Problem of Proving the Mandelstam Representation*

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It is shown that sufficiency conditions for the validity of the Mandelstam representation of a collision amplitude can be stated in terms of the location of singularities of its physical branch when one of the energy variables is real and positive and the second independent energy variable is complex. It is also shown that in perturbation theory the real singularities of the physical branch must correspond to positive Feynman parameters provided that none of the curves of singularities identified in this way have turning points in positive spectral regions. The same condition will exclude complex singularities in the physical sheet that are connected to the curves of singularities on its real boundary. If there are any disconnected complex singularities they must correspond to complex Feynman parameters.

It is concluded that sufficiency conditions for the Mandelstam representation are the absence of turning points in positive spectral regions and the absence of disconnected complex singularities in the physical sheet. In the equal-mass case these conditions are both necessary and sufficient, and real singularities in the physical sheet can be identified by requiring that the Feynman parameters shall be positive. The conditions are satisfied by ladder diagrams when there are no anomalous thresholds. [It has now been proved that the representation is correct for every order in perturbation theory. See Phys. Rev. Letters **5**, 213 (September 1, 1960), letter by the author. The proof requires some of the results of this paper.]

I. INTRODUCTION

THE aim of this paper is to obtain sufficiency conditions for the validity of the Mandelstam representation[1] in a form that is convenient for study in perturbation theory. The Mandelstam representation implies that a scattering amplitude has a branch that is analytic in the complex planes of the invariant energies, s, t, and u except for branch cuts and poles on the positive real axes. These are defined by the equations

$$s=(p_1+p_2)^2,\quad t=(p_1+p_4)^2,\quad u=(p_2+p_3)^2, \quad (1.1)$$

and

$$\sum_1^4 p_i=0,\quad p_i^2=M_i^2,\quad s+t+u=\sum_1^4 M_i^2, \quad (1.2)$$

where p_i are the external four-momenta of the colliding particles. In Sec. 2, it is shown that sufficiency conditions for the representation can be stated in terms of the location of singularities of the physical branch of the amplitude $A(s,t,u)$ when one of the variables s, t, u is real and positive.

In order to study whether these conditions are in fact satisfied by an amplitude defined by a perturbation solution in quantum field theory, we must have a method for locating singularities of the physical branch without considering also those of the more complicated unphysical branches. The physical branch is defined so that it satisfies the causality requirements of quantum field theory in the physical scattering regions of the real, s, t, u plane which forms the boundary of the physical sheets of the complex planes. This definition leads to the usual prescription of associating every internal mass with a small negative imaginary part to give $(m^2-i\epsilon)$ in the Feynman integrals of the perturbation series. Singularities of these integrals can always be associated with critical values of the Feynman parameters in their integrands. It is shown in Sec. 3 that under certain conditions the real singularities of the physical branch can be identified by the requirement that the critical values of the Feynman parameters must be positive. The conditions used are the absence of turning points in those curves of singularities for which the Feynman parameters are positive. These conditions are sufficient, but probably are not always necessary in the general mass case. They also ensure the absence of complex singularities that connect to real curves of singularities on the boundary of the physical sheet. These conclusions are not affected by the possible existence of disconnected complex singularities. If these do exist, the Mandelstam representation would have to be modified by the addition of complex contours of integration.

The restriction to positive Feynman parameters greatly simplifies the discussion of curves of singularities. The problem of determining whether an amplitude given by a perturbation series satisfies these conditions has been discussed elsewhere.[2] Some results of that paper (hereinafter denoted as I) will be used in Sec. 3.

II. SUFFICIENCY CONDITIONS

In this section we obtain conditions on the location of the singularities of a branch of a collision amplitude for it to satisfy the Mandelstam representation. The representation will be established by a double application of Cauchy's theorem by making explicit assumptions about the location of singularities. These

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] R. J. Eden, Phys. Rev. **119**, 1763 (1960).

assumptions then provide the sufficiency conditions that we require.

For simplicity we use the equal-mass problem to illustrate our assumptions. The general mass case requires special consideration in the problem of proving these assumptions, but in establishing their sufficiency it has no special significance.

Assumption 2A

There is a region of the real s, t plane and a complex neighborhood of this region in which the amplitude $A(s,t)$ has no singularities or branch cuts.

Assumption 2B

With t real and within the bounds set by the analytic region of assumption 2A, the amplitude $A(s,t)$ has no singularities in the complex s plane except for branch points or poles on the real axis.

We will not explicitly include poles in our formulas, since they give only trivial changes. The region of analyticity in the equal-mass case has been obtained in I. A single application of Cauchy's theorem in the complex s plane gives,

$$A(s,t)=\frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{[A(s'+i\epsilon,t)-A(s'-i\epsilon,t)]}{(s'-s)}ds' \\ -\frac{1}{2\pi i}\int_{-t}^{-\infty}\frac{[A(s'+i\epsilon,t)-A(s'-i\epsilon,t)]}{(s'-s)}ds'. \quad (2.1)$$

The region of analyticity and the two integration contours are illustrated in Fig. 1, where the contours are denoted (a) and (b).

Assumption 2C

For all real and positive values of s except for a discrete set of points, the amplitude $A(s,t)$ has no singularities in the complex t plane except for branch points on the real axis in the ranges $4m^2\leq t$, and $t\leq -s$.

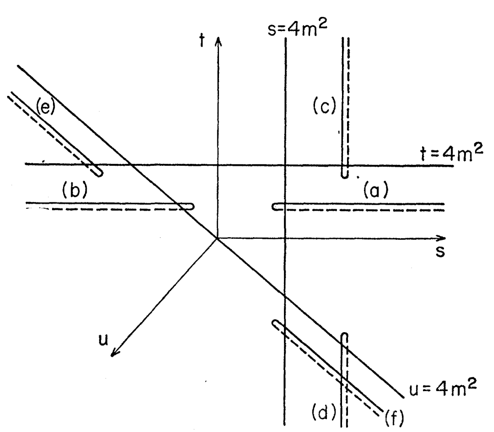

FIG. 1. The real s, t, u plane ($u=4m^2-s-t$). Branch points and contours of integration in the complex planes of s, t, u are indicated.

We can allow ϵ in Eq. (2.1) to tend to zero except in a negligible part of the contours of integration. The assumption 2C permits a second application of Cauchy's theorem, which transforms the first integral on the right of Eq. (2.1) into a double-dispersion integral. The appropriate contours are denoted (c) and (d) in Fig. 1, and the double integral is

$$\frac{1}{(2\pi i)^2}\int_{4m^2}^{\infty}ds'\int_{4m^2}^{\infty}dt'\frac{\rho_1(s',t')}{(s'-s)(t'-t)} \\ +\frac{1}{(2\pi i)^2}\int_{4m^2}^{\infty}du'\int_{m^2}^{\infty}ds'\frac{\rho_2(u',s')}{(u'-u)(s'-s)}, \quad (2.2)$$

where

$$\rho_1(s',t')=\lim_{\epsilon\to 0}[A(s'+i\epsilon,t'+i\epsilon)-A(s'-i\epsilon,t'+i\epsilon) \\ -A(s'+i\epsilon,t'-i\epsilon)+A(s'-i\epsilon,t'-i\epsilon)] \quad (2.3)$$

and $\rho_2(u',s')$ is defined from Eq. (2.3) by the substitution

$$t'=4m^2-s'-u'. \quad (2.4)$$

If we made an assumption similar to assumption 2C but for s real and negative, the integrand of the second integral on the right of Eq. (2.1) could be transformed by Cauchy's theorem, and we could formally obtain a double integral. However this would not be in a useful form. This is clear from Fig. 1, which shows that the values of t in the integrand that we wish to transform lie on the cut in the complex t plane. Hence there would be an essential contribution to the integral from the neighborhood of these values.

A useful transformation of the integrand of the second integral in Eq. (2.1) can be obtained by using oblique axes. These are defined by a contour along which u is fixed and real. We require a further assumption.

Assumption 2D

For all real and positive values of u except for a discrete set of points, the amplitude $A(4m^2-t-u,u)$ has no singularities in the complex t plane except for branch points on the real axis in the ranges $4m^2\leq t$, and $t\leq -u$.

With this assumption, the integrand of the second integral in Eq. (2.1) can be transformed into a dispersion integral along the contours (e) and (f) in Fig. 1. This leads to double-dispersion integrals

$$-\frac{1}{(2\pi i)^2}\int_{-t}^{-\infty}ds_1\int_{4m^2}^{\infty}dt_2\frac{\rho_3(t',u')}{(t-t)(u'-u)} \\ -\frac{1}{(2\pi i)^2}\int_{-t}^{-\infty}ds_1\int_{4m^2}^{\infty}ds_2\frac{\rho_2(u',s')}{(u'-u)(s'-s)}. \quad (2.5)$$

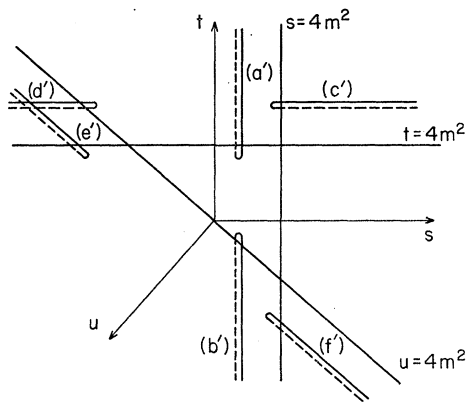

FIG. 2. The real s, t, u plane showing contours used in the second double application of Cauchy's theorem.

The integration elements ds_1 are taken with t' fixed, and dt_2 and ds_2 are taken with u' fixed. The function $\rho_3(t',u')$ is obtained from Eq. (2.3) by using Eq. (2.4).

By repeating the above procedure with the first transformation made in the complex t plane, we obtain integrals along the paths (a'), (b'), (c'), (d'), (e'), and (f') in Fig. 2. Clearly the double integrals are over the same regions as those in expressions (2.2) and (2.5). An additional assumption has to be made.

Assumption 2E

For all real and positive values of t except for a discrete set of points, the amplitude $A(s,t)$ has no singularities in the complex s plane except for branch points on the real axis in the ranges $4m^2 \leq s$, and $s \leq -t$.

The integrals over oblique axes in this second case are the same as in expression (2.5), but the integration elements are now dt_1 taken with s' fixed, and dt_2 and ds_2 taken with u' fixed as before. Using Eq. (2.4), we have

$$du' = -ds' - dt' = -ds_1 - dt_1. \tag{2.6}$$

This permits us to combine the double integrals. After dividing by two, we obtain the Mandelstam representation for $A(s,t)$, namely, the two terms shown in expression (2.2) and the term

$$\frac{1}{(2\pi i)^2} \int_{4m^2}^{\infty} dt' \int_{4m^2}^{\infty} du' \frac{\rho_3(t',u')}{(t'-t)(u'-u)}. \tag{2.7}$$

For the complete amplitude, ρ_1, ρ_2, and ρ_3 will be the same functions in the equal-mass case. For parts of the amplitude, such as terms in the perturbation series, they will not in general be the same functions.

We conclude that sufficiency conditions for the Mandelstam representation require (a) a real region and its complex neighborhood in which a branch of the amplitude $A(s,t,u)$ is free from singularities, and (b) analyticity when each variable in turn is real and positive and the others are complex but satisfy Eq. (1.2).

III. THE PHYSICAL BRANCH OF A COLLISION AMPLITUDE

The Mandelstam representation assumes it is the physical branch of a collision amplitude that has the simple analyticity properties that were discussed in Sec. 2. In this section we show that the definition of the physical branch in perturbation theory in terms of the Feynman amplitude in the real scattering regions sufficient under certain conditions to define the amplitude unambiguously throughout the physical sheet. It is shown further that real singularities of the physical branch under the same conditions can be identified by requiring critical values of Feynman parameters to be real and positive.

A. The Real Boundary of the Physical Sheet

A scattering amplitude satisfying causality conditions and giving the correct scattering in physical parts of the real s, t plane can be expressed as a sum of terms corresponding to the Feynman diagrams for the perturbation series. Each term has the form

$$F(s,t) = \underset{\epsilon\to 0}{\text{limit}}\; c \int dk_1 \cdots \int dk_l \prod_{i=1}^{n} \frac{1}{(q_i^2 - m_i^2 + i\epsilon)} \tag{3.1}$$

in which q_i is the four-momentum in an internal line and depends linearly on the internal momenta k_j and the external momenta p_k.

The integral on the right of Eq. (3.1) can be transformed, giving

$$F_\epsilon(s,t) = c_1 \int dk_1 \cdots \int dk_l \int_0^1 d\alpha_1 \cdots \times \int_0^1 d\alpha_n \frac{\delta(1-\sum \alpha_i)}{[\psi(k,\alpha,p)]^{n+1}}, \tag{3.2}$$

where

$$\psi(k,\alpha,p) = \sum_{i=1}^{n} \alpha_i (q_i^2 - m_i^2 + i\epsilon). \tag{3.3}$$

In the physical scattering regions, each four-momentum q_i is a real Minkowski vector. Hence for $\epsilon > 0$ we have

$$q_i^2 - m_i^2 + i\epsilon \neq 0. \tag{3.4}$$

This proves that for $\epsilon > 0$ in the physical scattering regions, $F_\epsilon(s,t)$ is nonsingular. It is known also that as ϵ tends to zero in these regions the amplitude becomes singular at normal thresholds, and its component F may become singular at these points. The amplitude may also have singularities below the first normal threshold if certain mass inequalities hold. These singularities give anomalous thresholds. If there are no anomalous thresholds, the Euclidean region of the real s, real t plane is free from singularities (see I, Sec. 4 and 8).

The real s, real t plane is defined as the boundary of

the physical sheets in s, t, and u by taking branch cuts in the complex planes of these variables along their real axes from the leading real branch point to infinity. In order to make this definition meaningful, we must show that the physical branch of the amplitude can be defined near the real boundary of the physical sheet.

B. Near the Real Boundary of the Physical Sheet

When ϵ is positive, the integration in Eq. (3.2) over the internal momenta can be carried out unambiguously. This gives

$$F_\epsilon(s,t)=c_2\int_0^1 d\alpha_1\cdots\int_0^1 d\alpha_n\frac{\delta(1-\sum\alpha_i)C(\alpha)^{n-2l-1}}{D_\epsilon(\alpha,s,t)^{n-2l}},\quad(3.5)$$

where D_ϵ is the discriminant of ψ as a function of k, and C is the discriminant of ψ_0 obtained from ψ by putting $m_i=0$, $s=0$, and $t=0$. The discriminant D_ϵ has the form

$$D_\epsilon(\alpha,s,t)=sf(\alpha)+tg(\alpha)-K_\epsilon(\alpha),\quad(3.6)$$

where $K_\epsilon(\alpha)$ is a function of the external masses M_r^2, the internal masses $m_i^2-i\epsilon$, and the variables α. The coefficient of $i\epsilon$ is $C(\alpha)$, which is positive when the variables α are real and positive since it is then the discriminant of a positive definite quadratic form. It follows that with $\epsilon>0$, and s and t both real, we have

$$D_\epsilon(\alpha,s,t)\neq 0\quad(3.7)$$

along the path of integration in the α variables. This result shows that analytic continuation through the whole of the real s, real t plane can be carried out for $F_\epsilon(s,t)$.

For certain real values of s and t, $F_\epsilon(s,t)$ becomes singular as ϵ goes to zero. At these values the singularities of the integrand of Eq. (3.5) either pinch the contour of integration or occur at the end point, $\alpha_i=0$. These singularities are zeros of $D(\alpha,s,t)$ as a function of the α variables. They must also occur at real positive values of each α_i, since, when ϵ is positive, Eq. (3.7) holds and it is never necessary to distort the contour of integration from the real path, 0 to 1.

If there are no turning points in the curves of singularities, $f(\alpha)$ and $g(\alpha)$ are both positive at the critical α values that correspond to a singular point of the amplitude (see I, particularly Sec. 7). We will assume for this discussion that this condition is valid. Then near the critical values of α we can write

$$D_\epsilon(\alpha,s,t)=(s+i\epsilon')f(\alpha)+(t+i\epsilon'')g(\alpha)-K(\alpha)\quad(3.8)$$

$$=D(\alpha,\,s+i\epsilon',\,t+i\epsilon'')\quad(3.9)$$

where ϵ' and ϵ'' are positive, and

$$K(\alpha)=K_{(\epsilon=0)}(\alpha).\quad(3.10)$$

The relation (3.8) is not valid in general at all positive values of α, since f and g do not always remain positive away from the critical values of α. However, away from these values, at least one of the α_i will give singularities that are not coincident, and they can therefore be avoided by a suitable small distortion of the contour of integration. This distortion is not required near the critical values of α, where the imaginary part of s or t suffices to prevent the singularities being on the contour of integration.

At a normal threshold in s, for example, $g(\alpha)$ is zero. Then the singularity can be avoided by giving s a small positive imaginary part. The above discussion of curves of singularities shows that, in the absence of turning points, a small positive imaginary part in s will also avoid these singularities in the region where s is positive. For a discrete set of real values of t (the normal thresholds), the location of singularities is independent of s, but analytic continuation past them is defined by giving t a small positive imaginary part. The discussion can be extended in an obvious manner to regions where the variable u is positive. These results are summarized in the following theorem.

Theorem 3A

Provided there are no turning points in the curves of singularities of the physical branch of the amplitude in positive spectral regions, the amplitude can be analytically continued through the complex neighborhood of the real boundary of the physical sheet.

C. Positive Feynman Parameters and the Physical Branch of the Amplitude

We consider first the situation in which the curves of singularities have no turning points. From Theorem 3A the amplitude is single-valued under analytic continuation near the entire real boundary of the physical sheet. In the limit as the real boundary is approached there may be singularities. These singularities must correspond to positive values of the Feynman parameters α, since the path of integration does not need to be distorted in any significant way in the complex neighborhood of the boundary. It follows that negative Feynman parameters are important on or near the real boundary only if the amplitude is not single-valued on the physical sheet. It can be multivalued only if there are complex singularities. From Theorem 3A these cannot be connected to singularities on the real boundary. Hence they must be disconnected complex singularities. Even if these disconnected singularities exist, we can still choose that branch of the function whose singularities are given by the positive α condition and make it single-valued by introducing suitable branch cuts in the complex part of the physical sheet. This will be the physical branch of the function.

Theorem 3B

Provided there are no turning points in curves of singularities obtained with the positive condition on

the Feynman parameters, the physical branch of the amplitude can be defined so that (a) it is single-valued in the physical sheet, and (b) all its singularities on the real boundary of the physical sheet are given by the positive condition on the Feynman parameters.

This discussion assumes the absence of turning points. If there are spurious turning points (see I, Sec. 7D), there will be complex branch cuts which prevent the simple analytic continuation obtained in Theorem 3A. With anomalous turning points, there are two possibilities. Either the resulting complex singularities go into a nonphysical sheet, in which case Theorem 3A will still be valid, or they remain in the physical sheet and connect to real singularities on another branch of the curve of singularities. In the latter case, the conditions of Theorem 3A do not hold, and Feynman parameters may not always be positive for the physical branch of the amplitude. Barmawi has pointed out that this happens in fourth-order perturbation theory with superanomalous thresholds and the physical branch has some singularities corresponding to negative Feynman parameters.[3]

In the equal-mass case, there are no anomalous thresholds. Hence any turning points that exist must be spurious, and if they occur in positive spectral regions, they will cause a breakdown of the Mandelstam representation. Hence in this case the absence of turning points in positive spectral regions is also a necessary condition for the validity of the representation.

D. Disconnected Complex Singularities

When s is real and t is complex with a positive imaginary part, we can see from Eq. (3.8) that $D(\alpha,s,t)$ is nonzero provided α is real and $g(\alpha)$ is nonzero. Thus the zeros of D do not cross the real α axis except where $g(\alpha)$ is zero. In the simple case of ladder diagrams, or diagrams formed by reducing ladders, the functions $f(\alpha)$ and $g(\alpha)$ consist only of positive terms on the path of integration. Then unless there are anomalous thresholds, D will not vanish for s real and t complex, and there will be no disconnected complex singularities. For the equal-mass case, ladder diagrams have no anomalous thresholds and hence no disconnected singularities, nor do they have spurious turning points. It follows that these ladder diagrams satisfy the Mandelstam representation. This result has also been proved by Cutkowski[4] and by Wanders[5] using the Bethe-Salpeter equation.

In general, in a region where $g(\alpha)$ is negative, some of the zeros of D will cross the real α axis as t goes complex, and will lead to distortion of the contour of integration. When t is nearly real, this distortion is unimportant, as was seen in Theorem 3A. As the imaginary part of t increases, the variables α become complex and the condition that $g(\alpha)$ vanishes is no longer necessary for other zeros of D to cross the real α axis. There may then be new coincident singularities of the integrand on the right of Eq. (3.5). It is still necessary for D as well as its derivatives with respect to each α_i to be zero if the integral in Eq. (3.5) is also to be singular. The latter condition cannot be satisfied unless some of the α_i are complex.

If there are such disconnected complex singularities, they will cause a breakdown of the Mandelstam representation by the introduction of additional branch cuts in the physical sheet. The difficulty in eliminating the possibility that they exist lies in their very tenuous relation to the positive condition on Feynman parameters, which holds only for real singularities under the conditions discussed earlier. This obscures the identification of the Riemann sheets in which disconnected complex singularities occur, since in all sheets including the physical one, they involve Feynman parameters that are complex and have no simple relation to their values for real singularities.

Note added in proof. It has now been proved that every term in the perturbation expansion of the scattering amplitude satisfies the Mandelstam representation. The proof will be described in a paper by the author to be submitted to the Physical Review. The proof was outlined at the Rochester Conference 1960, and in Phys. Rev. Letters **5**, 213 (September 1, 1960).

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[3] M. Barmawi, Physics Department, University of Chicago (private communication).

[4] R. Cutkowski, Physics Department, Carnegie Institute of Technology (private communication).

[5] G. Wanders, Institut für Theoretische Physik der Universität, Hamburg, 1960 (unpublished).