PHYSICAL REVIEW VOLUME 120, NUMBER 5 DECEMBER 1, 1960

High-Energy Pion-Nucleus Scattering*

Mirza A. Baqi Bég†
Department of Mathematical Physics, The University of Birmingham, Birmingham, England
(Received July 18, 1960)

It is suggested that an inconsistency in the customary treatment of Pauli principle effects is responsible for the discrepancy between the experimentally observed scattering cross sections of high-energy pions on nuclei and the values calculated from two-body amplitudes in the impulse approximation. The effect considered, and briefly discussed within the framework of a t-matrix theory, is the inverse eclipse effect noted by Glauber. Results are stated for the optical potential and the effective cross section and numerical values are quoted for negative pions on C^{12}.

I. INTRODUCTION

In discussing their data on high-energy pion-nucleus scattering, Cronin, Cool, and Abashian,[1] presented an optical-model analysis based on the following well-known expression for the optical potential (in suitable units)

$$V_0=(2\pi/E_\pi)\langle\rho f(0)\rangle. \tag{1.1}$$

E_π is the energy of the incident pion and $f(0)$ the forward pion-nucleon amplitude, both in the laboratory frame. Also, ρ is the nucleon density and the brackets denote an average over isobaric spin.[2] The real part of $f(0)$ was determined from the imaginary part by use of the ordinary dispersion relation and the density function used was, apart from a small difference accountable in terms of finite-range effects, consistent with the electron scattering data. These authors noted that while they could fit the absorption cross sections, the calculated diffraction cross sections were 20–30% smaller than the measured values.[1]

It is difficult to say how seriously one should treat this discrepancy; however, it does not seem entirely uninteresting to see if it can be resolved by using a more refined connection between the optical potential and the two-body scattering amplitudes than is afforded by Eq. (1). This is the primary motivation for the present paper. It is hoped that the mechanism we suggest will serve to draw attention to the physical consequences of a phenomenon noticed before but regarded more or less as a curiosity, namely, the anomalous action of the Pauli principle effect at high energies. By this we imply that the Pauli principle becomes responsible for an actual increase in the "effective" cross section per nucleon,[3] contrary to what one expects on usual phase-space considerations.[4] We mention at this point that the two effects considered in this paper, viz., (a) Pauli principle, (b) real absorption by pairs of nucleons, were in fact noted in reference 1. However, it was assumed that only the imaginary part of V_0 was affected and was *reduced* by the Pauli principle. The enhancement due to real absorption was thereby cancelled, the two effects being of roughly the same magnitude; hence, the use of Eq. (1) was considered justifiable.

2. THEORY OF THE PAULI PRINCIPLE EFFECT

The usual treatments follow the intuitive notions of Goldberger[4] visualizing the nucleus as a Fermi gas at zero temperature in which the incident particle scatters off each nucleon separately; the effect of other nucleons being simply to restrict the available phase space for the struck nucleon. Since no account is taken of double (or higher order multiple) scattering, the above picture is strictly valid only in the limit of extreme dilution—when the Pauli principle is in any case irrelevant.

Our purpose in this section is twofold; first, to outline a consistent description of the Pauli principle effect in which the role of double scattering is directly manifest and the connection with Goldberger's intuitive approach immediately established; second, to obtain an expression for the optical potential seen by high-energy negative pions. To these ends we use (with minor modifications) Watson's t-matrix formalism.[5] The advantages of this procedure are well known. While in this paper we simply take note of the approximation under which the results reduce to those obtainable in a semiclassical context,[3] inconsistencies arising from a fundamental limitation of the semiclassical formalism are nevertheless clearly underlined. Quantitative treatment of the corrections involved will be given in a paper on "off-shell scattering" in course of preparation.

An existing discussion[6] which in some respects comes close to the present one is, however, incomplete and contains some errors.

For any projectile incident on a target nucleus of mass number A, the optical potential (defined so as to

* Part of this work was done when the author was a guest of the Brookhaven National Laboratory.
† Present address: Brookhaven National Laboratory, Upton, New York.
[1] J. W. Cronin, R. Cool, and A. Abashian, Phys. Rev. **107**, 1121 (1957).
[2] See, for example, R. Frank, J. Gammel, and K. Watson, Phys. Rev. **101**, 891 (1956).
[3] R. J. Glauber, mimeographed notes of lectures delivered at the summer Institute for Theoretical Physics, University of Colorado, 1958 [*Lectures in Theoretical Physics* (Interscience Publishers, New York, 1958), Vol. 1, p. 315].
[4] See, for example, M. L. Goldberger, Phys. Rev. **74**, 1269 (1948). When the incident particle is not a nucleon, the relevant modification is given, for example, by R. M. Sternheimer, Phys. Rev. **106**, 1027 (1957).
[5] K. M. Watson, Phys. Rev. **105**, 1388 (1957), and earlier references cited therein.
[6] K. M. Watson, Revs. Modern Phys. **30**, 565 (1958).

simulate the elastic scattering) may be written as

$$V_0=-\langle 0|\theta|0\rangle, \tag{2.1}$$

where $|0\rangle$ is the nuclear ground state and, to order $1/A$,

$$\theta=\sum_{l=1}^{A} T_l+\sum_{l\neq m} T_lPGT_m+\cdots. \tag{2.2}$$

In this equation $G=(E_N+E_P+i\epsilon-\mathcal{H}_N-\mathcal{H}_P)^{-1}$, $\mathcal{H}_N$ the Hamiltonian for the target nucleus, $\mathcal{H}_P$ for the incident particle, E_N and E_P being the respective eigenvalues in the initial state. The interaction, $\mathcal{H}_{PN}$, is presumed to be a sum of the two-body interactions alone,

$$\mathcal{H}_{PN}=\sum_{l=1}^{A} v_l, \tag{2.3}$$

and the T_l are defined by

$$T_l=v_l(1-Gv_l)^{-1}. \tag{2.4}$$

P is a projection operator defined by

$$P=1-|0\rangle\langle 0|.$$

Equation (2.1) differs from the corresponding expression derived by Watson,[5] in that it involves only free propagators.[7] Its proof is straightforward and will not be considered here.

We wish to express θ in terms of two-body collision operators defined by

$$\begin{aligned} t_l(z_l)&=v_l(1-g_lv_l)^{-1},\\ g_l&=(z_l+E_P+i\epsilon-\mathcal{H}_P-\mathcal{H}_l)^{-1}. \end{aligned} \tag{2.5}$$

For an independent particle nucleus (i.e., containing Pauli principle correlations only) this can be accomplished *exactly* up to the second term in Eq. (2.2), provided one does not antisymmetrize the nuclear states in the spectral resolution of G.[8] Denoting the single-particle states by $|\alpha_j\rangle$, one gets after some routine algebra

$$\sum_l \langle 0|T_l|0\rangle=\sum_l \langle\alpha_l|t_l|\alpha_l\rangle, \tag{2.6}$$

$$\sum_{l\neq m} \langle 0|T_lPGT_m|0\rangle=-\sum_l \langle\alpha_l|t_lQ_lg_lt_l|\alpha_l\rangle, \tag{2.7}$$

where z_l implicit on the right-hand side is to be identified with $E(\alpha_l)$, the energy of the single-particle state $|\alpha_l\rangle$; Q_l is a projection operator annihilating $|\alpha_l\rangle$ as well as all *unoccupied* states. Equation (2.7) gives the Pauli principle correction to the optical potential which thus arises as a pure double scattering effect. It is evident that both the real and imaginary parts of V_0 are affected; furthermore, since $\mathrm{Im}V_0$ can be interpreted in terms of effective cross sections (Sec. 3), it is clear that these cross sections are not necessarily smaller than the free cross sections.

Consideration of the effective cross sections serves to clarify the limit in which Goldberger's picture is consistent and exactly valid. For the effective cross section in this picture is

$$\bar{\sigma}=\frac{2\pi}{vA}\sum_{\mathbf{k}}\sum_{\beta,\alpha}|\langle\beta,\mathbf{k}|t(\alpha)|\alpha,\mathbf{k}_0\rangle|^2 \times\delta(E(\alpha)-E(\beta)+E_P(\mathbf{k}_0)-E_P(\mathbf{k})), \tag{2.8}$$

where $\mathbf{k}_0$ and v are the momentum and the velocity of the incident particle and α, β enumerate the occupied and unoccupied nucleon states, respectively. It is easy to verify that if $|f\rangle$ is any nuclear state, (2.8) is identical to

$$\bar{\sigma}=\frac{2\pi}{vA}\sum_{\mathbf{k}}\sum_f|\langle f,\mathbf{k}|\sum_l T_l|0,\mathbf{k}_0\rangle|^2 \times\delta(E_f-E_0+E_P(\mathbf{k})-E_P(\mathbf{k}_0)), \tag{2.9}$$

and thus, as one would naturally expect, gives the cross section in the impulse approximation. The total nuclear cross section $\bar{\sigma}A$ can, however, be obtained directly, and with proper cognizance of multiple scattering from the total collision operator $\sum_l T_l+\sum_{l\neq m} T_lGT_m+\cdots$ by using the optical theorem.

Equation (2.9) and hence (2.8) are then seen to be correct (to second order in T matrices) only if $T_l=T_l^{\dagger}$. This condition is exactly satisfied only in the Born approximation.[9]

To go back to our discussion of the optical potential, we note that if the de Broglie wavelength of the incident particle is small compared to the free path between successive scatterings, we may put each scattering on the energy shell[6,10] and write, to second order in the t matrices,

$$-V_0=\sum_l\langle\alpha_l|t_l|\alpha_l\rangle +i\pi\sum_l\langle\alpha_l|t_lQ_l\delta(E_l+E_P-\mathcal{H}_l-\mathcal{H}_P)t_l|\alpha_l\rangle. \tag{2.10}$$

Explicit evaluation of (2.10) is easy for a Fermi gas model of the nucleus. Without further ado we quote the result for the case of negative pions,

$$V_0=\frac{2\pi}{E_\pi}(\rho_pf_1+\rho_nf_2)-i\frac{2\pi}{kE_\pi}\frac{1}{5}\left[\rho_pK_{Fp}^2\left\{f_1^2+\frac{5K_{Fn}^2-K_{Fp}^2}{4K_{Fp}^2}f_3^2\right\}+\rho_nK_{Fn}^2f_2^2\right]+i\theta_a, \tag{2.11}$$

where ρ_p and ρ_n are the proton and neutron densities; K_{Fp} and K_{Fn}, the Fermi momenta, respectively. The quantities f_1, f_2, and f_3 are the forward amplitudes for

[7] In analogy to the nomenclature in perturbation theory one may refer to Eq. (2.1) as the Rayleigh-Schrödinger expansion for the optical potential; Watson's result being then the Brillouin-Wigner expansion.

[8] Of course we have the freedom to define G either in terms of a complete set of projection operators or the restricted set which defines the space of antisymmetric eigenfunctions. See W. Tobocman, Phys. Rev. **107**, 203 (1957).

[9] See K. A. Brueckner, Phys. Rev. **89**, 834 (1953).

[10] The intermediate state is visualized as long-lived compared to a period of the incident wave (but short-lived compared to a period of the bound nucleon).

the processes $\pi^-(p,p)\pi^-$, $\pi^-(n,n)\pi^-$, and $\pi^-(p,n)\pi^0$, respectively. These amplitudes (as well as k and E_π, the momentum and energy of the incident pion) are in the laboratory frame.

The third term added on is the enhancement in the imaginary part of V_0 due to true absorption by nucleon pairs ($p-n$ or $p-p$). In the semiphenomenological model of Brueckner, Serber, and Watson,[11] θ_a is proportional to σ_a, the absorption cross section in deuterium. We may write it in the form

$$\theta_a=(2\pi/E_\pi)\rho_p(\Gamma k\sigma_a/4\pi), \tag{2.12}$$

the proportionality factor Γ being inferred as in reference 1.

Similar results for ΔV_{Pp}, the Pauli principle correction to V_0, are quoted by Watson[12] and Watson and Zemach.[13] These results do not agree in detail with ours; they lead to $\Delta V_{\mathrm{Pp}}\sim\langle f(0)\rangle^2$ and hence, as is obvious from an inspection of (2.10), are not properly averaged over isobaric spin. Correct evaluation leads, of course, to a (weighted) average of the form $\langle f(0)^2\rangle$ and thus for projectiles such as pions (or antinucleons) the result, while independent of any direct interference between f_1 and f_2, involves the charge-exchange amplitude as well. In the limit $f_1\rightarrow f_2$, the result we quote for ΔV_{Pp} still disagrees, in numerical factors, with those quoted in references 6 and 13. This disagreement is, however, trivial; it is due to approximate evaluations, by these authors, of the correlation integral R_c (as defined in reference 13).

It has been indicated before that the effect of Pauli principle correlations on the optical potential was first studied by Glauber on semiclassical considerations.[3]

The multiple scattering nature of the effect is somewhat concealed in Glauber's deduction. The result derived, is for the simple case of a nonrelativistic projectile, devoid of spin and isobaric spin; our result, in this limit, is in explicit agreement.

Apart from the central approximation $k\gg k_F$ ($k_F=K_{Fp}$ or K_{Fn}), which permits us to define the scattering amplitude at energy E_π, the derivation of Eq. (2.11) entails the additional approximation $k_Fa\ll1$, a being the range of the pion-nucleon interaction. It is this approximation which permits us to express ΔV_{Pp} in terms of forward amplitudes alone; furthermore, if $k_Fa\sim1$, the concept of "free path" loses its meaning and "off-shell scattering" can no longer be neglected. What we wish to emphasize is that in the absence of an unambiguous procedure for calculating the off-shell contribution, it is not possible to obtain a consistent generalization of the correlation correction to the optical potential for arbitrary values of k_Fa.[14]

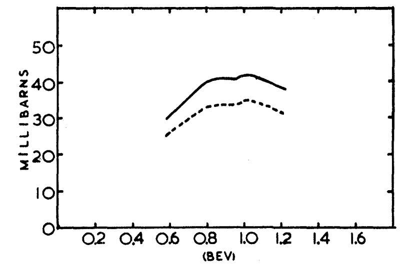

FIG. 1. Total cross sections for pion-nucleon scattering averaged over isobaric spin. The dashed curve shows the free $\bar{\sigma}$ inferred from the measurements of Cool, Piccioni, and Clark; the continuous curve shows the effective $\bar{\sigma}$ in carbon.

We add a few words about the applicability of Eq. (2.11) to the realistic case of a finite nonuniform nucleus. For the first term the appropriate generalization is well known; the result remains unchanged except that ρ_p and ρ_n are now functions of position. However, the dependence on only forward scattering does not arise naturally as in the case of nuclear matter, it arises on separate incorporation of the physical hypothesis of predominantly forward scattering. The terms thereby neglected are of order a^2/R^2, R being the nuclear radius. The generalized form of the second term is quite tedious, for purposes of explicit evaluations; however, it seems a good approximation to use it as it stands; it varies as $\rho^{5/3}$, hence the predominant contribution comes from regions of high density where the characteristics of nuclear matter most closely obtain.

3. EFFECTIVE CROSS SECTIONS

We define f_1' and f_2', the effective forward amplitudes for pion-nucleon scattering in the nucleus, by writing Eq. (2.11) as

$$V_0=(2\pi/E_\pi)[\rho_pf_1'+\rho_nf_2'], \tag{3.1}$$

whence

$$f_1'=f_1-i\frac{K_{Fp}^2}{5k}\left[f_1^2+\frac{5K_{Fn}^2-K_{Fp}^2}{4K_{Fp}^2}f_3^2\right]+i\Gamma\frac{k\sigma_a}{4\pi}, \tag{3.2}$$

$$f_2'=f_2-i\frac{K_{Fn}^2}{5k}f_2^2. \tag{3.3}$$

The effective cross section per nucleon is then defined by

$$\sigma_{i\mathrm{eff}}=(4\pi/k)\,\mathrm{Im}f_i';\; i=1,\,2. \tag{3.4}$$

In the high-energy region where the imaginary parts of the forward amplitude are greater than the real parts, the Pauli principle will enhance the cross sections; the effective (π^-,p) cross section is thus substantially larger than the free cross section. Average effective cross sections (for negative pions in C^{12}) are plotted in Fig. 1.

[11] K. A. Brueckner, R. Serber, and K. M. Watson, Phys. Rev. 84, 258 (1951).

[12] Equation (8) of reference 6 cited above. The correlation correction to the cross section given in Eq. (9) of this paper is, however, identical with that in Eq. (2.9) and thus of dubious value for studying the pair correlation function at high energies.

[13] K. M. Watson and C. Zemach, Nuovo cimento 10, 452 (1958).

[14] Semiclassical treatment of finite range [Eq. (268) of reference 3] suffers from just this defect even though it is claimed that the treatment is valid for arbitrary ratios of a to the correlation length ($\sim1/k_F$). The difficulty is pronounced for antinucleons.

TABLE I. Parameters and cross sections for 0.97-Bev pions on carbon. The letters in parentheses indicate the density distribution (modified Gaussian and tapered Fermi).

Quantity	Computed value using free amplitudes	Computed value using effective amplitudes	Experiment
ζ	0.10	0.08	...
$\bar{\sigma}$ (mb)	34	41	...
σ_A (mb)	232 (MG)	256 (MG)	252±13
	244 (TF)	...	
σ_D (mb)	67 (MG)	84 (MG)	78±21
	61 (TF)	...	

4. ABSORPTION AND DIFFRACTION CROSS SECTIONS OF π^- ON C^{12}

For calculational purposes we rewrite (2.11) in the form

$$V_0=(k/2E_\pi)\rho(\xi)(\zeta+i)\bar{\sigma}. \tag{4.1}$$

ζ is, of course, the ratio of the real to imaginary part and $\bar{\sigma}$ is the average effective cross section defined in the previous section. The appropriate formulas connecting these parameters with σ_A and σ_D are quoted, for example, in reference 1 and need not be reproduced here.

To determine ζ and $\bar{\sigma}$ we require first the amplitudes f_1 and f_2. The imaginary parts of these amplitudes are direct observables; the real parts have been calculated by Sternheimer.[15] We used the values given by Cool, Piccioni, and Clark.[16] Secondly, we need to know the Fermi momentum $K_F=K_{Fp}=K_{Fn}$. This was evaluated in a uniform model in which the effective nuclear radius can be expressed as $R=r_0A^{\frac{1}{3}}$ fermi, so that

$$K_F=(9\pi/8)^{\frac{1}{3}}(1/r_0)\,(\text{fermi})^{-1}. \tag{4.2}$$

Thirdly, we require the values of Γ and σ_a. In accordance with the value quoted in reference 1, we took $\Gamma=10$. The cross section σ_a can be deduced by detailed balancing from that for the production reaction $p+p\rightarrow\pi^++d$. The data upon this reaction are still quite crude. From the work of Block *et al.* (3-Bev protons), Cronin *et al.*[1] deduced $\sigma_a\cong 0\cdot 5$ mb for 1.37-Bev pions. From the recent work of Batson *et al.*[17] (0.97-Bev protons), we find $\sigma_a=(0.44\pm0.22)$ mb for 0.32-Bev pions. Ignoring any possible resonances, we took $\sigma_a=0.5$ mb over the entire energy range (0.6 to 1.2 Bev).

Finally, the density distribution chosen[18] was

$$\rho(\xi)=\rho_0(1+\tfrac{4}{3}\xi^2/a^2)\exp(-\xi^2/a^2),\quad \rho_0=1/3(a\sqrt{\pi})^3, \tag{4.3}$$

with the radial parameter $a=1.732$ f, this value being 6% larger than the value $a=1.635$ f for electron scattering. [This takes some account of the finite range of the pion-nucleon interaction (*vide supra*).] The corresponding r_0 is 1.41 f.[18]

We have summarized in Table I the values of the parameters ζ and $\bar{\sigma}$ and the calculated and observed σ_A and σ_D at 970 Mev. For purposes of comparison and for gauging the importance of the correction terms in (2.11), we also quote the values of these entities without benefit of these corrections. Since we chose a different density distribution, the computed values of Cronin *et al.* are also listed. While the extremely neat agreement at this particular energy is fortuitous, it is obvious from an inspection of Fig. 2 that the agreement in the entire energy range of (0.6 to 1.2 Bev) can be described as satisfactory. The diffraction cross sections are ~30% larger than the computed values of Cronin *et al.* (Table II).

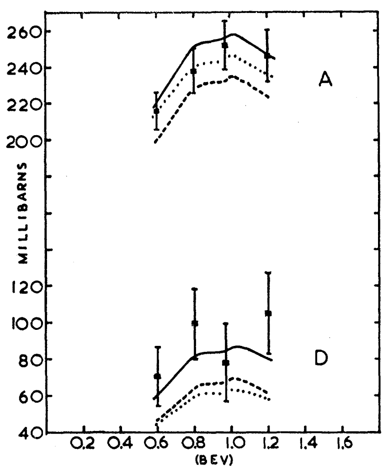

FIG. 2. Absorption and diffraction cross sections of negative pions on carbon. The experimental points are those of Cronin, Cool, and Abashian and the dotted curves indicate their theoretical analysis. The dashed curves are the results of an analysis using a modified Gaussian density distribution but using the same values of ζ and $\bar{\sigma}$; while in the continuous curves the effective values, as defined in the text, are used.

TABLE II. Diffraction cross sections of pions on carbon at various energies.

Kinetic energy in lab frame (Mev)	Computed Cronin *et al.* (mb)	Computed present work (mb)	Experiment (mb)
600	45	60	70±16
800	60	81	99±19
970	61	84	78±21
1000	...	87	...
1200	59	79	105±22

ACKNOWLEDGMENTS

It is a pleasure to acknowledge my debt to Professor G. E. Brown for a number of stimulating and instructive discussions and for bringing the work of Glauber to my attention. I am indebted to Dr. C. J. Batty for machine programming of the computations of Sec. 4. Finally, I wish to thank Professor R. E. Peierls for his kind interest in this work, perusal of the manuscript, and some useful suggestions.

[15] R. M. Sternheimer, Phys. Rev. **101**, 384 (1956).
[16] R. Cool, O. Piccioni, and D. Clark, Phys. Rev. **103**, 1082 (1956).
[17] A. P. Batson *et al.*, Proc. Roy. Soc. (London) **A251**, 218 (1959).
[18] J. H. Fregeau, Phys. Rev. **104**, 225 (1956).