assume that the ratio of the second maximum to the primary maximum is approximately the same as in $Fe^{56}(d,t)Fe^{55}$. The shapes of the angular distributions for all $l=1$ groups, including those from the Fe^{56} target, are very similar. Estimating the intensity of the primary maximum on the basis of that of the secondary maximum is believed to introduce a possible error of no more than 10% into the resulting reduced width.

Another possible source of error in the reduced widths results from the choice of the value of Λ. The value of Λ is obtained from comparison of (d,p), (p,d), and (d,t) reactions which proceed between the same two levels. On the basis of data which extend up to $A=25$, Macfarlane[1] finds $\Lambda=195\pm35$. This is the value used in the present calculations. There may be a large deviation from this value of Λ in the region of A investigated in the present work, but in the absence of any additional information no better estimate can be made. The reduced widths given in Table I may be simply corrected for any improved value of Λ by keeping the product $\Lambda\theta^2$ constant.

Apart from the errors involved in determining Λ, all possible sources of experimental error introduce less than 20% error into the reduced widths obtained here. The reduced widths for the various levels and groups are given in Table I, where $\theta^2(1)$ refers to angular distributions corresponding to $l=1$ transitions and $\theta^2(3)$ refers to angular distributions corresponding to $l=3$ angular distributions. The groups are listed according to residual nuclides and excitation energy in the nucleus. The "less than" symbol indicates the absence of an observed $l=3$ transition when one is possible, the listed value for the reduced width then being an upper limit on the $l=3$ contribution as set by the cross section at the minimum of the observed $l=1$ angular distribution.

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Analysis of the Structure of Nuclei from (d,t) Reactions*

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The following work illustrates the use of experimentally determined reduced widths from (d,t) reactions in analyzing the structure of the ground state of the target nuclei. The (d,t) reaction is an especially sensitive and almost unique experimental technique for measuring small components of nuclear wave functions. In this role it becomes a valuable tool in investigating the presence of strong pairing forces in nuclei. This analysis demonstrates the strong mixture of the $2p_{3/2}$ and $1f_{5/2}$ neutron states in the region around $A=60$. In contrast to this, nuclei with 28 neutrons show no observable mixing of states. Special attention is given to Fe^{57} and a shell-model wave function is derived that gives the observed magnetic moment as well as the observed (d,t) reduced widths. In this connection a simple general formula is presented for the magnetic moment of any shell-model wave function.

I. INTRODUCTION

THE Butler theory of nuclear reactions[1] has been a most productive theory in the fields of (*a*) the mechanism of nuclear reactions, and (*b*) nuclear structure and spectroscopy. In this paper the primary interest is in area *b*. The Butler theory is used as an empirical tool in analyzing (d,t) pickup experiments in order to extract information about nuclear wave functions.

This application of Butler theory was first used by Bethe and Butler[2] to estimate the l value admixtures in certain nuclear wave functions. Since that time several authors[3] have extended this technique and demonstrated its usefulness in determining nuclear structure. The most recent work in this field is that of Macfarlane and French,[4] who have made a complete

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951). See also S. T. Butler, *Nuclear Stripping Reactions* (John Wiley & Sons, Inc., New York, 1957) for a complete list of references to the early work in this field.

[2] H. A. Bethe and S. T. Butler, Phys. Rev. **85**, 1045 (1952).

[3] See, for example, J. B. French and B. J. Raz, Phys. Rev. **104**, 1411 (1956); T. Auerbach and J. B. French, Phys. Rev. **98**, 1276 (1955); A. M. Lane, Proc. Phys. Soc. (London) **A66**, 977 (1953); and S. Okai and M. Sano, Progr. Theoret. Phys. (Kyoto) **14**, 399 (1955), and **15**, 203 (1956).

[4] M. H. Macfarlane and J. B. French, Revs. Modern Phys. **32**, 567 (1960).

survey of all relevant stripping and pickup experiments and have analyzed this data in terms of nuclear wave functions. The techniques used in much of the present paper follows the formalism developed by Macfarlane and French and the real justification for the procedure rests on the consistent picture that emerges from their analysis of the experimental information.

The Butler formula for the differential cross section for neutron pickup via a (d,t) reaction may be expressed as

$$\frac{d\sigma}{d\omega}=\Lambda\frac{A(A+1)}{(A+3)^2}\frac{K_t}{K_d}\frac{R}{\gamma_l^2}G_l^2\theta^2(l), \tag{1}$$

where: (1) Λ is an empirically determined constant that gives the normalization of (d,t) experiments relative to (d,p) experiments.[1,4,5] Theoretically this constant is a measure of the overlap of the deuteron and triton wave functions. (2) A is the mass of the residual nucleus. (3) K_t is the wave number of the triton in the center-of-mass system of the residual nucleus after emission of the triton. (4) K_d is the wave number of the deuteron in the center-of-mass system of the target nucleus before it was struck by the deuteron. (5) R is the appropriate stripping radius for this process. (6) γ_l is equal to $h_l(ikR)/[ikh_{l+1}(ikR)]$, in which (a) h_l is the spherical Hankel function of order l, and (b) k is related to the binding energy of the picked-up neutron and is defined by

$$k^2=\frac{2M}{\hbar^2}\frac{A}{A+1}(Q \text{ value of reaction}+6.258 \text{ Mev}).$$

(7) G_l^2 is a function defined by

$$G_l^2=\frac{1}{(Q^2+k^2)^2}[j_l(QR)-\gamma_l Q j_{l+1}(QR)]^2,$$

in which (a) $\mathbf{Q}$ is the momentum transferred by the reaction and is given by

$$\mathbf{Q}=\mathbf{K}_t-\frac{A}{A+1}\mathbf{K}_d,$$

(b) j_l is the spherical Bessel function of order l, (c) G_l^2 is related to the tabulated values of Lubitz[6] by the expression

$$G_l^2=(R\gamma_l)^2[1+0.008(x^2+y^2)]^2\sigma_l(\theta)^{\text{Lubitz}},$$

where $x=QR$ and $y=kR$. Hence, G_l^2 may also be written

$$G_l^2=\left|\frac{1}{ikh_{l+1}(ikR)}\right|^2 \times\left\{\frac{[j_l(Qr)\partial h_l(ikr)/\partial r-h_l(ikr)\partial j_l(Qr)/\partial r]_{r=R}}{Q^2+k^2}\right\}^2.$$

(8) $\theta^2(l)$ is the dimensionless reduced width to be extracted from the experimental data.

By use of the above relationships and formula (1), the reduced width may be determined from the absolute cross section of any (d,t) experiment.

The reduced width $\theta^2(l)$, is the basic piece of information that is obtained from the analysis of the (d,t) reaction work. This constant, $\theta^2(l)$ is related to the structure of the nuclear levels involved, by the relationship $\theta^2(l)=\mathcal{S}(l)\theta_0^2(l)$. Here $\theta_0^2(l)$ is the pure single-particle reduced width that is determined from experiments involving single-particle states in a variety of nuclei. Examples of these levels would be the ground states and some of the excited states in O^{17} and Ca^{41}. If the single-particle states are considered as states in some static potential, then $\theta_0^2(l)$ is just equal to $\frac{1}{3}R_l^2(r_0)r_0^3$ where $R_l(r_0)$ is the value of the radial part of the wave function, evaluated at the nuclear surface $r=r_0$.

In practice formula (1) is used to extract values of $\theta^2(l)$ and thus $\mathcal{S}(l)$ for the various levels seen in the (d,t) reaction. This procedure, at first glance, seems to imply that the Butler theory gives the correct relationship of cross section to the variables of the problem: incident energy, excitation energy of the final nucleus and charge of the particles involved. However, it is clearly unwise to assume that such a simple theory as the Butler theory gives the exact dependence of cross section. Detailed calculations[7] including distorted waves, Coulomb corrections, and other refinements indicate the limitations of the Butler theory, but these limitations do not destroy the usefulness of the procedure. These variations of cross section may be incorporated into a dependence of the single-particle reduced width $\theta_0(l)$ and the normalizing constant Λ on the Q value of the reaction, on the value of Z, and on the incident energy. This dependence has been investigated[4,5] in the region of the atomic weight less than 30, and is under investigation[8] experimentally in the region around $A=50$–60. The results[4,5] indicate that both these parameters, Λ and $\theta_0^2(l)$, vary only slowly with the parameters of the problem, so that this procedure is practical and the absolute value of the cross section given by Butler theory is easily normalized correctly by performing the necessary experiments in the region of interest.

For definiteness, the following values were taken from reference 4: $\Lambda=200$, $\theta_0^2(2p)=2.2\times10^{-2}$, and $\theta_0^2(1f)=1.1\times10^{-2}$. These values have uncertainties in them but relevant experiments are under way[8] to reduce this ambiguity in the region of the periodic table about $A=50$. Future more accurate values of these parameters may easily be incorporated into the present results. The over-all consistency of the results

[5] A. I. Hamburger, Phys. Rev. **118**, 1271 (1960).
[6] C. R. Lubitz, University of Michigan Report, 1957 (unpublished).
[7] W. Tobocman, Phys. Rev. **115**, 98 (1959).
[8] M. Macfarlane (private communication).

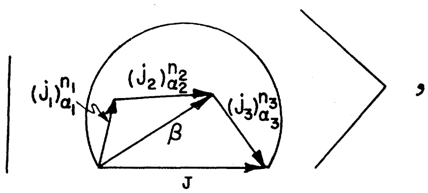

FIG. 1. The representation of a nuclear wave function in terms of a vector diagram.

presented here[9] suggests that the values of the parameters taken are valid choices.

II. DETERMINATION OF SPECTROSCOPIC FACTORS

The spectroscopic factor $\mathcal{S}(l)$ measures the degree of overlap between the wave functions of the target nucleus and the final state.[10] In particular, if the wave functions are described in terms of a jj coupling shell model, the $\mathcal{S}$ factor is proportional to the square of the overlap integral between the target state and a wave function described by the total angular momentum vector of the picked-up neutron coupled to that vector of the final state to yield a total spin equal to that of the target state.

To be more precise, the nuclear wave functions may be described in the representation in Fig. 1, where the symbol represents a shell-model state formed by three groups of n_i nucleons having j value j_i and total angular momentum α_i, where $i=1$, 2, or 3. In addition, $\alpha_1+\alpha_2=\beta$ and $\beta+\alpha_3=J$, where J is the spin of the entire state. The arc between the ends of J is to symbolize that the entire wave function is to be antisymmetric under interchange of either two protons or two neutrons. Isotopic spin is ignored in this particular use although the generalization is trivial and is given in complete detail in Macfarlane and French.[4]

For pure nuclear states of the above type, $\mathcal{S}(l)$ becomes $\mathcal{S}(j)$ for each j value since different j values do not interfere. Then $\mathcal{S}(j_3)$ is displayed in Fig. 2, where the last symbol in the overlap integral represents the vector coupling of the picked-up nucleon to the final state.

$$\mathcal{S}(j_3)=n_3\langle j_3^{n_3}\alpha_3|j_3^{n_3-1}\alpha_3'\rangle^2U^2(\beta'\alpha_3'Jj_3;J_f\alpha_3)\times\delta(\alpha_1\alpha_2\beta;\alpha_1'\alpha_2'\beta'),\quad(3)$$

where U is a normalized Racah[11] coefficient, and $\langle j_3^{n_3}\alpha_3|j_3^{n_3-1}\alpha_3'\rangle$ is the coefficient of fractional parentage between the states α_3 and α_3'.

In the (d,t) reactions it is often most convenient to look at the sum of $\mathcal{S}(j_3)$ over α_3 and J_f rather than $\mathcal{S}(j_3)$ for each individual level. The value of $\mathcal{S}$ fall off rapidly with excitation since, in general, highly excited states of the final nucleus do not contain large parts of the ground-state wave function of the target. This effect is seen quite clearly in the experimental results which show that θ^2 decreases rapidly with the excitation energy of the final nucleus.

The sum of all $\mathcal{S}$ values associated with a given j in the nucleus simplifies both the theoretical and experimental situation. Experimentally it is not necessary to resolve individual levels but only to find the total contribution of a given j for all levels. Theoretically, the value of the sum of $\mathcal{S}(j_3)$ becomes just n_3 by making use of the orthonormality of both the coefficients of fractional parentage and the normalized Racah coefficients.[4]

Even if the wave functions are described by a linear superposition of the states of the above representation, the above sum rule is still very useful. In particular, if the target is represented as

$$\Psi_J=\sum_s K_s{}^J\psi_J(n_1j_1,n_2j_2,n_3j_3,\gamma),\quad(4)$$

where $\sum_s$ indicates a sum over all possible variations in n_1, n_2, n_3, α_1, α_2, α_3, and β that is consistent with the properties of ψ_J, and γ stands for all other quantum numbers that are not explicitly stated.

The final state is represented by

$$\Phi_\Gamma=\sum_r K_r{}^\Gamma\phi(r),\quad(5)$$

where Γ stands for all identifying quantum numbers of the final state and r stands for all the quantum numbers necessary to characterize the $\phi(r)$.

When the general sum over r is performed, the result obtained for the $\mathcal{S}$ factor is

$$\sum_\Gamma \mathcal{S}_\Gamma(j_3)=\sum_s n_3|K_s{}^J(n_1n_2n_3\gamma)|^2,\quad(6)$$

or, in general,

$$\sum_\Gamma \mathcal{S}_\Gamma(j_i)=\sum_s n_i|K_s{}^J(n_1n_2n_3\gamma)|^2.\quad(7)$$

This result is based on normalization of $K_s{}^J$ and K^Γ's as well as the orthonormality of the coefficients of fractional parentage and the normalized Racah coefficients. With this general sum rule the composition of the wave functions that describe the ground state of the target nucleus may easily be investigated.

A simple example will illustrate the procedure. In the reaction $Fe^{56}(d,t)Fe^{55}$, five distinct triton groups

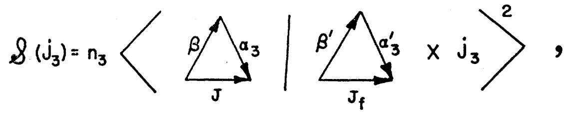

FIG. 2. The spectroscopic factor $\mathcal{S}$ defined in terms of the representation in Fig. 1.

[9] All the experimental (d,t) reaction data used in this paper is from B. Zeidman, J. L. Yntema, and B. J. Raz, preceding paper [Phys. Rev. **120**, 1723 (1960)].

[10] This factor $\mathcal{S}$ is the same as the factor S in Eqs. (1) and (2) in French and Raz, reference 3. Equation (2) in that work should read:

$$S=n\sum_{j_n}|\sum_{\alpha\beta}K_\alpha(n)K_\beta(n-1)\langle\psi_\alpha(n)|\psi_\beta(n-1)\times j_n\rangle|^2.$$

[11] See, for example, G. Racah and U. Fano, *Irreducible Tensorial Sets* (Academic Press, New York, New York, 1959); A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

TABLE I. Reduced widths,[a] θ^2, and spectroscopic factors $\mathcal{S}(l)$ for strong groups. The groups are listed according to excitation in the final nucleus. The symbols $\theta^2(1)$ and $\theta^2(3)$ designate reduced widths and $\mathcal{S}(1)$ and $\mathcal{S}(3)$ designate spectroscopic factors for $l=1$ and $l=3$, respectively. The "less than" symbol indicates an upper limit for an unobserved $l=3$ contribution. In the use of Eq. (1) for these results Λ was set equal to 200, $\theta_0^2(1)$ was set equal to 2.2×10^{-2} and $\theta_0^2(3)$ was set equal to 1.1×10^{-2}.

Target nucleus	Final nucleus	Level in final nucleus	$\theta^2(1)\times10^2$	$\mathcal{S}(1)$	$\theta^2(3)\times10^2$	$\mathcal{S}(3)$	Comments
$_{23}V_{28}^{51}$	V^{50}	0.3			1.8	1.63	$\Sigma\,\mathcal{S}(3)=4.27$
		1.1			1.6	1.46	No $p_{3/2}$ excitation seen
		3.1			1.3	1.18	of the $(f_{3/2}{}^8)$ closed shell.
$_{24}Cr_{28}^{52}$	Cr^{51}	0			2.9	2.64	$\Sigma\,\mathcal{S}(3)=3.00$
		0.75			0.4	0.36	No $p_{3/2}$ excitation seen.
$_{25}Mn_{30}^{55}$	Mn^{54}	0	2.9	1.32			
		1.1	0.5	0.23	1.0	0.91	
		2.7	0.5	0.23			
		4.0			0.2	0.18	
$_{26}Fe_{30}^{56}$	Fe^{55}	0	1.6	0.73			
		0.42	0.4	0.18			
		1.4	0.2	0.09	1.4	1.27	
		2.0	0.1	0.04			
		2.5			0.1	0.09	
$_{26}Fe_{31}^{57}$	Fe^{56}	0	0.38	0.17			$\Sigma\,\mathcal{S}(f_{5/2})=1.47$
		0.85	0.76	0.35	0.70	0.63	$\Sigma\,\mathcal{S}(p_{3/2})=1.18$
		2.9	1.82	0.83			$\Sigma\,\mathcal{S}(p_{1/2})=0.17$
		4.0			0.93	0.84	
$_{27}Co_{32}^{59}$	Co^{58}	0.3	5.3	2.41	<2.2	<2.0	
$_{29}Cu_{34}^{63}$	Cu^{62}	0.4	3.8	1.73	<2.0	<1.8	
		1.4	0.7	0.32	<0.4	<0.4	
$_{29}Cu_{36}^{65}$	Cu^{64}	0.4	6.4	2.92	<4.0	<3.6	
$_{30}Zn_{34}^{64}$	Zn^{63}	0	3.3	1.50	<1.5	<1.4	
		0.64	0.9	0.41	<0.3	<0.3	
		1.1	0.4	0.18	<0.2	<0.2	
$_{30}Zn_{36}^{66}$	Zn^{65}	0	3.85	1.75	<2.35	<2.1	
		0.86	0.75	0.34	<0.50	<0.5	
$_{30}Zn_{37}^{67}$	Zn^{66}	0			0.29	0.26	$\Sigma\,\mathcal{S}(1)=1.13$
		1.0	0.35	0.16	0.20	0.18	$\Sigma\,\mathcal{S}(3)=2.18$
		2.7	0.64	0.29	0.91	0.83	
		3.7	1.5	0.68	1.0	0.91	
$_{30}Zn_{38}^{68}$	Zn^{67}	0.3	4.5	2.04	3.7	3.4	

[a] See reference 9.

are observed and the reduced widths and the $\mathcal{S}$ factors for each are displayed in Table I. The ground state of Fe^{56} is composed of 26 protons and 30 neutrons; and in terms of a shell-model picture, the configuration may be represented by $(f_{7/2}{}^{-2})_0$, $(p_{3/2}{}^2)_0$ or $(f_{7/2}{}^{-2})_0$, $(f_{5/2}{}^2)_0$ where the first parenthesis indicates the proton configuration and the second parenthesis, the neutron configuration. For simplicity in this example the $l=3$ transitions are assumed to come only from $f_{5/2}$ pick up rather than $f_{7/2}$ pick up from the closed neutron shell. The wave function may be described by

$$\psi=\alpha(f_{7/2}{}^{-2})_0(p_{3/2}{}^2)_0+\beta(f_{7/2}{}^{-2})_0(f_{5/2}{}^2)_0.$$

From Eq. (7) and the observed values of $\mathcal{S}(l=1)$ and $\mathcal{S}(l=3)$, it follows that

$$2\alpha^2=\sum\mathcal{S}(l=1)=0.73+0.18+0.09+0.04=1.04,$$

$$2\beta^2=\sum\mathcal{S}(l=3)=1.27+0.09=1.36.$$

Since the wave functions are assumed to be normalized, $\alpha^2+\beta^2$ should be equal to one so $2\alpha^2+2\beta^2=2$; but use of the above data leads to a value of $2\alpha^2+2\beta^2$ that is 20% too high. This merely reflects the possible errors in absolute normalization. The sources of these errors include (a) the measurement of the absolute cross section, (b) the extraction of the reduced width from the data, (c) the normalization factor Λ relating (*d*,*p*) and (*d*,*t*) cross sections, and (d) the precise value of the single-particle cross sections for $2p$ and $1f$ states. In view of these experimental and theoretical problems, the 20% difference in normalization is not disturbing.

The above relations lead to $\alpha^2/\beta^2=1.04/1.36=0.76$, a value which suggests that the $f_{5/2}$ state has a higher probability than the $p_{3/2}$ state. This same type of analysis is applied to all the nuclei, with special attention given to Fe^{57}.

III. DISCUSSION OF OBSERVED $\mathcal{S}$ FACTORS FOR EVEN GROUPS OF NEUTRONS

The experimental results and the corresponding $\mathcal{S}$ values are displayed in Table I. Several general comments are necessary before discussing each individual case.

A. If the target has a spin of 0, then conservation of parity and angular momentum insure that each state of the final nucleus is formed by pickup of a nucleon having a unique j value. Thus any triton group that has both $l=3$ and $l=1$ must go to at least two unresolved levels of different spin in the final state.

B. If the ground state of the final nucleus has zero spin a similar type of limitation occurs. A neutron having a j value equal to the spin of the target must be picked up in the reaction to the final ground state. An example of this occurs in Fe^{57} for which a $p_{1/2}$ neutron is picked up in the (d,t) reaction to the ground state of Fe^{56}.

C. Weak $l=3$ levels are hard to locate experimentally and are often masked by $l=1$ pickup.

D. As the number of particles outside a closed shell increases, $\sum \mathcal{S}$ increases but often large contributions to this sum occur at excitation energies higher than the energies investigated in these experiments. For example in Cr^{52} with $8f_{7/2}$ particles, the value of $\sum \mathcal{S}(l=3)$ is 8 while experimentally this turns out to be only 3.0 up to 0.75 Mev. The remaining $l=3$ contributions should be found in levels at higher excitation energy in Cr^{51}.

Some features of Table I will now be examined, with the above four comments in mind. In V^{51} and Cr^{52}, the 28 neutrons are in a closed-shell configuration of $(f_{7/2}{}^8)$. Any trace of $(f_{7/2}{}^{-2})(p_{3/2}{}^2)$ would show up as an $l=1$ transition which could not be missed. Judging from the weakest $l=1$ that was observed in the other nuclei, it is quite plain that the admixture of two-particle excitation configuration $(f_{7/2}{}^{-2})(p_{3/2}{}^2)$ in 28-neutron ground states of V^{51} and Cr^{52} has less than a 4% probability.

This technique is a very sensitive one for detecting two-particle excitation in an even group of nucleons. The presence of a strong pairing force between pairs of particles coupled to give spin zero[12] could be detected by observing two-particle excitation in the ground states of nuclei having an even neutron number. If the (d,t) experiment were performed on a target nucleus containing 22, 24, or 26 neutrons, any $p_{3/2}$ admixture could be easily detected and would indicate two-particle excitation of the $f_{7/2}$ particles to the $p_{3/2}$ level. The isotopes Ca^{42}, Ca^{44}, Sc^{45}, Ti^{46}, and Cr^{50} are targets that might show this effect. Preliminary results on Sc^{45} did not show this effect.

The neutron number for the rest of the targets is above 29 so that the $f_{7/2}$ shell is closed in these cases. The next shells to be filled are the $p_{3/2}$ and $f_{5/2}$ shells, which are observed to be almost degenerate in many odd-neutron nuclei in this region. An unambiguous example of this near degeneracy is Cr^{53}, whose $p_{3/2}$ and $f_{5/2}$ levels are separated by only 0.97 Mev. Therefore, these two levels are expected to mix strongly and all $l=3$ reduced widths are assumed to come from $f_{5/2}$ pickup rather than a breaking of the closed $f_{7/2}$ shell.

Note added in proof. This assumption is probably not valid in the case of the 1.4 Mev group in $Fe^{56}(d,t)Fe^{55}$. Therefore the value of β given for Fe^{56} is to be regarded as an upper limit. A more detailed discussion of this point will be given in a future paper.

The results for the remaining targets with even numbers of neutrons are quite similar and they are discussed as a group. The ratio of $\sum \mathcal{S}(l=3)$ to $\sum \mathcal{S}(l=1)$ measures the probability of picking up a $f_{5/2}$ or a $p_{3/2}$ particle from the target nucleus, and this probability in turn is a direct measure of the relative amount of each j value in the wave function of the target.

In Table II the values of $\sum \mathcal{S}(l=1)$, $\sum \mathcal{S}(l=3)$, and the ratio of $\sum \mathcal{S}(l=3)$ to $\sum \mathcal{S}(l=1)$ are tabulated for the targets with an even number of neutrons in this region of the periodic table. The gradual filling of both the $p_{3/2}$ and the $f_{5/2}$ shell is observed in the values of $\sum \mathcal{S}(l=1)$ and $\sum \mathcal{S}(l=3)$.

IV. Fe^{57} WAVE FUNCTION

The $Fe^{57}(d,t)Fe^{56}$ reaction can be analyzed in more detail since the $p_{1/2}$ contribution may be identified in this reaction. The investigation of the ground state of Fe^{57} is of current interest because of the present use of the 14-kev gamma ray in Fe^{57} for precise measurements of frequency shifts.[13] This Fe^{57} ground state does not fit the conventional shell model.[14] Its spin and magnetic moment are both anomalous and are therefore the first features to be discussed. Next the analysis of the (d,t) reduced widths is discussed to provide the necessary information to determine the wave function. Finally, for completeness, the properties of the $\frac{3}{2}^-$ 14-kev level are discussed.

The general formula for the magnetic moment of any state described by jj-coupling wave functions may be evaluated immediately either by using a semiclassical vector-model picture or the more general tensor algebra

TABLE II. Sums of spectroscopic factors for nuclei with even number of neutrons in the $p_{3/2}f_{5/2}$ region. The ratio $\Sigma\,\mathcal{S}(3)/\Sigma\,\mathcal{S}(1)$ is equal to the ratio of the probability of picking up a $1f$ neutron to the probability of picking up a $2p$ neutron. $\theta_0{}^2(1)$ and $\theta_0{}^2(3)$ were set equal to 2.2×10^{-2} and 1.1×10^{-2}, respectively, and Λ was set equal to 200.

Number of neutrons Total	Outside closed shell	Elements	$\Sigma\,\mathcal{S}(1)$	$\Sigma\,\mathcal{S}(3)$	$\Sigma\,\mathcal{S}(l=3)/\mathcal{S}(l=1)$
30	2	Mn^{55}	1.78	1.09	0.61
30	2	Fe^{56}	1.04	1.36	1.31
32	4	Co^{59}	2.41	<2.0	<0.8
34	6	Cu^{63}	2.05	<2.2	<1.1
34	6	Zn^{64}	2.09	<1.9	<0.9
36	8	Cu^{65}	2.92	<3.6	<1.2
36	8	Zn^{66}	2.09	<2.6	<1.2
38	10	Zn^{68}	2.04	3.4	1.67

[12] See, for example, S. T. Belyaev, Kgl. Danske Videnskab Selskab, Mat-fys. Medd. **31**, no. 11 (1959).

[13] J. P. Schiffer and W. Marshall, Phys. Rev. Letters **3**, 556 (1959); R. V. Pound and G. A. Rebka, Jr., Phys. Rev. Letters **3**, 554 (1959).

[14] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Structure* (John Wiley & Sons, Inc., New York, 1955).

of Racah.[11,15] The general formula[16] is found to be

$$\langle (j)^N J,(j')^{N'} J' I|\mu|(j)^N J,(j')^{N'} I\rangle$$
$$=\frac{J(J+1)+I(I+1)-J'(J'+1)}{2J(I+1)}\langle (j)_J{}^N|\mu|(j)_J{}^N\rangle$$
$$+\frac{J'(J'+1)+I(I+1)-J(J+1)}{2J'(I+1)}$$
$$\times\langle (j')_{J'}{}^{N'}|\mu|(j')_{J'}{}^{N'}\rangle, \quad (8)$$

where

$$|(j)^N J,(j')^{N'} J' I\rangle$$

represents the wave function formed by vector coupling the wave function of N equivalent particles of spin j and total spin J to the wave function of N′ equivalent particles of spin j' and total spin J' to give the final spin I. Since the magnetic moment operator is a symmetric one-particle operator, it is unnecessary to explicitly antisymmetrize wave functions in this matrix element.

One finds immediately (see, for example, Mayer and Jensen[17]) that

$$\langle (j)_J{}^N|\mu|(j)_J{}^N\rangle=(J/j)\langle j|\mu|j\rangle,$$

where $\langle j|\mu|j\rangle$ is the single-particle (Schmidt) value of the magnetic moment,[18] given by

$$\langle j|\mu|j\rangle=j\left[g_l\pm\frac{1}{2l+1}(g_s-g_l)\right], \quad j=l\pm\tfrac{1}{2}, \quad (9)$$

$$g_s=2\mu_p \quad \text{and} \quad g_l=1 \text{ nm for protons},$$
$$g_s=2\mu_n \quad \text{and} \quad g_l=0 \text{ for neutrons},$$

where

$$\mu_p=2.792 \text{ nm} \quad \text{and} \quad \mu_n=-1.913 \text{ nm}.$$

By using Eqs. (8) and (9), the magnetic moment for any shell-model configuration may be easily calculated.

If core excitation is neglected, the possible configurations for the three last neutrons in Fe^{57} are

$$[(p_{3/2}{}^2)_0 p_{1/2}]_{1/2}, \quad [(f_{5/2}{}^2)_0 p_{1/2}]_{1/2}, \quad [(p_{3/2}{}^2)_2 f_{5/2}]_{1/2},$$
$$[(f_{5/2}{}^2)_2 p_{3/2}]_{1/2}.$$

(Only particles in the same oscillator level are included in this discussion.)

Unfortunately, each of these configurations has a large positive magnetic moment when coupled to a proton group having zero spin so that there is no way to fit the very small value of the magnetic moment[19]

[15] G. Racah, Phys. Rev. **62**, 438 (1942).

[16] See also V. W. Hughes in *Recent Research in Molecular Beams*, edited by I. Estermann (Academic Press, New York, 1959), p. 69.

[17] See reference 14, p. 246.

[18] Contrary to common usage the g factors in this formula are given in units of the nuclear magneton. It turns out to be very convenient to express all values in terms of μ_n and μ_p, the magnetic moment of the neutron and proton, and this is the reason for this unusual choice of units for g factors.

[19] G. W. Ludwig and H. H. Woodbury, Phys. Rev. **117**, 1286 (1960).

TABLE III. The shell-model configurations used to describe the ground state of Fe^{57} and the corresponding contributions to the magnetic moment.

Coefficient	Protons	Neutrons	Magnetic moment
α	$(f_{7/2}{}^{-2})_0$	$(p_{3/2}{}^2)_0 p_{1/2}$	$-\frac{1}{3}\mu_n$
β	$(f_{7/2}{}^{-2})_0$	$(f_{5/2}{}^2)_0 p_{1/2}$	$-\frac{1}{3}\mu_n$
γ	$(f_{7/2}{}^{-2})_0$	$(p_{3/2}{}^2)_2 f_{5/2}$	$-(7/9)\mu_n$
δ	$(f_{7/2}{}^{-2})_0$	$(f_{5/2}{}^2)_2 p_{3/2}$	$-(13/21)\mu_n$
ϵ	$(f_{7/2}{}^{-2})_2$	$(p_{3/2}{}^3)_{3/2}$	$(2/7)(3+\mu_p)-\frac{1}{3}\mu_n$
η	$(f_{7/2}{}^{-2})_2$	$(p_{3/2}{}^2)_0 f_{5/2}$	$-(4/21)(3+\mu_p)-\frac{1}{3}\mu_n$
μ	$(f_{7/2}{}^{-2})_2$	$(f_{5/2}{}^3)_{5/2}$	$-(4/21)(3+\mu_p)-\frac{1}{3}\mu_n$
κ	$(f_{7/2}{}^{-2})_2$	$(f_{5/2}{}^2)_0 p_{3/2}$	$(2/7)(3+\mu_p)-\frac{1}{3}\mu_n$

by use of these states alone. This is no longer true if proton excitation is included. Table III gives a list of configurations for Fe^{57} and the magnetic moment associated with each. The Greek letter associated with each coefficient represents the amplitude of this part of the wave function. Only proton excitation to spin 2 with neutron states in the $p_{3/2}f_{5/2}$ subshell with lowest seniority are included. All the listed levels should be quite comparable in energy and represent a reasonable selection of shell-model states.

There are only two states in Table III that have a negative magnetic moment, so the final answer must contain a considerable fraction of these two states in order to form a wave function that will match the small observed magnetic moment. This may be seen by the formula for the magnetic moment. The magnetic moment operator for neutrons connects only states of the same l value so that there are no cross terms that contribute to the magnetic moment and the equation for this magnetic moment in terms of the amplitudes for the wave function from Table III is

$$[\tfrac{1}{3}(\alpha^2+\beta^2+\epsilon^2+\eta^2+\mu^2+\kappa^2)+(7/9)\gamma^2+(13/21)\delta^2](-\mu_n)$$
$$+[(2/7)(\epsilon^2+\kappa^2)-(4/21)(\eta^2+\mu^2)](\mu_p+3)$$
$$=+0.090 \text{ nm}=\mu(Fe^{57}),$$

where

$$-\mu_n\equiv+1.913 \text{ nm}, \quad (\mu_p+3)=5.792 \text{ nm}.$$

Because of the normalization of the wave function, this becomes

$$(\tfrac{1}{3}+(4/9)\gamma^2+(6/21)\delta^2)1.91$$
$$+[2(\epsilon^2+\kappa^2)-\tfrac{4}{3}(\eta^2+\mu^2)]0.826=+0.090.$$

Now the results of the $Fe^{57}(d,t)Fe^{56}$ experiment may be used to supply additional information about the composition of this wave function. The $\mathcal{S}$ values for $Fe^{57}(d,t)Fe^{56}$ are given in Table I. Of special interest is the transition to the ground state of Fe^{56}. This transition is a pure $p_{1/2}$ transition and immediately gives a measure of the amount of $p_{1/2}$ admixture in the Fe^{57} ground state.

The levels[20] in Fe^{56} at 0.840 Mev, 2.65 Mev, and 2.98 Mev all have a spin of 2 and thus cannot be reached by $p_{1/2}$ pickup. Therefore the total $\mathcal{S}(p_{1/2})$ contribution observed is found in the ground-state reaction.

[20] R. W. Bauer and M. Deutsch, Phys. Rev. **117**, 519 (1960).

In this analysis, no $f_{7/2}$ pickup is included. While it is perfectly possible that some of the $\sum \mathcal{S}(l=3)$ observed is contributed by picking up an $f_{7/2}$ neutron from the closed $f_{7/2}$ shell in Fe^{57}, there seem to be good theoretical reasons for excluding this possibility from the analysis. These are:

(1) The lowest group of core-excitation levels in Fe^{56} would be levels described by $(p_{3/2}{}^3)_{3/2}f_{7/2}{}^{-1}$, $(p_{3/2}{}^2)_0 f_{5/2}f_{7/2}{}^{-1}$, $(f_{5/2}{}^2)_0 p_{3/2}f_{7/2}{}^{-1}$, and $(f_{5/2}{}^3)_{5/2}f_{7/2}{}^{-1}$. The excitation energy of these levels would be expected to exceed the sum of the single-particle splitting between the $p_{3/2}$ and $f_{7/2}$ states (about 2 Mev) and the pairing energy necessary to break the $(f_{7/2}{}^8)_0$ core (more than 1 Mev). A level from this group might be the 4.10-Mev 4^+ level in Fe^{56}.

(2) The next group of core-excitation levels would be described by

$$(p_{3/2}{}^2)_2 f_{5/2}f_{7/2}{}^{-1},\quad (f_{5/2}{}^2)_2 p_{3/2}f_{7/2}{}^{-1},\quad \text{and}\quad (f_{5/2}{}^3)_J f_{7/2}{}^{-1}$$

(where $J=3/2$ or $9/2$), and perhaps proton core-excitation levels such as $(f_{7/2}{}^{-2})_2$ for protons and $(p_{3/2}{}^3)f_{7/2}{}^{-1}$ for neutrons. These levels are expected to be at even higher excitations than the first group and are most likely above 5 Mev.

(3) The (d,t) reaction from the $\frac{1}{2}^+$ ground state of Fe^{57} could go only to some of the levels in group 2 since adding an $f_{7/2}$ neutron to the states in group 1 will not give a spin of $\frac{1}{2}^+$ for Fe^{57}. With this in mind, it seems unlikely that any $f_{7/2}$ pickup is seen in this experiment.

By using the assumption of no core excitation, the sum rules for the different j values may be applied to the wave function in Table III. This results in the following relationships:

$$\alpha^2+\beta^2=\sum \mathcal{S}(p_{1/2}),$$

$$2\alpha^2+2\gamma^2+\delta^2+3\epsilon^2+2\eta^2+\kappa^2=\sum \mathcal{S}(p_{3/2}),$$

$$2\beta^2+\gamma^2+2\delta^2+\eta^2+3\mu^2+2\kappa^2=\sum \mathcal{S}(f_{5/2}),$$

where

$$\sum \mathcal{S}(p_{1/2})=0.17,\quad \sum \mathcal{S}(p_{3/2})=1.17,\quad \sum \mathcal{S}(f_{5/2})=1.47.$$

The total $\sum \mathcal{S}$ should equal 3 (the number of neutrons outside the closed shell of 28 neutrons). In this analysis this sum is 2.81, in close agreement with the value 3 predicted for this total. Therefore, only very small contributions to the various $\mathcal{S}$ sums have been missed in this experiment. To compensate for this, the values of $\sum \mathcal{S}$ are all increased by 7% for the actual calculations.

The equation for the magnetic moment and these three equations do not supply enough information to give a unique solution for this eight-component wave function. Two assumptions, however, do seem reasonable and these will be used in order to get a solution. These assumptions are:

(a) The ratio of $(p_{3/2}{}^2)_0 p_{1/2}$ to $(f_{5/2}{}^2)_0 p_{1/2}$ in the Fe^{57} ground-state wave function is the same as the ratio of $(p_{3/2}{}^2)_0$ to $(f_{5/2}{}^2)_0$ in the Fe^{56} ground-state wave function. With this assumption the contribution to the $\sum \mathcal{S}(p_{1/2})$ from the higher state of zero spin would not be observable.

(b) $\epsilon^2+\kappa^2=0$. This is assumed in order to minimize the two-proton excitation states in the wave function.

These two assumptions coupled with the normalization of the wave function, the expression for the magnetic moment, and the three relationships from the (d,t) results are sufficient to uniquely determine the magnitude of the six remaining coefficients. The values obtained for these coefficients are listed in Table IV.

The first excited state of Fe^{57} will now be examined because the measured transition probability for the transition to the ground state offers a check on the wave functions of the ground state. The magnetic moment of the excited state has recently been measured[21] and this gives information about the composition of the wave function for this level. The three simplest neutron configurations for this 3/2 level are $(p_{3/2}{}^3)_{3/2}$, $(f_{5/2}{}^2)_0 p_{3/2}$ and $(f_{5/2}{}^3)_{3/2}$, in which the protons are in the $(f_{7/2}{}^{-2})_0$ state. The contribution to the observed magnetic moment of $-(0.153\pm0.005)$ nm are μ_n, μ_n, and $-(3/7)\mu_n$, respectively, for these states. If the wave function for this state is given by

$$\psi_{3/2}=A(p_{3/2}{}^3)_{3/2}+B(f_{5/2}{}^3)_{3/2}+C(f_{5/2}{}^2)_0 p_{3/2},$$

then the equation for the magnetic moment is

$$[(A^2+C^2)-(3/7)B^2]\mu_n=\mu(Fe^{57*})=-0.153\ \bar{n}\bar{m},$$

thus

$$A^2+C^2-(3/7)B^2=0.080.$$

This equation coupled with the normalization leads immediately to $B^2=0.644$ and $A^2+C^2=0.356$.

The $M1$ γ-ray transition may now be investigated. The theoretical relationship for the reduced transition probability $B(M1)$ is determined in terms of the coefficients describing the two states involved. It is

$$B(M1)=[(2\mu_n)^2/4\pi](\alpha A+\beta C)^2.$$

The mean life of this 14.4-kev state has been measured to be 1.7×10^{-7} sec.[22] The conversion coefficient[22] has been measured to be about 15 so that the mean life τ for γ emission is about 27×10^{-7} sec. With this value

TABLE IV. Deduced values for the coefficients of these configurations. The use of three significant figures is a matter of arithmetical consistency and should not be taken to imply reliability to this accuracy.

$\alpha^2=0.078$	$\epsilon^2=0^a$
$\beta^2=0.102$	$\eta^2=0.364$
$\gamma^2=0.176$	$\mu^2=0.270$
$\delta^2=0.009$	$\kappa^2=0^a$

[a] Assumed to be equal to zero.

[21] S. S. Hanna, J. Heberle, C. Littlejohn, G. J. Perlow, R. S. Preston, and D. H. Vincent, Phys. Rev. Letters 4, 179 (1960).

[22] H. R. Lemmer, O. J. A. Segaert, and M. A. Grace, Proc. Phys. Soc. (London) A68, 701 (1955).

of τ, the term $(\alpha A+\beta C)^2$ in the expression for $B(M1)$ is about 0.006.

By use of the relationships $A^2+C^2=0.356$, $(\alpha A+\beta C)^2=0.006$, and the values of α^2 and β^2 from Table IV, values for A and C are obtained. If α and β are assumed to have the same sign, then there are two sets of solutions: $A=-0.308$, $C=+0.511$; and $A=+0.547$, $C=-0.236$. This leads to two sets of solutions for the 14-kev level: Set (1), $A^2=0.095$, $B^2=0.644$, and $C^2=0.261$; and Set (2), $A^2=0.300$, $B^2=0.644$, and $C^2=0.056$.

These results give a consistent picture of the $\frac{1}{2}^-$ and $\frac{3}{2}^-$ levels in Fe^{57} that fits all the observed experimental results. Results for $Fe^{56}(d,p)$ and $Fe^{57}(d,p)$ would provide independent checks on these results.

The quadrupole moment predicted by these coefficients is essentially zero since no proton excitation is involved. Hence, even small collective effects would contribute greatly to this quadrupole moment and therefore could be investigated by a measurement of the quadrupole moment of the $\frac{3}{2}^-$ state.

As is demonstrated above, the (d,t) experiment provided the major information in determining the wave functions for the $\frac{1}{2}^-$ and $\frac{3}{2}^-$ levels in Fe^{57}. This is a clear illustration of the power of this experimental method of determining nuclear properties.

V. ZINC-67

In $Zn^{67}(d,t)Zn^{66}$, the ground-state transition is pure $l=3$, which is consistent with the measured $\frac{5}{2}^-$ spin for Zn^{67}. The transitions to the higher levels show both $l=1$ and $l=3$. The values $\sum \mathcal{S}(l=3)=2.18$ and $\sum \mathcal{S}(l=1)=1.13$ for all the levels lead to the ratio

$$\sum \mathcal{S}(l=3)/\sum \mathcal{S}(l=1)=1.93.$$

The total of all the $\mathcal{S}$ values should equal the number of neutrons (9 in this case) in the $f_{5/2}$, $p_{3/2}$, and $p_{1/2}$ shells. The actual value is only 3.31. This may be due to the existence of large $\mathcal{S}$ values for levels with excitation energies above those in the region explored.

Because of this uncertainty, an analysis similar to that of Fe^{57} cannot be performed and only a few comments will be given concerning the data.

(a) The value of $\mathcal{S}(l=3)$ is 0.26 for the ground state. A pure $(f_{5/2}{}^5)_{5/2}$ level would have a value of $\frac{1}{3}$ for this quantity. Thus the results for the ground-state reaction seems to be quite reasonable.

(b) The magnetic moment for the ground state if $+0.874$ nm. A pure $(f_{5/2}{}^5)_{5/2}$ level has a magnetic moment of $-(5/7)\mu n=1.36$ nm. Thus the qualitative gross features of the ground state of Zn^{67} are understandable with a neutron wave function composed mainly of $(p_{3/2}{}^4)_0(f_{5/2}{}^5)_{5/2}$. Further experimental information is needed before a more detailed analysis is possible.

VI. CONCLUSIONS

The above work demonstrates the use of experimentally determined reduced widths from (d,t) reactions in analyzing the structure of the target nuclei. This is an especially sensitive method of detecting individual components of nuclear wave functions and indicates clearly the mixing of the $p_{3/2}$ and $f_{5/2}$ neutron shells in the ground states of the nuclei having more than 28 neutrons. This also clearly indicates the lack of core excitation in the ground states having a closed shell of 28 neutrons. A study of the situation with 22, 24, and 26 neutrons in the target nucleus would be most valuable in determining the amount of two-particle excitation in an even group of neutrons and thereby indicating the strength of the pairing force between pairs of neutrons coupled to give zero spin.

The analysis of the experimental information on Fe^{57} demonstrates how reduced widths may be used with other experimental data to provide nuclear wave functions that fit the observed results. In fact, with the use of two quite reasonable assumptions, a unique wave function was determined for the ground state of Fe^{57}. It would be most interesting to determine the nuclear interactions that could reproduce this wave function but this is beyond the scope of the present work.

This paper clearly indicates the usefulness of more detailed stripping and pickup experiments (both angular distributions and absolute cross sections) throughout the periodic table.

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