Complex Singularities of Partial-Wave Amplitudes in Perturbation Theory*

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We show that the complex singularities which invalidate Mandelstam's representation do not cause complex singularities of partial wave amplitudes for two-particle scattering processes, except for the expected "kinematic" complex branch points.

RECENTLY, a general method has been developed for locating the complex singularities of contributions from Feynman diagrams.[1] In particular, Tarski[2] has studied in detail the fourth-order square diagram of Fig. 1, obtaining, in a simple manner, the condition on the masses under which the two-dimensional representation of Mandelstam[3] is no longer valid. Using Tarski's analysis, we show that the complex singularities which invalidate Mandelstam's representation do not cause complex singularities of the partial-wave amplitudes, $C_l(q^2)$. This result is of importance in that the most useful analyticity property for scattering amplitudes is, in practice, a cut plane of analyticity for the partial-wave amplitudes. Of course, if the outgoing particles differ from the incoming particles, there will be the expected "kinematic" complex branch points, due to the fact that q^2 (the squared center-of-mass momentum of an incoming particle) is not then real for all real values of the invariants $s=(p_1+p_2)^2$, $t=(p_1+p_4)^2$, $u=(p_1+p_3)^2$.

Neglecting irrelevant factors, the contribution from Fig. 1 is

$$F(s,t)=\int_0^1 \Big(\prod_{i=1}^4 dx_i\Big)\delta(1-\Sigma x_i)/D^2, \tag{1}$$

while

$$C_l(q^2)=\int_{-1}^1 d(\cos\theta)F(q^2,\cos\theta)P_l(\cos\theta), \tag{2}$$

where θ is a center-of-mass scattering angle. D is quadratic in the parameters x_i, but linear in $\cos\theta$.

Following the method of references 1 and 2 we see that $C_l(q^2)$ can have a singularity only when F has a singularity at an end point of the $\cos\theta$ integration, i.e., at $\cos^2\theta=1$. (There cannot be a "coincident" singularity since D is linear in $\cos\theta$.) Such points lie on the curve C in the s, t plane given by

$$\begin{vmatrix} p_4^2 & p_1p_4 & p_3p_4 \\ p_4p_1 & p_1^2 & p_3p_1 \\ p_4p_3 & p_1p_3 & p_3^2 \end{vmatrix}=0. \tag{3}$$

The complex singularities of $F(s,t)$ lie[2] on the curve Γ given by

$$\begin{vmatrix} 2m_1^2 & m_1^2+m_2^2-p_1^2 & m_1^2+m_3^2-s & m_1^2+m_4^2-p_4^2 \\ m_1^2+m_2^2-p_1^2 & 2m_2^2 & m_2^2+m_3^2-p_2^2 & m_2^2+m_4^2-t \\ m_1^2+m_3^2-s & m_2^2+m_3^2-p_2^2 & 2m_3^2 & m_3^2+m_4^2-p_3^2 \\ m_1^2+m_4^2-p_4^2 & m_2^2+m_4^2-t & m_3^2+m_4^2-p_3^2 & 2m_4^2 \end{vmatrix}=0,$$

which may be written in the form

$$\begin{vmatrix} p_4^2 & p_1p_4 & p_3p_4 & (m_1^2-m_4^2)-p_4^2 \\ p_4p_1 & p_1^2 & p_3p_1 & (m_2^2-m_1^2)+p_4^2-t \\ p_4p_3 & p_1p_3 & p_3^2 & (m_4^2-m_3^2)+p_3^2 \\ (m_1^2-m_4^2)-p_4^2 & (m_2^2-m_1^2)+p_4^2-t & (m_4^2-m_3^2)+p_3^2 & 4m_4^2 \end{vmatrix}=0. \tag{4}$$

There are complex singularities in the physical sheet of F (defined by certain cuts over real values of s and t, and hence u) only if

$$\theta_1+\theta_2+\theta_3+\theta_4>2\pi, \tag{5}$$

where

$$\theta_i=\cos^{-1}[(m_i^2+m_{i+1}^2-p_i^2)/2m_im_{i+1}] \tag{6}$$

(which are real by stability conditions on the internal and external masses), and then they lie on the surface Σ_1 generated by the complex intersections of Γ with real lines of the form

$$\alpha s+\beta t=\gamma, \tag{7}$$

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[1] J. C. Polkinghorne and G. R. Screaton, Nuovo cimento **15**, 289 (1960).
[2] J. Tarski, J. Math. Phys. (to be published).
[3] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

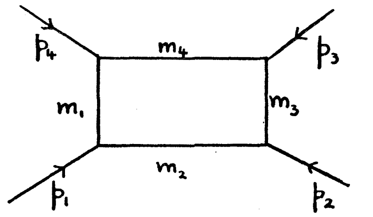

FIG. 1. Fourth-order diagram whose complex singularities are discussed in the text.

where $\alpha>0$, $\beta>0$, and γ is such that the line passes between the branches Γ_5 and Γ_1 of Γ (Fig. 2).

Suppose P is a complex singularity of F lying on C. Then, by (3) and (4), the coordinates of P depend on the m_i^2 only as differences: If we add an amount x to each m_i^2, the resultant $\Gamma(x)$ still passes through P, which lies on a line of the form (7) which is independent of x. As x increases, by (6), the θ_i remain real and tend to zero, and, as $\theta_1+\theta_2+\theta_3+\theta_4 \rightarrow 2\pi+0$, the branches $\Gamma_1(x)$ and $\Gamma_5(x)$ join. Thus no line of the form (7) can meet all $\Gamma(x)$ in the same complex point, and so P is nonexistent, and $C_l(q^2)$ cannot have complex singularities in its physical sheet (defined by cuts on the real q^2 axis), except for the above-mentioned "kinematic" singularities.

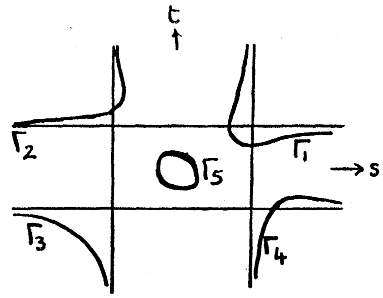

FIG. 2. Curves of real singularities of the fourth-order diagram of Fig. 1 in the s, t plane.

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Oscillatory Character of Reissner-Nordström Metric for an Ideal Charged Wormhole*

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A transformation is presented to remove coordinate ("pseudo") singularities from metrics of a certain class, a special case of which is the transformation of Kruskal, extending the Schwarzschild metric beyond its pseudosingularity. The transformation is applied to the Reissner-Nordström metric, which describes a concentration of charge and mass in general relativity. On an initial surface this metric shows the same general behavior as the Schwarzschild metric, describing a "wormhole," or bridge, between two asymptotically flat spaces, but with electric flux flowing through the wormhole. It is found that the region of minimum radius, the so-called "throat" of the wormhole, begins to contract, but reaches a minimum and re-expands after a finite proper time, rather than pinching off as in the Schwarzschild-Kruskal case: *the raduis of the throat pulsates periodically in time*, "cushioned" by Maxwell pressure of the electric field through the throat. The motion of charged particles in this metric is investigated, and it is shown that no particle can hit the geometric singularity at $r=0$; (1) quite in general, provided only that the mass of the test particle exceeds the value associated in general relativity with its charge, and (2) in particular when the test particle has no charge at all, but (3) such collisions are *not* avoided when the throat itself is not endowed with any electric flux.

I. INTRODUCTION

MANY of the well-known solutions of Einstein's equations contain apparent singularities which have not been understood until recently. Such singularities seem to be a characteristic feature of solutions of the free-space Einstein and Einstein-Maxwell equations; they can often be prevented from occurring if one grants that near the singularity some field of nonzero rest-mass contributes to the curvature of space. However, it has been suggested[1] that in the domain of free-space gravitation and electromagnetism all "properly closed" spaces must *always* develop an intrinsic geometrical singularity as time evolves. It is therefore interesting to examine some of the known exact solutions of the Einstein-Maxwell equations for singularities, even in cases where they describe spaces which—instead of being closed—are asymptotically flat at great distances.

To do this, one must distinguish between true geometric singularities at which invariants of the Riemann curvature tensor become singular, and "pseudosingularities," which are due to an unfortunate choice of coordinate system. The well-known Schwarzschild solution provides a good example of both types of singularity. In the two most common coordinate systems, some of the metric coefficients vanish or

* Based in part on Chapters 2 and 6–10 of a thesis submitted by the first author in partial fulfillment of the requirements for the B.A. degree at Princeton University.

[1] J. A. Wheeler, Nuovo cimento.