Determination of the Forward Scattering Amplitude from the Optical-Model Potential in the High-Energy Limit*

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A relation between the two-particle forward scattering amplitude, $f_k(0)$, and the volume integral over the optical potential, $V_k(\mathbf{x})$, is given in the high-energy limit, which is valid at least to second order in the two-body interactions, neglecting correlation and exchange terms. The relation is

$$\int V_k(\mathbf{x})d\mathbf{x} = -(2\pi\hbar^2/m)Af_k(0),$$

where A is the number of nucleons in the target, and m is the reduced mass of the two-particle system. The connection between the shape of the optical-model potential and the density distribution of the target nucleus is discussed.

IN the high-energy limit, the optical-model potential for the scattering of a spinless particle by a complex nucleus is usually written in the local form[1]

$$V_k(\mathbf{x}) = -(2\pi\hbar^2/m)Af_k(0)\rho(\mathbf{x}), \tag{1}$$

where $f_k(0)$ is the two-particle forward scattering amplitude for the energy $\hbar^2k^2/2m$, A is the number of particles in the target nucleus, m is the reduced mass of the two-particle system, and $\rho(\mathbf{x})$ is the density of the target nucleus in the ground state, $\int\rho(\mathbf{x})d\mathbf{x}=1$. According to formula (1), $f_k(0)$ determines the strength and $\rho(\mathbf{x})$ the shape of the potential.

Still within the framework of local potentials, we have found that a more exact expression for $V_k(\mathbf{x})$ is[2]

$$V_k(\mathbf{x}) = -\frac{2\pi\hbar^2}{m}A\frac{1}{(2\pi)^3}\int d\mathbf{k}'C(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}}f(\mathbf{k}+\mathbf{k}', \mathbf{k}), \tag{2}$$

where

$$C(\mathbf{k}) = \int\rho(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}, \quad C(0)=1, \tag{3}$$

and $f(\mathbf{k}+\mathbf{k}', \mathbf{k})$ is an off-the-energy-shell scattering amplitude:

$$f(\mathbf{k}+\mathbf{k}', \mathbf{k}) = -\frac{m}{2\pi\hbar^2}\int e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}v(\mathbf{x})\psi_{\mathbf{k}}(\mathbf{x})d\mathbf{x}, \tag{4}$$

$$f(\mathbf{k},\mathbf{k}) \equiv f_k(0).$$

Expression (2) is valid at least to second order in the two-body interactions, but correlation and exchange terms have been neglected. Equation (1) follows from Eq. (2) only under the somewhat unrealistic assumption that $f(\mathbf{k}+\mathbf{k}', \mathbf{k})$ is independent of $\mathbf{k}'$ for the range of momenta for which $C(\mathbf{k}')$ differs appreciably from zero.

Equation (2) can be written in a form similar to Eq. (1).

$$V_k(\mathbf{x}) = -(2\pi\hbar^2/m)Af_k(0)\rho_k^{\text{eff}}(\mathbf{x}), \tag{5}$$

where

$$\rho_k^{\text{eff}}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}'C(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}}\frac{f(\mathbf{k}+\mathbf{k}', \mathbf{k})}{f_k(0)}. \tag{6}$$

It is easily seen that, just as for the actual density,

$$\int\rho_k^{\text{eff}}(\mathbf{x})d\mathbf{x}=1. \tag{7}$$

In formula (5) the shape of the potential is no longer the same as that of the actual density $\rho(\mathbf{x})$. It is determined by the unknown $\rho_k^{\text{eff}}(\mathbf{x})$ which depends implicitly on the two-body interaction and on the energy. In general, $\rho_k^{\text{eff}}(\mathbf{x})$ is complex, and the real and imaginary parts have different shape. Use of a local optical-model potential with a particular shape thus amounts to an estimate of this effective density function.

We may note that ρ^{eff} becomes real and energy independent if the two-particle scattering amplitudes which appear in Eq. (6) only in the form of a ratio are treated in Born approximation:

$$\rho^{\text{eff}}(\mathbf{x}) = \left[\int v(\mathbf{y})d\mathbf{y}\right]^{-1}\int\rho(\mathbf{x}-\mathbf{y})v(\mathbf{y})d\mathbf{y}.$$

The purpose of this note is to point out that the volume integral of the optical model potential has special significance, since, as a consequence of Eq. (7),

$$\int V_k(\mathbf{x})d\mathbf{x} = -(2\pi\hbar^2/m)Af_k(0). \tag{8}$$

Equation (8) may be useful in the understanding of the optical-model potential in the high-energy limit as determined by the analysis of experimental cross sections.

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[1] See, e.g., K. M. Watson, Revs. Modern Phys. **30**, 565 (1958).
[2] This result follows from a self-consistent treatment of optical model potentials for elastic and inelastic scattering which is currently under investigation. The same result was also obtained by R. J. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1959), Vol. 1.

For the case of nucleon-nucleus scattering, where $f_k(0)$ is a well established quantity,[3] Eq. (8) serves as an additional requirement which may distinguish between different phenomenological optical potentials giving equally good fits to the cross-section curves at a particular energy. For a square well, for example, Eq. (8) implies that

$$\tfrac{4}{3}\pi R^3(V+iW)=(2\pi\hbar^2/m)Af_k(0),$$

where V and W are the depths of the real and imaginary parts of the potential and R is the radius of the well. On the other hand, for a more realistic rounded well of the form

$$V_{\mathrm{opt}}(r)=-(V+iW)[1+e^{(r-R)/a}]^{-1},$$

Eq. (8) implies the restriction[4]

$$\tfrac{4}{3}\pi R^3[1+9.88a^2/R^2](V+iW)\simeq(2\pi\hbar^2/m)Af_k(0).$$

Consequences of this relation for the scattering of nucleons are being investigated,[4a] but our present concern is the scattering of K mesons by complex nuclei. Here $f_k(0)$ is not known and, indeed, the optical-model analysis is used in attempts to determine it. It has been shown that a large family of "reasonable" phenomenological potentials can be found which fit the data.[5] It turns out, however, that all of these have approximately the same value of the volume integral.[6] To the extent that this volume integral is thereby determined, we conclude that

$$f_k(0)=-(m/2\pi\hbar^2A)\int V_k(\mathbf{x})d\mathbf{x},$$

where $V_k(\mathbf{x})$ is any one of the phenomenological optical-model potentials which fit the data. In this sense the forward scattering amplitude appears to be better determined than the optical-model potential itself, at least to the extent that the omission of exchange and correlation terms is justified.

We also remark that these considerations are not restricted to local potentials. The analogous result for nonlocal potentials is

$$\int\int V_k(\mathbf{x},\mathbf{x}')d\mathbf{x}d\mathbf{x}'=-\frac{2\pi\hbar^2}{m}Af_k(0). \qquad (9)$$

This follows immediately from the Watson-Riesenfeld expression for the nonlocal optical model potential, which is most conveniently written in momentum space[1]:

$$(\mathbf{p}|V_k|\mathbf{p})=-\frac{2\pi\hbar^2}{m}Af(k\hat{p},k\hat{p}')\frac{1}{(2\pi)^3}C(\mathbf{p}-\mathbf{p}'), \qquad (10)$$

where $\hat{p}$ and $\hat{p}'$ are unit vectors in the direction of $\mathbf{p}$ and $\mathbf{p}'$. Then

$$\int\int V_k(\mathbf{x},\mathbf{x}')d\mathbf{x}d\mathbf{x}' = \lim_{\mathbf{p}=\mathbf{p}'\to 0}(2\pi)^3(\mathbf{p}|V_k|\mathbf{p}')=-(2\pi\hbar^2/m)Af_k(0).$$

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[3] Here we mean by $f_k(0)$ the spin-independent part of the forward scattering amplitude which is related to the central part of the optical model potential.

[4] G. Igo, D. G. Ravenhall, J. J. Tiemann, W. W. Chupp, G. Goldhaber, S. Goldhaber, J. E. Lannutti, and R. M. Thaler, Phys. Rev. **109**, 2133 (1958), Eq. (8).

[4a] Such an investigation has in fact been carried out by A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. **8**, 551 (1959), who also include the spin-dependent terms in the potential.

[5] M. A. Melkanoff, D. R. Price, D. H. Stork, and H. K. Ticho, Phys. Rev. **113**, 1303 (1959); D. H. Stork, *Proceedings of the International Conference on the Nuclear Optical Model, Florida State University Studies, No. 32* (The Florida State University, Tallahassee, Florida, 1959).

[6] M. A. Melkanoff, D. J. Prowse, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **5**, 108 (1960).