nu(\mathbf{x})=\exp(i\mathbf{t}\cdot\mathbf{x})u_{\mathrm{t}},\quad\psi_{\bar{\nu}}(\mathbf{x})=\exp(-i\mathbf{s}\cdot\mathbf{x})v_{\mathrm{s}},\quad(4)$$

where u_{t} and v_{s} are the appropriate Dirac spinors. Therefore, using the momentum delta function we have

$$M=\int d^3\mathbf{x}\,\bar{\psi}_e(\mathbf{x})\gamma_\sigma a\psi_\mu(\mathbf{x})e^{i(\mathbf{p}+\mathbf{k})\cdot\mathbf{x}}(u_{\mathrm{t}}\gamma_\sigma a v_{\mathrm{s}}).\quad(5)$$

[3] D. D. Yovanovitch, Phys. Rev. **117**, 1580 (1960).

[4] H. Überall, Phys. Rev. **119**, 365 (1960).

[5] We use the conventions $\hbar=c=1$ and $x^2=x_4^2-x_1^2-x_2^2-x_3^2$. The γ matrices are given by

$$\gamma_4=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad\text{and}\quad\gamma_a=\begin{pmatrix}0&\sigma_a\\-\sigma_a&0\end{pmatrix}\quad\text{for } a=1,2,3.$$

The spin projection operator is $a=\frac{1}{2}(1+i\gamma_5)$ where $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$. The spinor u_0 will denote either

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\quad\text{or}\quad\begin{bmatrix}0\\1\\0\\0\end{bmatrix}.$$

All spinors will be normalized by $u_{\mathrm{k}}^*u_{\mathrm{k}}=1$. We also take $m_\mu=1$ and let $Z=Z_e/137$ where Z_e is the atomic number.

It is convenient to define another quantity N_σ by $M=N_\sigma(\bar{u}_t\gamma_\sigma a v_s)$. Then

$$\int\frac{d^3\mathbf{s}}{(2\pi)^3}\int\frac{d^3\mathbf{t}}{(2\pi)^3}|M|^2(2\pi)^4\delta^4(Q-p-k-s-t)$$
$$=N_\sigma\bar{N}_\lambda(1/12\pi)(H_\sigma H_\lambda-H^2\delta_{\sigma\lambda}),\quad(6)$$

where $H=Q-p-k$. We have made use of the identity

$$\int\frac{d^3\mathbf{s}}{(2\pi)^3}\int\frac{d^3\mathbf{t}}{(2\pi)^3}\frac{s\cdot at\cdot b}{2s2t}(2\pi)^4\delta^4(H-s-t)$$
$$=\frac{1}{96\pi}(2H\cdot aH\cdot b+H^2a\cdot b).\quad(7)$$

Let us denote the available energy Q_4-p_4 by W. Then $H=(W-k,\,-(\mathbf{p}+\mathbf{k}))$.

The total rate for the decay of polarized muons can now be written

$$R=8G^2\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{(2\pi)^3}$$
$$\times\sum_{\text{spins}}N_\sigma\bar{N}_\lambda\frac{1}{12\pi}(H_\sigma H_\lambda-H^2\delta_{\sigma\lambda}).\quad(8)$$

B. Simple Example

It is instructive to carry through the calculation with simple wave functions to see how things go and especially to see which regions contribute most to the integral.

The simplest wave functions we can choose are

$$\psi_e(\mathbf{x})=e^{i\mathbf{k}\cdot\mathbf{x}}u_{\mathbf{k}},\quad\psi_\mu(\mathbf{x})=(\mu^{\frac{3}{2}}/\pi^{\frac{1}{2}})e^{-\mu r}u_0,\quad(9)$$

where $\mu=Z_\mu/137$.

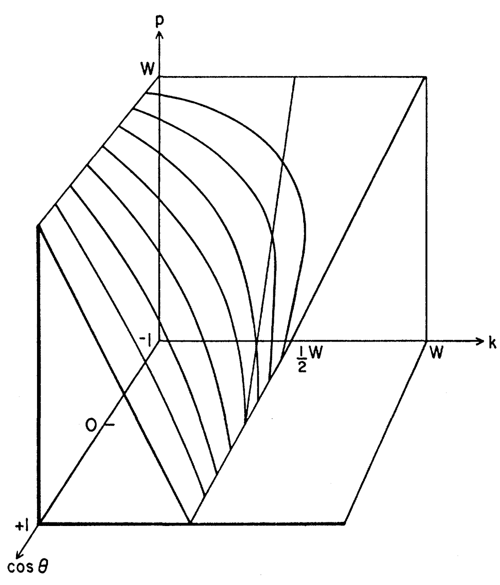

FIG. 2. The region of integration in k, p, $\cos\theta$ space. It is bounded on the right by the surface $k_m=\frac{1}{2}(W^2-p^2)/(W+p\cos\theta)$.

The result of performing the space integrals and the sum over spins is

$$R=\frac{G^2}{192\pi^3}\frac{256\mu^5}{\pi}\frac{1}{2}\int d\cos\alpha\int_0^W dp$$
$$\times\int_{-1}^{+1}d\cos\theta\int_0^{k_{\max}}dk\frac{p^2k^2}{(p^2+k^2)^4}$$
$$\times\{[3W^2-4kW-p^2+2p(W-2k)\cos\theta]$$
$$+[W^2-4kW-p^2+2p(W+2k)\cos\theta$$
$$-2p^2\cos^2\theta]\cos\alpha\},\quad(10)$$

where $\cos\alpha=\hat{k}\cdot\hat{z}$, $\cos\theta=\hat{p}\cdot\hat{k}$, and

$$k_{\max}=\frac{W^2-p^2}{2(W+p\cos\theta)}.\quad(11)$$

The muon spin is along $\hat{z}$ so the second term in the brackets gives the angular distribution. The shape of the region in phase space is shown in Fig. 2. The region of integration depends, of course, only on the kinematics and will be the same for all subsequent integrations.

The remaining integrals can be performed analytically to yield

$$R=\frac{G^2W^5}{192\pi^3}\frac{2}{3\pi}\left[\frac{\lambda}{1+\lambda^2}(3-4\lambda^2-15\lambda^4)\right.$$
$$\left.+3(1-2\lambda^2+5\lambda^4)\tan^{-1}(1/\lambda)\right],\quad(12)$$

where $\lambda=\mu/W$. For small μ we have

$$R\approx\frac{G^2W^5}{192\pi^3}(1-2\mu^2).\quad(13)$$

These results are well known.[6] Notice that for small μ the integral over p peaks sharply in the neighborhood of $p=\mu/\sqrt{3}$. Now let us return to the radial integral in N_σ:

$$4\pi\int dr\,r^2e^{-\mu r}\frac{\sin pr}{pr}.$$

If we put $p=\mu/\sqrt{3}$ we find that 0.69 of the integral comes from the region $1<r\mu<3$.

3. WAVE FUNCTIONS

A. Muon Wave Function

The muon is in a state of positive parity and $j=\frac{1}{2}$. The wave function can be written in the form

$$\psi_\mu(\mathbf{x})=\frac{1}{(4\pi)^{\frac{1}{2}}}[g(r)+f(r)i\boldsymbol{\alpha}\cdot\hat{x}]u_0.\quad(14)$$

[6] C. E. Porter and H. Primakoff, Phys. Rev. 83, 849 (1951).

For the case of a point nucleus we have[7]

$$g(r)=(1+\gamma)^{\frac{1}{2}}\frac{2^{\gamma}}{[\Gamma(2\gamma+1)]^{\frac{1}{2}}}\mu^{\frac{3}{2}}e^{-\mu r}(\mu r)^{\gamma-1},$$
$$f(r)=\left(\frac{1-\gamma}{1+\gamma}\right)^{\frac{1}{2}}g(r), \tag{15}$$

with $\gamma=(1-\mu^2)^{\frac{1}{2}}$.

Even for the lightest nuclei the singularity at the origin is not realistic and for heavier elements the finite size of the nucleus becomes important. For example, at iron the nuclear radius is about one-half the muon Bohr radius so that the effect of a finite nucleus must be considered.

For this calculation we chose $g(r)$ and $f(r)$ of the form

$$g(r)=\frac{2}{(1+\Lambda^2)^{\frac{1}{2}}}\mu^{\frac{3}{2}}e^{-\mu r},\quad f(r)=\Lambda g(r). \tag{16}$$

We obtained numerical solutions of the Dirac equation with the potential of a uniformly charged nucleus[8] and then adjusted μ and Λ to best fit the exact solutions. It is necessary to have a relatively simple analytic expression for the radial wave function so that the integrals remain tractable.

We can define an effective charge $Z_m\equiv 137\mu$. For instance at $Z_e=26$, $Z_m=24.2$.

One could obtain a slightly better approximate wave function by using a different effective charge for the small component since in a finite nucleus the small component starts off with one higher power of r than the large component. However, the effect of the small component on the rate is only about 3% at $Z_e=26$ so this improvement is not necessary.

B. The Electron Wave Function

The Dirac equation with a Coulomb potential cannot be solved in closed form for scattering states. Since the decay of bound muons proceeds through many angular momentum states, each of which is not especially easy to handle, it would be very nice to have a fairly simple approximate solution in closed form. Sommerfeld and Maue have worked out very useful wave functions.[9] They are most conveniently arrived at in the following way: Start with the Dirac equation for a massless electron in the field of a central potential $V(r)$

$$(k-V+i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla})\psi_e(\mathbf{x})=0, \tag{17}$$

and let $\psi_e(\mathbf{x})=(1/2k)(k-V-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla})\varphi(\mathbf{x})$. Then $\varphi(\mathbf{x})$ satisfies

$$[(k-V)^2+\nabla^2-i\boldsymbol{\alpha}\cdot(\boldsymbol{\nabla}V)]\varphi(\mathbf{x})=0. \tag{18}$$

The Coulomb potential is $-Z/r$ so we can write

$$\left(\nabla^2+k^2+2k\frac{Z}{r}\right)\varphi=-\left[i\boldsymbol{\alpha}\cdot\left(\boldsymbol{\nabla}\frac{Z}{r}\right)+\frac{Z^2}{r^2}\right]\varphi. \tag{19}$$

Consider the equation with the right-hand side set equal to zero. A solution of this truncated equation, with incoming scattered waves, is

$$\varphi_0(x)=e^{i\mathbf{k}\cdot\mathbf{x}}\,{}_1F_1(-iZ,1,-i(kr+\mathbf{k}\cdot\mathbf{x})). \tag{20}$$

It is now easy to show that the function

$$\chi(\mathbf{x})=e^{i\mathbf{k}\cdot\mathbf{x}}\left(1+\frac{1}{2k}i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right)Fu_{\mathbf{k}} \tag{21}$$

is a solution of (19) with the neglect of the terms

$$-e^{i\mathbf{k}\cdot\mathbf{x}}\left\{\frac{Z^2}{r^2}+\left[i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\left(\frac{Z}{r}\right)+\frac{Z^2}{r^2}\right]\frac{1}{2k}i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right\}Fu_{\mathbf{k}}. \tag{22}$$

The letter F stands for the confluent hypergeometric function in (20). The approximate solution to the Dirac equation is now

$$\psi(\mathbf{x})=e^{i\mathbf{k}\cdot\mathbf{x}}\left(1-\frac{1}{2k}i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right)Fu_{\mathbf{k}}+e^{i\mathbf{k}\cdot\mathbf{x}}\frac{Z}{r}\frac{1}{4k^2}i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}Fu_{\mathbf{k}}. \tag{23}$$

The first part, aside from a normalizing factor, is the celebrated Sommerfeld-Maue (S-M) wave function and it is this wave function that we shall use in our calculation. Correctly normalized it is

$$\psi_{\rm SM}(\mathbf{x})=e^{(\pi/2)Z}\Gamma(1+iZ)e^{i\mathbf{k}\cdot\mathbf{x}}\left(1-\frac{1}{2k}i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right)Fu_{\mathbf{k}}. \tag{24}$$

Notice that $\boldsymbol{\nabla}\,{}_1F_1(-iZ,1,-i(kr+\mathbf{k}\cdot\mathbf{x}))$ is $O(Z/r)$ asymptotically, and $O(Zk)$ at the origin. The terms we neglected in (19), i.e. (22), are then $O(Z^2/(kr)^2)$ compared with the leading term. Since, at least for small Z_e, the spatial region of importance is in the neighborhood of two muon Bohr radii, $O(Z^2/(kr)^2)$ is roughly equivalent to $O(Z^4/k^2)$ for the purpose of bound muon decay. The smallest term we have kept in the differential equation is $O(Z/(kr)^2)$.

In choosing the Sommerfeld-Maue wave function we have also dropped the last term in (23). Notice, however, that it is smaller than the second term by a factor $Z/(2kr)$. The second term is $O(Z/(2kr))$ asymptotically, and $O(Z/2)$ at the origin. We expect, then, that in the calculation of the muon decay rate the contributions of the neglected terms, i.e., the error

[7] A. I. Akhiezer and V. B. Berestetsky, *Quantum Electrodynamics* (AEC-tr-2876), p. 110. Note that Eq. (10.13) is wrong and there is an error in sign in all subsequent work on the Dirac equation in a central potential.

[8] The nuclear radius is obtained from $R=R_0A^{\frac{1}{3}}$ with $R_0=1.2\times10^{-13}$ cm.

[9] A. Sommerfeld and A. W. Maue, Ann. Physik **22**, 629 (1935). See also L. Bess, Phys. Rev. **77**, 550 (1950). H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954).

involved in using S-M wave functions, is slightly larger than the square of the contribution of the second term in the S-M wave function.

The normalization of the electron wave function depends only on the leading term because all the other terms vanish in the limit of large distances.

The wave function $\psi_{\text{SM}}(\mathbf{x})$ was obtained by approximating the Dirac equation for an electron in the field of a point charge but it is really a slightly better approximation for an electron in the field of a finite nucleus. Finite-size effects will be estimated later by expanding $\psi_{\text{SM}}(\mathbf{x})$ in angular states and comparing with the exact solution.

While we have been careful to use an electron state with incoming scattered waves we could equally well have used a state with outgoing scattered waves if we had treated it as an initial state. This is a consequence of time reversal invariance.

4. CALCULATIONS AND RESULTS

To obtain the rate of decay it now remains to do the integrals in N_σ, to sum over electron spins, and to do the momentum integrals. The space integrals are performed analytically and for muons with spin along the z axis we are left with an expression of the form

$$R=\frac{1}{2}\int d\cos\alpha\int dk\int dp\int d\cos\theta \times[R(k,p,\cos\theta)+S(k,p,\cos\theta)\cos\alpha]. \quad (25)$$

The three nontrivial integrations over k, p, $\cos\theta$ were done numerically to give rates and asymmetry coefficients. We dropped a number of terms and to make this clear let us make the following definition. Let

$$\psi_{\text{G}}(\mathbf{x})=e^{(\pi/2)Z}\Gamma(1+iZ)e^{i\mathbf{k}\cdot\mathbf{x}} \times{}_1F_1(-iZ,1,-i(kr+\mathbf{k}\cdot\mathbf{x}))u_{\mathbf{k}}, \quad (26)$$

where the subscript G stands for Gordon who first used this wave function. If we also let ψ_0 denote the nonrelativistic muon wave function times a suitable spinor u_0, we can write

$$\begin{aligned}\psi_{\text{SM}}&=\psi_{\text{G}}+\Delta\psi_{\text{G}},\\ \psi_\mu&=\psi_0+\Delta\psi_0.\end{aligned} \quad (27)$$

The matrix element clearly contains four types of terms. We dropped the piece of the matrix element containing $\Delta\psi_{\text{G}}$ and $\Delta\psi_0$. In the square of the matrix element we neglected all terms which involved the product of two small terms. That is, we kept the square of the leading term and cross terms with the leading term. This will be discussed at the end of this section.

Let $R_1(k,p,\cos\theta)$ and $S_1(k,p,\cos\theta)$ refer to the leading terms in the rate and angular distribution, i.e., the terms arising from the product of ψ_{G} and ψ_0.

$$\begin{aligned}R_1(k,p,\cos\theta)&=\Omega(A^2+B^2)[3W^2-4Wk-p^2+2(W-2k)p\cos\theta],\\ S_1(k,p,\cos\theta)&=\Omega(A^2+B^2)[W^2-4Wk-p^2-2(W+2k)p\cos\theta-2p^2\cos^2\theta],\end{aligned} \quad (28)$$

where

$$A=(p^2+\mu^2)(\mu+Zk)+2\mu(kp\cos\theta-Z\mu k),$$

$$B=-2\mu(Zkp\cos\theta+\mu k),$$

$$\Omega=\frac{G^2}{192\pi^3}\frac{256\mu^3}{\pi}\frac{2\pi Z}{1-e^{-2\pi Z}}\frac{k^2p^2}{(p^2+\mu^2)^4}\frac{e^{-2Z\varphi}}{(q^2-k^2+\mu^2)^2+4\mu^2k^2},$$

and

$$\varphi=\tan^{-1}\left(\frac{2\mu k}{q^2-k^2+\mu^2}\right),\quad q^2=k^2+p^2+2kp\cos\theta.$$

Now we give the corrections arising from the cross term of the leading term in the matrix element with the piece involving ψ_0 and $\Delta\psi_{\text{G}}$.

$$\begin{aligned}\Delta R_1(k,p,\cos\theta)&=2\Omega BD[3W^2-4Wk-p^2+2(W-2k)p\cos\theta]\\ &\quad-2\Omega AE\{p\cos\theta[3W^2-4Wk-p^2+2(W-2k)p\cos\theta]+2(W-k)p^2(1-\cos^2\theta)\},\end{aligned} \quad (29)$$

$$\begin{aligned}\Delta S_1(k,p,\cos\theta)&=2\Omega BD[W^2-4Wk-p^2-2(W+2k)p\cos\theta-2p^2\cos^2\theta]\\ &\quad-2\Omega AE\{p\cos\theta[W^2-4Wk-p^2-2(W+2k)p\cos\theta-2p^2\cos^2\theta]-2p^2(k+p\cos\theta)(1-\cos^2\theta)\},\end{aligned}$$

with

$$D=-\tfrac{1}{2}Z\mu(p^2+\mu^2),$$
$$E=-\tfrac{1}{2}Z(p^2+\mu^2).$$

The following are the additional corrections to $R_1(k,p,\cos\theta)$ and $S_1(k,p,\cos\theta)$ arising from the cross terms of the leading term with the term involving ψ_{G} and $\Delta\psi_0$.

$$\begin{aligned}\Delta R_1(k,p,\cos\theta)&=2\Lambda\Omega(AF+BG)p[-2Wp+(W^2-4Wk-3p^2)\cos\theta-4kp\cos^2\theta]-2\Lambda\Omega BJk[W^2-4Wk-p^2\\ &\quad-2p(W+2k)\cos\theta-2p^2\cos^2\theta],\\ \Delta S_1(k,p,\cos\theta)&=2\Lambda\Omega(AF+BG)p\cos\theta[3W^2-4Wk-p^2+2(W-2k)p\cos\theta]-2\Lambda\Omega BJk[3W^2-4Wk-p^2\\ &\quad+2(W-2k)p\cos\theta],\end{aligned} \quad (30)$$

where

$$F=q^2-k^2+\mu^2-2Z\mu k,$$
$$G=-2(Zkp\cos\theta+\mu k),$$
$$J=-Z(p^2+\mu^2).$$

We have neglected the small component of ψ_μ in the normalization. This introduces an error of less than

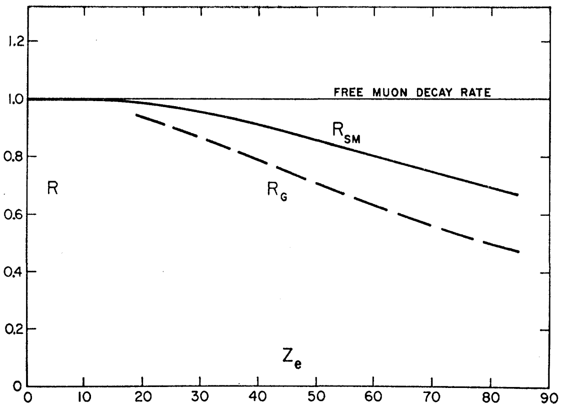

FIG. 3. The decay rate of bound muons in units of the free decay rate. The rates R_G and R_{SM} are obtained by using ψ_G and ψ_{SM}, respectively, for the electron wave function.

1% at $Z_e=26$ and about 2% at $Z_e=82$. Recall that the parameters in the muon wave function are empirical and, for instance, Λ for $Z_e=82$ is 0.15. In any case the error is easily corrected.

The final results are presented in Table I and Figs. 3–6. The rates R_G and R_{SM} are obtained by using ψ_G and ψ_{SM}, respectively, for the electron wave function.

Note that the difference between using ψ_G and ψ_{SM}, or the effect of $\Delta\psi_G$, is about 7% in the rate at $Z_e=26$. The relativistic correction $\Delta\psi_0$ to the muon wave produced about a 3% change in the rate at $Z_e=26$. In the square of the matrix element we have only kept the square of the leading term and the cross terms of the small terms with the leading term. The small size of these extra pieces justifies, at least near $Z_e=26$, the neglect of squares of the small terms in the rate, and the neglect of the smallest term in the matrix element.

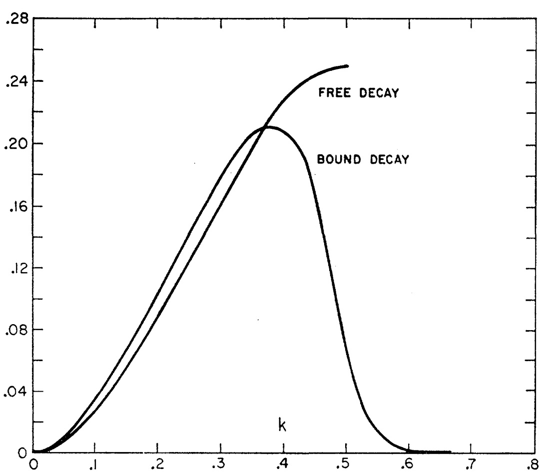

FIG. 4. The spectrum of decay electron from bound muons in iron ($Z_e=26$). The electron is represented by an S−M wave function. The free muon decay spectrum is shown for comparison. The energy k is in units of the muon mass.

A more serious possible source of error is our approximate electron wave function. The next section is devoted to providing a reliable estimate of this error.

5. PARTIAL RATE

Since the Dirac equation separates in spherical coordinates the muon decay problem can be solved exactly for a given angular state of the electron. We can also pick out the corresponding piece of the S-M wave function, perform the same calculation and then compare the results.

A. Expansion of ψ_{SM}

Let us restrict our attention to the $j=\frac{1}{2}$ positive parity state since the error will be greatest in the lowest angular momentum states. Any such state can be written in the form

$$\psi_{\frac{1}{2}^+}(\mathbf{x})=\frac{1}{(4\pi)^{\frac{1}{2}}}[a(r)-b(r)\boldsymbol{\alpha}\cdot\hat{x}]u_0, \tag{31}$$

where the radial functions remain to be determined.

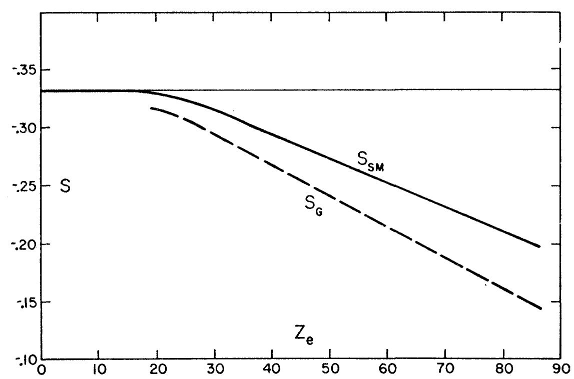

FIG. 5. The integrated asymmetry parameter of decay electrons from bound muons. The asymmetry parameter S is defined by $R(\alpha)=R+S\cos\alpha$. For free muon decay S is $-\frac{1}{3}$. The subscripts G and SM have the same meaning as in Fig. 3.

The expansion of ψ_{SM} is simplified by the use of the well-known identity proved by Gordon.[10]

$$e^{(\pi/2)Z}\Gamma(1+iZ)e^{ikz}\,{}_1F_1(-iZ,1,-i(kr+kz)) = (4\pi)^{\frac{1}{2}}\sum_l (2l+1)^{\frac{1}{2}}i^lL_l(r)Y_{l0}(\hat{x}), \tag{32}$$

where

$$L_l(r)=e^{(\pi/2)Z}\frac{\Gamma(l+1+iZ)}{(2l+1)!}(2kr)^le^{-ikr}\times{}_1F_1(l+1+iZ,2l+2,2ikr).$$

By a straightforward calculation we can obtain the $a(r)$ and $b(r)$ of the S-M wave function. It is interesting to compare them with the radial functions of a plane wave and of the exact solution to the Dirac equation for the case of a point nucleus. Let us write out the

[10] W. Gordon, Z. Physik 48, 180 (1928). See also H. A. Bethe and L. C. Maximon, Phys. Rev. 93, 768 (1954).

exact solution for a point nucleus.[11]

$$
\begin{aligned}
a_E = {} & (2\pi)^{\frac{1}{2}} e^{(\pi/2)Z} \frac{\Gamma(\gamma+iZ)}{\Gamma(2\gamma+1)} e^{i(\pi/2)(\gamma-1)} (2kr)^{\gamma-1} \\
& \times [(\gamma+iZ)e^{-ikr}\,{}_1F_1(\gamma+1+iZ,\, 2\gamma+1,\, 2ikr) \\
& \quad + e^{ikr}\,{}_1F_1(\gamma+1-iZ,\, 2\gamma+1,\, -2ikr)], \qquad (33) \\
b_E = {} & -(2\pi)^{\frac{1}{2}} e^{(\pi/2)Z} \frac{\Gamma(\gamma+iZ)}{\Gamma(2\gamma+1)} e^{i(\pi/2)(\gamma-1)} (2kr)^{\gamma-1} \\
& \times [(\gamma+iZ)e^{-ikr}\,{}_1F_1(\gamma+1+iZ,\, 2\gamma+1,\, 2ikr) \\
& \quad - e^{ikr}\,{}_1F_1(\gamma+1-iZ,\, 2\gamma+1,\, -2ikr)],
\end{aligned}
$$

where $\gamma=(1-Z^2)^{\frac{1}{2}}$.

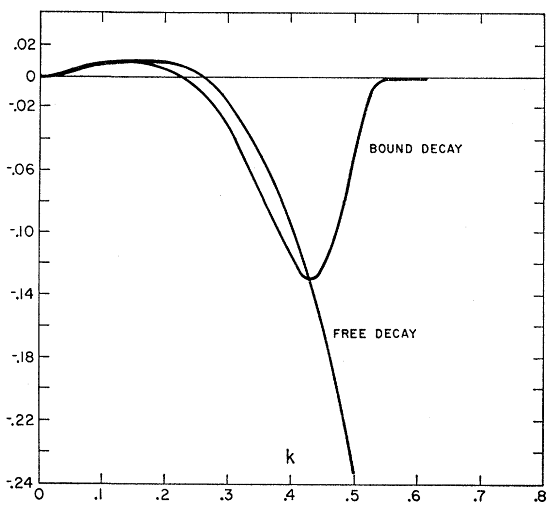

FIG. 6. The asymmetry parameter of decay electrons from bound muons in iron ($Z_e=26$). The electron is represented by an S−M wave function. The free muon asymmetry parameter is shown for comparison. The scale is the same as in Fig. 4.

The S-M radial wave functions for the $\frac{1}{2}^+$ state are obtained by replacing γ by 1 in the exact solution. In general, for higher angular momentum states the prescription for obtaining the S-M function from the exact function is to replace γ by $|\kappa|$ where κ is the eigenvalue of the operator $-\gamma_4(1+\mathbf{L}\cdot\mathbf{S})$. That is, we neglect the Z^2 inside the square root defining γ, but nowhere else. To get the radial functions of the plane wave expansion we simply set Z equal to zero everywhere.

We can obtain a state of $j=\frac{1}{2}$ and negative parity by multiplying $\psi_{\frac{1}{2}^+}$ by γ_5. This holds only for zero-mass particles but we have used this approximation throughout. The two states of opposite parity contribute equally to the rate. In each of these states we make an error of order $\frac{1}{2}Z^2$ in the electron wave function. However for higher angular states the error is of order $\frac{1}{2}Z^2/\kappa^2$ so it drops off rapidly with increasing $|\kappa|$.

[11] A. I. Akhiezer and V. B. Berestetsky, reference 7.

TABLE I. Bound muon decay rate. The decay rates R_G and R_{SM} were obtained by using ψ_G and ψ_{SM}, respectively, for the electron wave function, and are given in units of the free muon decay rate. The total available energy W is units of the muon mass. Z_m and Λ are the empirical parameters of the muon wave function.

Z_e	W	Z_m	Λ	R_G	R_{SM}
0	1.0000	0	0	1.000	1.000
16	0.9934	15.6	0.057	···	0.991
26	0.9836	24.2	0.089	0.889	0.962
35	0.9723	30.8	0.10	0.821	0.928
50	0.9507	39.6	0.14	0.703	0.853
82	0.9008	51.4	0.15	0.493	0.684

B. Rate to the $\frac{1}{2}^+$ Electron State

Using the $j=\frac{1}{2}$ part of the S-M wave function we have obtained the partial rate to that state. The rates $R_{SM}(\frac{1}{2}^+)$ for several Z_e are listed in Table II along with rates $R_P(\frac{1}{2}^+)$ obtained when the electron is represented by a plane wave. In calculating these rates we have ignored the small component of the muon wave function. We are no longer dealing with oriented muons.

The analytical expression for $R_{SM}(\frac{1}{2}^+)$ is

$$
\begin{aligned}
R_{SM}(\tfrac{1}{2}^+) = {} & \frac{G^2}{192\pi^3}\frac{32\mu^3}{\pi}\frac{2\pi Z}{1-e^{-2\pi Z}} \int dk \int dp \int d\cos\theta k^2 p^2 \\
& \times [(K^2+L^2+M^2+N^2)(3(W-k)^2-q^2) \\
& \quad -(KM+LN)4(W-k)q], \qquad (34)
\end{aligned}
$$

where

$$
\begin{aligned}
K+iL = {} & \frac{1}{8k^2q}\frac{e^{Z(\alpha+\beta-\pi)}}{1+Z^2}\{[ZC+(2+Z^2)D-2\sin\varphi] \\
& +i[-Z^2C-ZD+2Z\sin\varphi]\}, \\
M+iN = {} & \frac{1}{8k^2q}\frac{e^{Z(\alpha+\beta-\pi)}}{1+Z^2}\Bigg\{-\Bigg[ZE+(2+Z^2)F \\
& +\frac{k}{q}\frac{2}{Z}\left(Z+2\frac{\mu}{k}+Z^2\frac{\mu}{k}\right)\sin\varphi\Bigg] \\
& +i\left[Z^2E+ZF+2\frac{k}{q}\left(Z+\frac{\mu}{k}\right)\sin\varphi\right]\Bigg\},
\end{aligned}
$$

TABLE II. Bound muon decay rate to the $\frac{1}{2}^+$ electron state. The small component of the muon wave function has been neglected here. The rates R_P and R_{SM} are obtained by using plane waves and S−M wave functions, respectively. The last column gives the fraction of the total number of decays which proceed through the $\frac{1}{2}^+$ electron state. The same fraction proceeds through the $\frac{1}{2}^-$ state.

Z_e	$R_P(\frac{1}{2}^+)$	$R_{SM}(\frac{1}{2}^+)$	$R_{SM}(\frac{1}{2}^+)/R_{SM}$
0	0	0	0
16	0.094	0.101	0.102
26	0.151	0.176	0.181
35	0.179	0.223	0.240
50	0.191	0.266	0.312
82	0.159	0.274	0.400

and

$$q^2=k^2+p^2+2kp\cos\theta,$$

$$\alpha=\tan^{-1}\frac{\mu}{k+q},\quad \beta=\tan^{-1}\frac{\mu}{k-q},\quad 0<\beta<\pi$$

$$A=\frac{1}{2k}[(k+q)^2+\mu^2]^{\frac12},\quad B=\frac{1}{2k}[(k-q)^2+\mu^2]^{\frac12},$$

$$\varphi=Z\ln(B/A),$$

$$C=\frac{1}{B}\cos(\varphi+\beta)-\frac{1}{A}\cos(\varphi-\alpha),$$

$$D=\frac{1}{B}\sin(\varphi+\beta)+\frac{1}{A}\sin(\varphi-\alpha),$$

$$E=\frac{1}{B}\cos(\varphi+\beta)+\frac{1}{A}\cos(\varphi-\alpha),$$

$$F=\frac{1}{B}\sin(\varphi+\beta)-\frac{1}{A}\sin(\varphi-\alpha).$$

In the limit of $Z\rightarrow 0$ we have

$$R_P(\tfrac12{}^+)=\frac{G^2}{192\pi^3}\frac{32\mu^3}{\pi}\int dk\int dp\int d\cos\theta k^2p^2 \times[(K^2+M^2)(3(W-k)^2-q^2) -KM4(W-k)q],\quad (35)$$

where now

$$K=\frac{2\mu}{(q^2-k^2+\mu^2)^2+4\mu^2k^2},$$

$$M=\frac{\mu}{kq}\left[-\frac{q^2+k^2+\mu^2}{(q^2-k^2+\mu^2)^2+4\mu^2k^2}+\frac{1}{4qk}\ln\frac{(q+k)^2+\mu^2}{(q-k)^2+\mu^2}\right].$$

C. Error Estimate

We need to compare these results with the rates obtained with exact electron wave functions. In using S-M wave functions we make two types of errors: we neglect part of the Coulomb effect and we ignore the finite size of the nucleus. Fortunately, these errors cancel to some extent, but it is still interesting to separate them. Ultimately we want to know how well the S-M wave function approximates the exact solution for a finite nucleus.

Making this comparison presents a difficulty. The exact solutions for a point nucleus are known analytically but they are difficult to deal with, and the exact solutions for a uniformly charged finite nucleus are not known in closed form. However, it is easy to obtain exact numerical solutions for any central potential and then to perform the radial integrals. Unfortunately

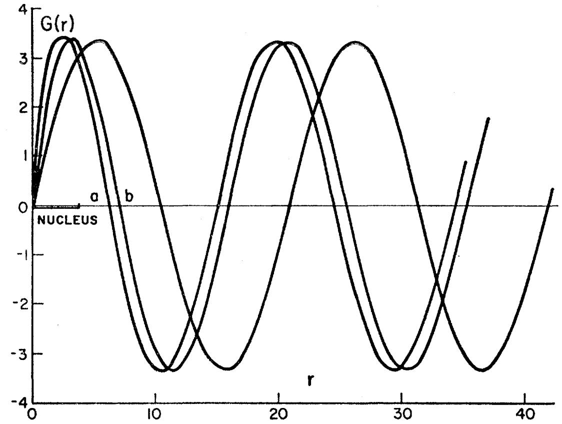

FIG. 7. The radial function $G(r)=rg(r)$ for a massless electron of energy $k=0.3m_\mu$ in a $\frac12{}^+$ state in the field of a nucleus of $Z_e=82$. The electron wave function is $\psi_e(\mathbf{x})=(4\pi)^{-\frac12}[g(r)+if(r)\boldsymbol{\alpha}\cdot\hat{\mathbf{x}}]u_0$. We show two cases: (a) a point nucleus, and (b) a uniformly charged finite nucleus. The corresponding radial function of a plane wave, i.e., $rj_0(kr)$, is shown for comparison. The distance is in units of m_μ^{-1}.

these integrations take too long for it to be practical to perform the integrals over k, p, and $\cos\theta$, but it *is* practical to compare the integrands at a number of points $(k,p,\cos\theta)$ in the neighborhood of the maximum. We have done this for $Z_e=26$ and $Z_e=82$. At $Z_e=26$ a generous estimate of the error in using the S-M wave function for the $\frac12{}^+$ state is about 3% when compared with the exact solution for a finite nucleus. The finite nuclear size has a rather small effect.

At large Z_e, say 82, the S-M approximation is also quite good because the neglected Coulomb effects and finite size effects almost cancel each other and the result is that the total decay rate is low by less than 10%. One can make a firm statement about the total rate from looking at the $j=\frac12$ state because about 80% of the electrons decay to this state. We do not reap the full benefits of this accuracy because we have not kept the higher cross terms in the numerical integration.

It is of some interest to see the effect of a potential on the continuum electron wave functions. Figures 7 and 8 show radial wave functions for the $\frac12{}^+$ state for $Z_e=82$.

The error in states of $j=\frac32$ will be about one-quarter of the error in the $j=\frac12$ states so, roughly speaking, the total percent error is approximately the percent error in the $j=\frac12$ states times the fraction of the rate which proceeds through these states plus one-quarter of the percent error in the $j=\frac12$ states times the fraction of the rate which proceeds through $j=\frac32$ states and higher. This is a generous estimate of the error.

6. COMPARISON WITH EXPERIMENTS

The results presented here are in marked disagreement with experimental results[3] both in the region $20<Z_e<30$ and for large Z_e. In the first region we do not obtain a sharp peak in the rate, and for large Z_e, say near lead, we obtain rates considerably higher than the experi-

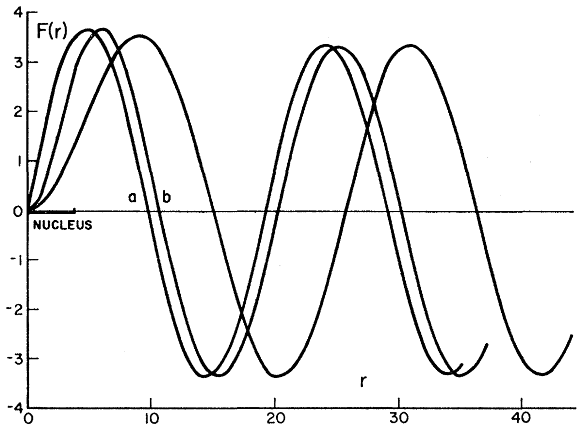

FIG. 8. The radial function $F(r)=rf(r)$ for a massless electron for the same cases as in Fig. 7. The corresponding radial function of a plane wave, i.e., $rj_1(kr)$, is shown for comparison.

mental rates. Both of these disagreements are far beyond our estimates of the error in the calculation.

The disagreement at large Z_e is perhaps less serious because the experiments are very difficult to perform and the calculation is less reliable. However at $Z_e=26$, the error in the calculation is less than 2%. The only possible conclusions seem to be: (a) there is some new process involving muons which has not been taken into account; (b) the experiments are wrong.

ACKNOWLEDGMENTS

We are indebted to Dr. S. Berman for bringing this problem to our attention, and to Professor R. F. Christy for suggesting the use of Sommerfeld-Maue wave functions. One of us (J. M.) wishes to acknowledge the support of the Radio Corporation of America.

APPENDIX

The space integrals over confluent hypergeometric functions can be performed in either of two ways.

The first is more or less standard. We use the contour integral representation[12]

$$ {}_1F_1(a,b,x)=(-1)^{b-1}(b-1)!x^{1-b}e^{x/2}\frac{1}{2\pi i} \times\oint dz\, e^{-xz}(z+\tfrac{1}{2})^{-a}(z-\tfrac{1}{2})^{a-b}, \quad \text{(A.1)} $$

where b is a positive integer, a an arbitrary number, and the contour encloses the two singularities $\pm\frac{1}{2}$ in a positive sense. Then the integrals over space, i.e., over x, are easily done and the contour integral is done last, using the method of residues.

The integrals were checked using another technique, which seems to be worth a brief discussion because of its wide applicability. We use the following property of the Laplace transformation[13]: Suppose

$$ \int_0^\infty dt\, e^{-st}F(t)=f(s), \qquad \int_0^\infty dt\, e^{-st}G(t)=g(s), \quad \text{(A.2)} $$

then

$$ \int_0^\infty dt\, e^{-st}F(t)G(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\, f(z)g(s-z), $$

where the path of integration on the right-hand side separates the singularities of $f(z)$ from those of $g(s-z)$.

This enables us to take Laplace transforms of products easily, provided the Laplace transform of at least one factor has just poles with no branch lines or other singularities. We merely deform the contour so that it encloses these poles and use the method of residues.

We illustrate the method by an example which arose in the course of this work. We wish to evaluate the integral

$$ I=\int_0^\infty dr\, r^2e^{-sr}j_0(qr)e^{-ikr}\,{}_1F_1(a,2,2ikr), \quad \text{(A.3)} $$

where a is a complex number.

The Laplace transform of $rj_0(qr)$ is elementary:

$$ \int_0^\infty dr\, e^{-sr}rj_0(qr)=\frac{1}{(s-iq)(s+iq)}=f_1(s). \quad \text{(A.4)} $$

Thus we write

$$ I=\int_0^\infty dr\, e^{-sr}[rj_0(qr)][re^{-ikr}\,{}_1F_1(a,2,2ikr)]. \quad \text{(A.5)} $$

The Laplace transform of the function in the second pair of square brackets is rather unpleasant.

$$ \int_0^\infty dr\, e^{-sr}re^{-ikr}\,{}_1F_1(a,2,2ikr) =\frac{1}{(s-ik)^a(s+ik)^{2-a}}=f_2(s). \quad \text{(A.6)} $$

However

$$ I=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\, f_1(z)f_2(s-z), \quad \text{(A.7)} $$

and we can deform the contour so it encloses the two simple poles of $f_1(s)$. The method of residues immediately gives the answer

$$ I=(2iq)^{-1}\{[s-i(q+k)]^{-a}[s-i(q-k)]^{a-2} -[s+i(q+k)]^{a-2}[s+i(q-k)]^{-a}\}. \quad \text{(A.8)} $$

[12] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic Press, New York, 1957), p. 30.

[13] G. Doetsch, *Theorie und Anwendung der Laplace-Transformation* (Verlag Julius Springer, Berlin, 1937), p. 167.