General Spin-Wave Dispersion Relations*

R. F. Soohoo
Lincoln Laboratory,† Massachusetts Institute of Technology, Lexington, Massachusetts
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The general spin-wave dispersion relation obtained by a simultaneous solution of the equation of motion of the magnetization and Maxwell's equations is given. In contrast to previous calculations, the effects of conductivity, relaxation, exchange, and propagation are all properly taken into account. The resulting algebraic equation, being biquadratic in the square of the wave number, k^2, has four possible pairs of solutions. Some of these solutions correspond to growing plane waves while others represent attenuated ones in the direction of propagation. Whereas an analytical solution for k could be easily obtained for the special case where the wave vector $\mathbf{k}$ is in the direction of the static magnetic field ($\theta_k=0$), the solution for the cases where $\theta_k\neq 0$ could be conveniently obtained only by numerical solution. The solutions for the latter cases have been obtained by using an IBM 709 computer and some of the representative results are given in this paper in graphical form. When relaxation and eddy current damping are neglected, our result reduces to that of Herring and Kittel in the static limit ($\omega\rightarrow 0$). Furthermore, it was found that the uniform precessional mode ($k=0$) can truly exist only under very special conditions, namely, under the condition of zero permeability for one of the two normal modes in a gyromagnetic medium.

I. INTRODUCTION

We shall derive the general dispersion relation for spin waves by a simultaneous solution of Maxwell's equations and the equation of motion of the magnetization with conductivity, relaxation, exchange, and propagation properly considered. The solutions for the real and imaginary components of the magnitude of the spin-wave vector k, obtained by numerical solution will be given in graphical form. A brief account of this work has previously been given elsewhere.[1]

Herring and Kittel,[2] neglecting relaxation damping, derived the dispersion relation for spin waves in an insulator. They assumed that the wavelengths of such spin waves are much smaller than the electromagnetic wavelength in the medium. For thermal spin waves, this condition is easily satisfied even at and above microwave frequencies. However, the situation is quite different in a conductor and for spin waves whose wavelengths are substantially longer than the thermal ones. The wavelengths of the spin waves of interest in a metal (e.g., those responsible for spin-wave resonance in thin Permalloy films) may be comparable to sample dimensions, skin depth, and the electromagnetic wavelength in the medium. Under these conditions, the spin-wave dispersion relation can be determined only by a simultaneous solution of Maxwell's equations and the equation of motion of the magnetization by taking proper account of the effects of conductivity, exchange, propagation, and relaxation. The dispersion relation obtained in this way would then be applicable to ferromagnets of any conductivity and for spin waves of any wavelength. The assumptions under which the special cases reported in the literature are valid can then be more easily understood.

Inasmuch as the magnetization and electromagnetic field distribution within a medium can, in principle at least, be explained in terms of plane waves, we shall examine in this note the plane wave solutions and shall only briefly discuss their application to the solutions of actual boundary value problems.

II. MAXWELL'S EQUATIONS

We shall start our derivation of the general spin-wave dispersion relation with a determination of the dipolar fields associated with the interaction of spins from Maxwell's equations. In Gaussian units, Maxwell's equations are

$$\nabla\times\mathbf{h}=\frac{\epsilon}{c}\frac{\partial\mathbf{e}}{\partial t}+\frac{4\pi\sigma}{c}\mathbf{e}, \tag{1}$$

$$\nabla\times\mathbf{e}=-\frac{1}{c}\frac{\partial\mathbf{h}}{\partial t}-\frac{4\pi}{c}\frac{\partial\mathbf{m}}{\partial t}, \tag{2}$$

where ϵ and σ are, respectively, the dielectric constant and the conductivity of the medium.

Taking the curl of Eq. (1) and substituting the resulting expression for $\nabla\times\mathbf{e}$ into Eq. (2), we obtain

$$\nabla^2\mathbf{h}-\frac{1}{c^2}\left(\epsilon\frac{\partial}{\partial t}+4\pi\sigma\right)\frac{\partial\mathbf{h}}{\partial t}=\frac{4\pi}{c^2}\left(\epsilon\frac{\partial}{\partial t}+4\pi\sigma\right)\frac{\partial\mathbf{m}}{\partial t}-4\pi\nabla(\nabla\cdot\mathbf{m}), \tag{3}$$

where we have made use of the constitutive Maxwell's equation $\nabla\cdot\mathbf{b}=\nabla\cdot(\mathbf{h}+4\pi\mathbf{m})=0$. Assuming that

$$\mathbf{h}=\mathbf{h}_0e^{i\omega t}+\sum_{k\neq 0}\mathbf{h}_ke^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}, \tag{4}$$

$$\mathbf{m}=\mathbf{m}_0e^{i\omega t}+\sum_{k\neq 0}\mathbf{m}_ke^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}, \tag{5}$$

* After this work was completed, the author became aware of a Letter by M. A. Gintsburg, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1047 (1958) [translation: Soviet Phys.-JETP **35**(8), 730 (1959)], in which he treated the special case of spin waves in an insulator including displacement currents.

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[1] R. F. Soohoo, Bull. Am. Phys. Soc. **5**, 356 (1960).

[2] C. Herring and C. Kittel, Phys. Rev. **81**, 869 (1951).

we find from Eq. (3) that for the spatial nonvarying part ($k=0$),

$$\mathbf{h}_0 = -4\pi\mathbf{m}_0. \tag{6}$$

For the spatial varying part ($k\neq 0$), we find from Eq. (3) that

$$\mathbf{h}_k = \frac{4\pi(\omega^2\epsilon_e/c^2)\mathbf{m}_k - 4\pi\mathbf{k}(\mathbf{k}\cdot\mathbf{m}_k)}{k^2-\omega^2\epsilon_e/c^2}, \tag{7}$$

where $\epsilon_e=\epsilon(1+4\pi\sigma/i\omega\epsilon)$ is the equivalent dielectric constant of the material with finite conductivity.

Equation (7) gives the general relationship between $\mathbf{h}_k$ and $\mathbf{m}_k$, and it can be combined with the equation of the magnetization to obtain the dispersion relation for the spin waves. We note that if $k^2\cos\phi \gg \omega^2\epsilon_e/c^2$ where ϕ is the angle between $\mathbf{m}$ and $\mathbf{k}$, then Eq. (7) reduces to $\mathbf{h}_k \simeq -4\pi\mathbf{k}(\mathbf{k}\cdot\mathbf{m})/k^2$, the often used expression for $\mathbf{h}_k$ in the so-called static approximation. It is worth noting, however, that if $\mathbf{k}\cdot\mathbf{m}=0$, then $\mathbf{h}_k$ is properly given by $4\pi\omega^2\epsilon_e/k^2c^2$ in the static approximation.

Returning now to Eq. (6), we find that the $k=0$ spin wave could exist if and only if $\mathbf{m}_0/\mathbf{h}_0=-1/4\pi$. Under this condition, the flux density $\mathbf{b}_0=\mathbf{h}_0+4\pi\mathbf{m}_0$ inside the medium is zero because $\mathbf{h}_0$ and $4\pi\mathbf{m}_0$ are equal in magnitude but oppositely directed.

III. EQUATION OF MOTION

A phenomenological equation of motion of the magnetization is:

$$\frac{d\mathbf{M}}{dt} = \gamma\mathbf{M}\times\left(\mathbf{H}+\frac{\alpha}{\gamma M_0}\frac{d\mathbf{M}}{dt}+\frac{2A}{M_0{}^2}\nabla^2\mathbf{M}\right), \tag{8}$$

where γ, α, and A are, respectively, the gyromagnetic ratio, the damping and exchange constants. Equation (8) is a modified form[3] of the equation first proposed by Landau and Lifshitz.[4] Several other equations[5–7] have been proposed for ferromagnetic resonance. Equation (8), however, displays the relaxation damping (α term) in the simplest form in our final results. Consequently, for simplicity, we have chosen to use Eq. (8) instead of any of the other equations.

For the spatial independent part, $\nabla^2\mathbf{M}=0$, so that if we assume that:

$$\mathbf{H}=\mathbf{l}_zH_0+\mathbf{h}_0e^{i\omega t},$$
$$\mathbf{M}=\mathbf{l}_zM_0+\mathbf{m}_0e^{i\omega t}. \tag{9}$$

Equation (8) in component form becomes

$$\begin{aligned}(i\omega/\gamma)m_{0x} &= (H_0-i\omega\alpha/\gamma)m_{0y}-M_0h_{0y},\\ (i\omega/\gamma)m_{0y} &= -(H_0-i\omega\alpha/\gamma)m_{0x}+M_0h_{0x},\end{aligned} \tag{10}$$

where we have assumed that $h_0 \ll H_0$ and $m_0 \ll M_0$ and H_0 and M_0 are the static field and static magnetization, respectively. Combining Eqs. (6) and (10), we obtain two algebraic equations for m_{0x} and m_{0y}. Setting the determinant of this equation equal to zero, we find the dispersion relation for the $k=0$ spin wave as

$$\omega_0{}^2=\gamma^2(H_0+4\pi M_0-i\omega\alpha/\gamma)^2. \tag{11}$$

Since ω_0 is in general real, Eq. (11) can be satisfied only if $\alpha=0$.

Additional insight may be gained by solving Eq. (10) directly for the ratio $\mathbf{m}_0/\mathbf{h}_0$ without invoking Eq. (6). In that case, we obtain the well-known Polder tensor $\boldsymbol{\chi}$ given by[8]

$$\boldsymbol{\chi}=\begin{vmatrix}\chi & -iK & 0\\ iK & \chi & 0\\ 0 & 0 & 0\end{vmatrix}, \tag{12}$$

where

$$\chi=\frac{\gamma^2(H_0-i\omega\alpha/\gamma)M_0}{\gamma^2(H_0-i\omega\alpha/\gamma)^2-\omega^2},$$

$$K=\frac{\gamma M_0\omega}{\gamma^2(H_0-i\omega\alpha/\gamma)^2-\omega^2}. \tag{13}$$

Diagonalizing tensor (12), we find the eigenvalues $\chi+K$ and $\chi-K$ corresponding to the susceptibilities of the two counter rotating circularly polarized modes. Whereas $\chi-K$ is always positive, $\chi+K$ is equal to $-1/4\pi$ at the frequency given by Eq. (11). Since the equivalent susceptibility $\chi+K$ is equal to the ratio $\mathbf{m}_0/\mathbf{h}_0$ for this normal mode, we see that condition (6) as imposed by Maxwell's equations is satisfied for this particular situation. However, it is interesting to note that at no other combination of ω and H_0 could the requirement of tensor (12) as imposed by the equation of motion (8) be compatible with the condition (6) derived from Maxwell's equations.

From the foregoing analysis, we conclude that if the frequency is finite, the uniform precessional mode ($k=0$) can exist only under very special conditions, namely, under the condition where the equivalent susceptibility is equal to $-1/4\pi$, i.e., at zero equivalent permeability. However, if $\omega\to 0$, the problem reduces to that of magnetostatics as far as Maxwell's equations are concerned. By combining the Laplace equation $\nabla^2\psi=0$ where ψ is a scalar magnetic potential and the equation of motion (8) neglecting relaxation and exchange, Walker[9] obtained the magnetostatic modes of an ellipsoid. He also found that the uniform mode of Kittel,[10] being related to demagnetization factors derived from entirely static considerations, is one of the normal modes of the ellipsoid.

If the permeability is other than zero, it appears that the wavelength of $\mathbf{h}$ and $\mathbf{m}$ are finite at a finite frequency

[3] T. A. Gilbert, Armour Research Institute (unpublished reports).
[4] L. Landau and E. Lifshitz, Physik Z. Sowjetunion **8**, 153 (1935).
[5] N. Bloembergen and S. Wang, Phys. Rev. **93**, 72 (1953).
[6] H. B. Callen, J. Phys. Chem. Solids **4**, 256 (1958).
[7] R. C. Fletcher, R. C. LeCraw, and E. G. Spencer, Phys. Rev. **117**, 955 (1960).
[8] D. Polder, Phil. Mag. **40**, 99–115 (1949).
[9] R. L. Walker, Phys. Rev. **105**, 390 (1957).
[10] C. Kittel, Phys. Rev. **73**, 155 (1948).

so that strictly speaking no uniform distribution of **h** and **m** within a finite sample is possible. The spatial distribution of the magnetization may then be expanded in terms of the various plane spin waves with arbitrary coefficients to be determined by the electromagnetic and exchange boundary conditions. Thus, their dispersion relation to be derived in the next section would be of general interest.

IV. SPIN WAVES

Combining Eqs. (7) and (8), we obtain two linear algebraic equations in the x and y components of **m**:

$$\begin{aligned}(i\omega+C_d\gamma M_0k_xk_y)m_x &\\ +(C_d\gamma M_0k_y^2-C_e\gamma M_0-\gamma H_0+i\alpha\omega)m_y&=0,\\ -(C_d\gamma M_0k_x^2-C_e\gamma M_0-\gamma H_0+i\alpha\omega)m_x &\\ +(i\omega-C_d\gamma M_0k_xk_y)m_y&=0,\end{aligned} \tag{14}$$

where

$$C_d=\frac{i2\pi\delta'^2}{1-i\frac{1}{2}\delta'^2k^2},$$

$$C_e=\frac{4\pi}{1-i\frac{1}{2}\delta'^2k^2}+\frac{2A}{M_0^2}k^2,$$

and $\delta'^2=-i2c^2/\omega^2\epsilon_e$ is a generalized skin depth of the medium. Setting the determinant of Eq. (14) equal to zero, we obtain an equation biquadratic in k^2:

$$(K^2)^4+A(K^2)^3+B(K^2)^2+CK^2+D=0, \tag{15}$$

where

$$A=2\eta'+\sin^2\theta_k+i4\xi^2,$$
$$B=\eta'(\eta'+\sin^2\theta_k)-\Omega^2+i4\xi^2(2\eta'+1+\tfrac{1}{2}\sin^2\theta_k)-4\xi^4,$$
$$C=i4\xi^2[\eta'(1+\tfrac{1}{2}\sin^2\theta_k)+\tfrac{1}{2}\sin^2\theta_k-\Omega^2+\eta'^2]-8\xi^4(\eta'+1),$$
$$D=4\xi^4[\Omega^2-(\eta'+1)^2],$$

and K, η', ξ^2, and Ω are dimensionless parameters given by

$$K=k\xi\delta,$$
$$\eta'=(H_0/4\pi M_0)i-\alpha\Omega,$$
$$\xi^2=A/2\pi M_0^2\delta'^2,$$
$$\Omega=\omega/\gamma4\pi M_0.$$

In the static approximation, as mentioned previously, the expression $-4\pi\mathbf{k}(\mathbf{k}\cdot\mathbf{m})/k^2$ instead of that given by Eq. (7) is used for $\mathbf{h}_k$. Correspondingly, $C_d\rightarrow-4\pi/k^2$ and $C_e\rightarrow 2Ak^2/M_0$. Within the framework of these approximations, Eq. (15) reduces to the Herring-Kittel formula which is quadratic in k^2. Thus, the other two pairs of possible solutions have been implicitly discarded in such an approximate calculation.

If $\theta_k\neq0$, the solution of Eq. (15) is quite complicated. We should like to first investigate two special cases, i.e., $\theta_k=0$ and $\theta_k=90°$. First consider the solution of Eq. (15) when $\theta_k=0$. In this case, it can be factored into two terms, each quadratic in K^2 having the solutions:

$$K_{1,2}^2=\tfrac{1}{2}\{\Omega-\eta'-i2\xi^2\pm[(\Omega-\eta'-i2\xi^2)^2+i8\xi^2(\Omega-\eta'-1)]^{\frac{1}{2}}\},$$

and

$$K_{3,4}^2=\tfrac{1}{2}\{-(\Omega+\eta'+i2\xi^2)\pm[(\Omega+\eta'+i2\xi^2)^2-i8\xi^2(\Omega+\eta'+1)]^{\frac{1}{2}}\}. \tag{16}$$

There are several interesting results to be noted in Eq. (16). First, if $\Omega=\eta'+1$, $K_2^2=0$ while $K_{1,3,4}^2$ remain finite. On the other hand, if $\Omega=-(\eta'+1)$, $K_3^2=0$ and $K_{1,2,4}^2$ remain finite. These conditions are exactly those required by Eq. (11) to give a uniform mode ($k=0$).

Secondly, we note that for Ω negative K_2 is always real while K_1 is imaginary but never complex for a perfect insulator ($\alpha\rightarrow0$, $\delta'^2\rightarrow-i2c^2/\omega^2\epsilon$). On the other hand, $K_{3,4}$ could both be real or imaginary also for an insulator. The characteristics of these K's are largely dependent upon the value of $\Omega+\eta'$ and $\Omega+\eta'+1$. These imaginary K's do not imply a violation of the law of conservation of energy as they merely represent reactive attenuation analogous to that of waves beyond cutoff in waveguiding structures.

In general, we find that if either the direction of the dc field or the sense of rotation of the circular polarization of the exciting radiation **h** is reversed, we obtain the complex conjugates of the original K's.

In the special case where $\theta=90°$, Eq. (15) becomes

$$\begin{aligned}(K^2+2i\xi^2)\{K^6+(2\eta'+1+i2\xi^2)K^4&\\ +[\eta'(\eta'+1)-\Omega^2+i4\xi^2(\eta'+1)]K^2&\\ -i2\xi^2[\Omega^2-(\eta'+1)^2]\}&=0.\end{aligned} \tag{17}$$

We see from Eq. (17) that one of the solutions for K^2 is $-2i\xi^2$. Equation (17) agrees with the results of Rado[11] except that in his case the Landau-Lifshitz[4] instead of the Gilbert[3] damping term was used.

V. NUMERICAL SOLUTION

In Figs. 1–3 and 4–6, we have plotted the real and imaginary components of the magnitude of K ($K'-iK''$), i.e., K' and K'' as a function of frequency for an insulator whose $\epsilon=12$ and $\alpha=0.05$ and for a conductor whose resistivity $1/\sigma=21\times10^{-6}$ ohm-cm and $\alpha=0$. The exchange constant was assumed to be equal to 10^{-6} erg/cm for both cases. For simplicity, we have assumed that α for the insulator case to be a constant independent of frequency. Curves for three different θ_k's are shown. It is seen from Figs. 4–6, that K_1'' is negative representing growing waves while $K_{2,3,4}''$ are in general positive representing decaying waves. We have shown here only plus K''s and associated K'''s. The positive K''s correspond to waves travelling in the positive **r** direction. Thus, there are a total of eight waves, four each travelling in opposite directions.

[11] G. T. Rado, Phys. Rev. **97**, 1559 (1955).

We notice that except for K_4', there are substantial differences between the K's of the insulator and conductor cases. Furthermore, we note from Figs. 4–6 that the curves are all well behaved except in the vicinity of $-\Omega=\eta'+1$ whereby $K_3\rightarrow 0$ as required by Eq. (11) and at $-\Omega=\eta'$.

VI. APPLICATIONS

Any magnetic distribution, in principle at least, could be represented as a linear combination of plane spin waves with arbitrary coefficients. The various coefficients can in turn be evaluated by imposing the appropriate electromagnetic and exchange boundary conditions. In practice, however, this procedure is not always easy to carry out. Fortunately, in the case of spin-wave resonance in Permalloy films, a problem of considerable interest, the required linear combination

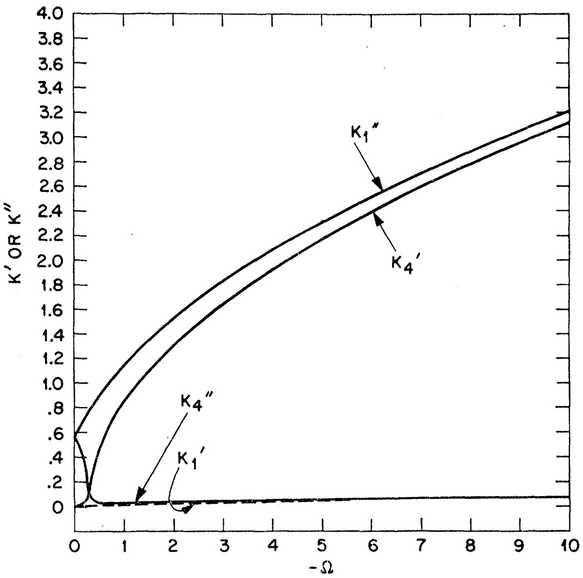

FIG. 1. Normalized components of **k** vs normalized frequency for an insulator with $\theta_k=0$ and $\xi=0$. $\eta=0.3$.

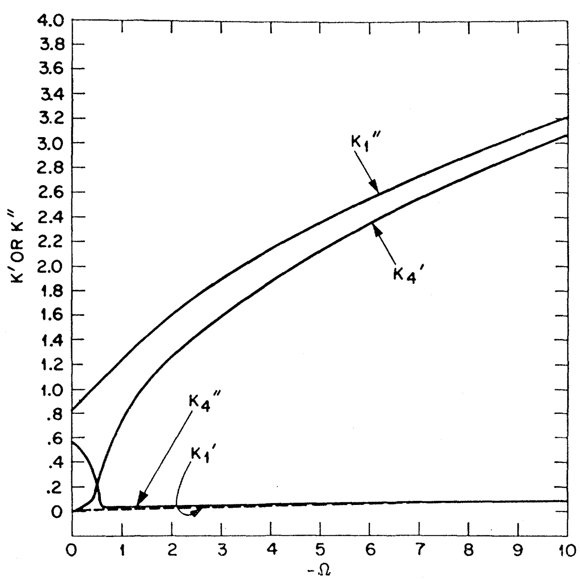

FIG. 2. Normalized components of **k** vs normalized frequency for an insulator with $\theta_k=40°$ and $\xi=0$. $\eta-0.3$.

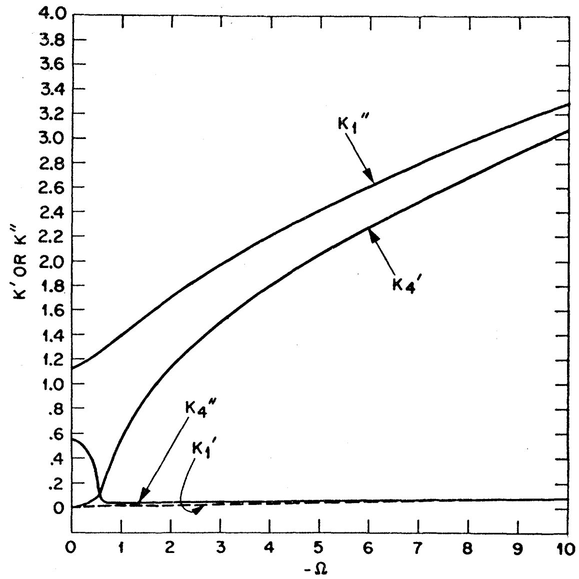

FIG. 3. Normalized components of **k** vs normalized frequency for an insulator with $\theta_k=90°$ and $\xi=0$. $\eta=0.3$.

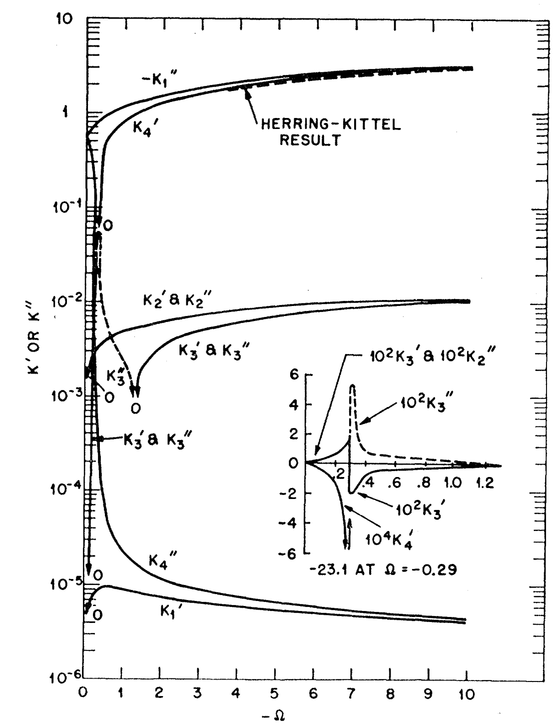

FIG. 4. Normalized components of **k** vs normalized frequency for a conductor with $\theta=0°$. $\eta=0.3$.

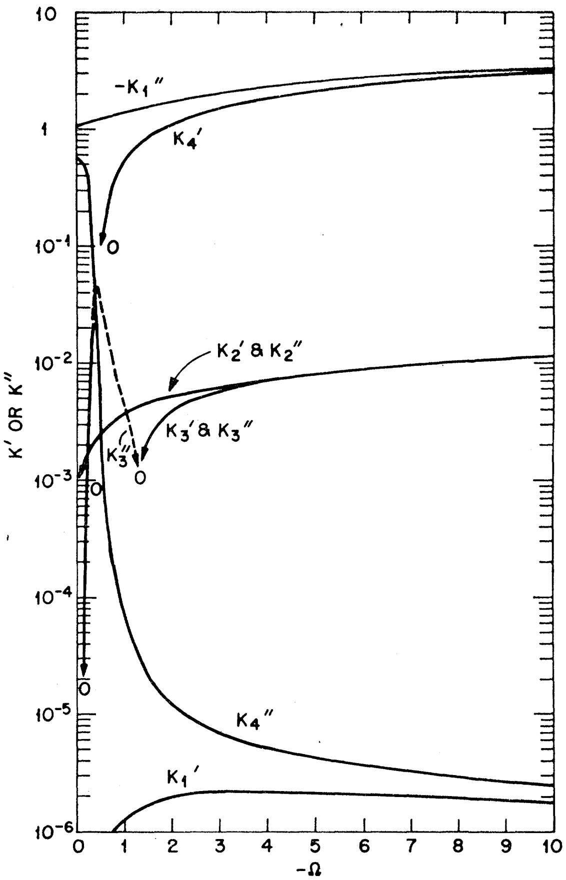

FIG. 5. Normalized components of **k** vs normalized frequency for a conductor with $\theta=40°$. $\eta=0.3$.

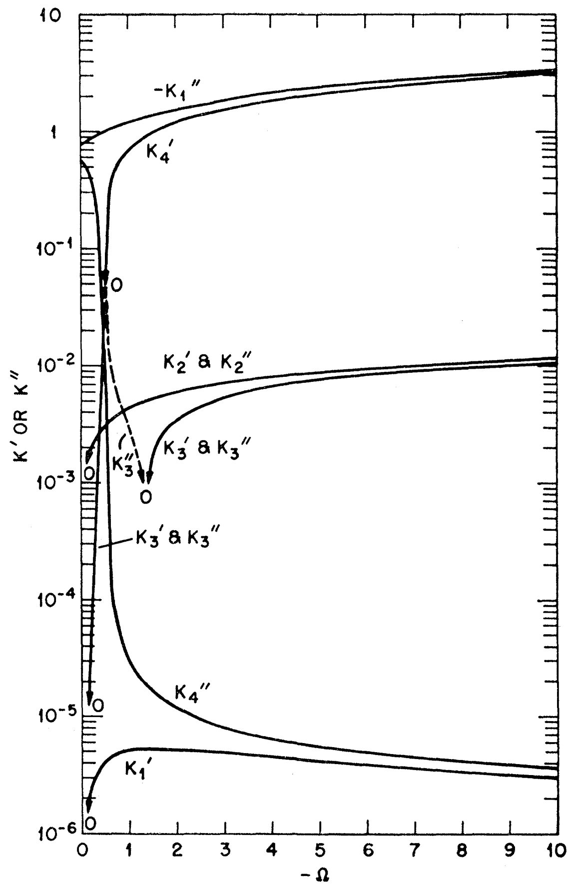

FIG. 6. Normalized components of **k** vs normalized frequency for a conductor with $\theta=90°$. $\eta=0.3$.

of plane spin waves can easily be found. Seavey and Tannenwald[12] and Pincus[13] have obtained solutions for the z-directed spin waves in Permalloy films while the author[14] has applied this method to obtain the general absorption spectrum for spin-wave propagation in different directions in Permalloy films.

VII. CONCLUDING REMARKS

In summary, we have derived the general spin-wave dispersion relations by properly taking into account the effects of conductivity, exchange, and propagation. In contrast to the Herring-Kittel spin-wave dispersion relation which is quadratic in k^2, our dispersion relation is biquadratic in k^2. The four new solutions did not appear in their calculation because they use the static approximation for the dipolar field $\mathbf{h}_k$. In Fig. 4, we have shown the $K'-\Omega$ curve as calculated from the Herring-Kittel relation for comparison. It is seen that this curve is quite similar to those corresponding to K_4'. We note also that $K_3 \to 0$ only in the vicinity of $-\Omega=+(\eta'+1)$; the $K=0$ mode corresponds to the well-known uniform precession. The results of this paper may be applied to the determination of magnetization distribution in a ferromagnetic ellipsoid of any size and conductivity.

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