Considerations on the Propagation and Generation of Magnetostatic Waves and Spin Waves

P. C. FLETCHER AND C. KITTEL*
Hughes Research Laboratories, Malibu, California
(Received August 3, 1960)

A discussion is given of the dispersion relation, magnetization distribution, and group velocity of magnetostatic waves in an infinite circular cylinder with the static magnetic field parallel to the cylinder axis. The dispersion relation of the modes with $e^{i\varphi}$ angular dependence is, for $kR\gg 1$,

$$\omega\cong\gamma H_0+\gamma 2\pi M_s(x_i/kR)^2+(D/\hbar)k^2,$$

where x_i is a root of $J_0(x)=0$; R is the cylinder radius; and D is the exchange constant. The group velocity of magnetostatic pulses at low wave vectors is shown to be considerably higher than magnon velocities.

IN view of the long magnetic relaxation times[1,2] in yttrium iron garnet, it is realistic to contemplate experiments in which spin waves are generated electromagnetically[3] or acoustically,[4] and their propagation characteristics studied in a single crystal rod of yttrium iron garnet. In this note we made several points which we believe are of central importance to the problem.

It is obvious that the applied static magnetic field H_0 must be chosen low enough so that the spin wave dispersion relation $\omega=\omega(k)$ will have solutions for which the wave vector k is largely real. The dispersion relation for a long thin circular cylinder with H_0 parallel to the axis is of the form,

$$\omega=\gamma H_0+F(k)+i\tau^{-1}, \tag{1}$$

for waves propagating parallel to the axis z of the cylinder; we write k for k_z. The function $F(k)$ will be positive real for real k.

The low-order magnetostatic modes have been discussed by Fletcher.[5] The quantity τ is the relaxation time for the relevant spin wave, and we assume that $\omega\tau\gg 1$. It is evident that

$$\gamma H_0<\omega \tag{2}$$

is a criterion for (1) to have a solution with k nearly entirely real. The inequality (2) is satisfied by the resonance condition $\gamma H_0=\omega-\gamma 2\pi M_s$ for the uniform mode, but there appears to be no special virtue in driving the system at this particular condition if our object is to excite spin waves. If we violate the inequality (2), the wave vector rapidly acquires a catastrophic imaginary component. We assume further that the wavelengths of interest are sufficiently short that in the propagation equations we may neglect coupling to the electromagnetic field—the criterion for the validity of this assumption will be examined later.

We now consider the function $F(k)$ in (1). For sufficiently high values of k the function is dominated by the exchange energy and approaches $\omega_{\mathrm{ex}}a^2k^2$, where a is the lattice constant. It has not always been recognized, however, that there is a wide region of k in which *magnetostatic waves* of quite simple form may propagate. For our geometry the Walker[6] equation for the magnetostatic modes has solutions for the magnetic scalar potential of the form, in cylindrical coordinates,

$$\psi_{\mathrm{in}}=A_{\mathrm{in}}J_n[ik\rho/(1+\kappa)^{\frac{1}{2}}]e^{ikz}e^{in\varphi}; \tag{3}$$

$$\psi_{\mathrm{out}}\propto H_n^{(1)}(ik\rho)e^{ikz}e^{in\varphi}; \tag{4}$$

where $H_n^{(1)}$ is a Hankel function; here

$$\kappa=\frac{4\pi M_sH_0}{H_0^2-(\omega/\gamma)^2}. \tag{5}$$

In the short-wavelength limit $k\rho\gg 1$ we may approximate the Hankel function by

$$H_n^{(1)}(ik\rho)\propto(k\rho)^{-\frac{1}{2}}e^{-k\rho}, \tag{6}$$

so that

$$\psi_{\mathrm{out}}\cong A_{\mathrm{out}}(k\rho)^{-\frac{1}{2}}e^{-k\rho}e^{in\varphi}e^{ikz}. \tag{7}$$

The boundary condition on the continuity of the tangential component of $\mathbf{H}$ at the surface $\rho=R$ is satisfied provided

$$A_{\mathrm{in}}J_n[ikR/(1+\kappa)^{\frac{1}{2}}]=A_{\mathrm{out}}(kR)^{-\frac{1}{2}}e^{-kR}. \tag{8}$$

The boundary condition on the normal component of $\mathbf{B}$ is

$$(1+\kappa)(\partial\psi_{\mathrm{in}}/\partial\rho)-(i\nu/R)(\partial\psi_{\mathrm{in}}/\partial\varphi)=\partial\psi_{\mathrm{out}}/\partial\rho, \tag{9}$$

* Permanent address: Department of Physics, University of California, Berkeley 4, California.

[1] E. G. Spencer and R. C. LeCraw, Phys. Rev. Letters 4, 130 (1960).

[2] M. Sparks and C. Kittel, Phys. Rev. Letters **4**, 232 (1960).

[3] E. Schlömann, Raytheon memorandum T-217 (unpublished).

[4] E. A. Turov and Yu P. Irkin, Phys. Metals Research **3**, 15 (1960); C. Kittel, Phys. Rev. **110**, 836 (1958); H. Bömmel and K. Dransfeld, Bull. Am. Phys. Soc. **5**, 58 (1960).

[5] P. C. Fletcher and E. H. Gregory (to be published); their results refer to prolate spheroids with high axial ratios.

[6] L. R. Walker, Phys. Rev. **105**, 390 (1957); R. Damon and J. Eshbach, J. Appl. Phys. **31**, 104S (1960) have solved the Walker equation for a flat plate and find wave-like solutions. The frequency of waves propagating along the field direction (taken in the plane of the plate) for a plate of thickness $2L$ is $\omega\cong\omega_0+2\pi\gamma M_s(n\pi/kL)^2$, for $kL\gg 1$, where n is a positive integer.

where

$$\nu=\frac{4\pi M_s(\omega/\gamma)}{H_0^2-(\omega/\gamma)^2}. \tag{10}$$

From (9) we have, for $kR\gg 1$,

$$A_{\text{in}}\{(1+\kappa)[ik/(1+\kappa)^{\frac{1}{2}}]J_n'[ikR/(1+\kappa)^{\frac{1}{2}}] + (n\nu/R)J_n[ikR/(1+\kappa)^{\frac{1}{2}}]\} = -A_{\text{out}}(k/R)^{\frac{1}{2}}e^{-kR}. \tag{11}$$

Combining (8) and (11),

$$i(1+\kappa)^{\frac{1}{2}}(J_n'/J_n)=-1-(n\nu/kR). \tag{12}$$

This is the characteristic equation, where the argument of the Bessel function is $ikR/(1+\kappa)^{\frac{1}{2}}$; our solution is just the magnetostatic limit of a problem treated by Kales.[7]

For excitation by an interaction which is uniform across the specimen we may set $n=1$. We set

$$(\omega-\gamma H_0)/4\pi\gamma M_s=\epsilon. \tag{13}$$

For $kR\gg 1$, it is seen that $\epsilon\ll 1$; then

$$\kappa\cong\nu\cong-1/2\epsilon, \tag{14}$$

and (12) becomes, for $n=1$, with $x=(2\epsilon)^{\frac{1}{2}}kR$,

$$J_1'(x)/J_1(x)=-1/x, \tag{15}$$

which may be reduced to

$$J_0(x)=0. \tag{16}$$

The three lowest roots are $x_i=2.405$; 5.520; and 8.654; the lowest corresponding eigenfrequency is

$$\omega_1\cong\gamma H_0+2\pi\gamma M_s(2.405/kR)^2. \tag{17}$$

The exchange energy may be taken approximately into account by adding to (17) a term $(D/\hbar)k^2$; this neglects the exchange energy associated with the radial and angular variation of the magnetization and is justified so long as the axial (z) variation is the shortest wavelength in the problem. Here D is the exchange constant. Thus

$$\omega_1\cong\gamma H_0+2\pi\gamma M_s(2.405/kR)^2+(D/\hbar)k^2. \tag{17a}$$

The group velocity of modes described by (17) is

$$v_g=\partial\omega_1/\partial k=-4\pi M_s(2.405/R)^2k^{-3}. \tag{18}$$

For yttrium iron garnet at helium temperatures and $R=0.1$ cm, we have

$$|v_g|\cong 2.5\times 10^{13}k^{-3}\text{ cm/sec}, \tag{19}$$

Thus for $kR=10$ and 100, we have $|v_g|\cong 2.5\times 10^7$ and 2.5×10^4 cm/sec. We note also from (17) that

$$\omega_1-\gamma H_0\cong 1.3\times 10^{11}/(kR)^2\text{ sec}^{-1}; \tag{20}$$

our approximations probably restrict the usefulness of (20) to $kR>5$, or $\omega_1-\gamma H_0<5\times 10^9$ sec^{-1}. This may be compared with $2\pi\gamma M_s=2.2\times 10^9$ sec^{-1}. For the limit $H_0\rightarrow 0$, see Appendix A.

The group velocity in the wave vector region where the exchange interaction is dominant is, for z-directed spin waves,

$$v_g\cong 0.1k\text{ cm/sec}, \tag{21}$$

where the constant is evaluated approximately for YIG at 0°K. The ratio of the magnetostatic wave velocity (19) to the exchange wave velocity (21) is (for $R=0.1$ cm)

$$|v_{\text{mag}}|/|v_{\text{ex}}|\cong 2.5\times 10^{14}/k^4,$$

which is equal to unity for $k=4\times 10^3$ cm^{-1}. This value of k may be considered as the demarcation line between exchange and magnetostatic modes.

We must now find a criterion for the neglect of the interaction of the electromagnetic field with our magnetostatic waves. From the Maxwell equations,

$$c^2k_p^2H^+=\epsilon\omega^2(H^++4\pi M^+), \tag{22}$$

where the Walker equation gives, for $k_sR\rightarrow k_mR$,

$$4\pi\chi^+=(\kappa-\nu)\cong\frac{\omega_s}{2\omega_0+Ck_m^{-2}}, \tag{23}$$

with $\chi^+=M^+/H^+$;

$$\omega_s=4\pi\gamma M_s,\quad\text{and}\quad C=2\pi\gamma M_s(2.405/R)^2.$$

The photon and magnon wave vectors are k_p and k_m, respectively. Under the usual conditions $4\pi\chi^+$ will be $\lesssim 1$, so that the electromagnetic wave vector k_p should be approximately by $\epsilon^{\frac{1}{2}}\omega=ck_p$, where ϵ is the dielectric constant. This neglects waveguide effects, which in any event only decrease k_p. So long as the wave vector k_m of the magnetostatic mode is very different from k_p, the coupling of the two dispersion relations should not be serious. Usually $ck_m\gg\epsilon^{\frac{1}{2}}\omega$. We can see that this implies weak coupling: in the Maxwell equation,

$$\text{curl}\mathbf{E}=-\partial\mathbf{B}/c\partial t, \tag{24}$$

the left side will involve k_sE; if $k_s/k_m\rightarrow k_m/k_p$, the value of E accompanying the magnetostatic wave will be roughly in the ratio k_p/k_m with respect to the value of E for the electromagnetic mode. The radiative part of H may thus be reduced in the ratio $(k_p/k_m)^2$ with respect to the magnetostatic part of H. The exact solutions of the radiative and magnetostatic problem as given by Kales[7] involve precisely this ratio.

We now consider briefly the magnitude of the coupling by an excitation field into the modes identified by the successive roots of $J_0(x)=0$, as in Eq. (16). We shall not enter into the details of any particular coupling process, but shall use

$$\mathcal{G}=\int_0^R m_\rho\rho d\rho\bigg/\int_0^R m_\rho^2\rho d\rho, \tag{25}$$

as a crude measure of the relative strength of the coupling into the several modes.

[7] J. Kales, J. Appl. Phys. 24, 604 (1953).

Now

$$4\pi m_\rho=\kappa(\partial\psi_{in}/\partial\rho)-(i\nu/\rho)(\partial\psi_{in}/\partial\varphi)$$

$$\propto\frac{ik\kappa}{(1+\kappa)^{\frac{1}{2}}}J_1'(x)+\frac{\nu}{\rho}J_1(x),\quad(26)$$

where we have dropped the factor $e^{i\varphi}e^{ikz}$; we suppose that the excitation has an appropriate angular dependence to give a nonzero value on angular integration with $e^{i\varphi}$, and that a similar requirement is satisfied by the z dependence. Using (13) and assuming $\epsilon\ll1$, we have, with $x=i\kappa\rho/(1+\kappa)^{\frac{1}{2}}$,

$$m_\rho\propto\frac{ik\kappa}{(1+\kappa)^{\frac{1}{2}}}[J_1'(x)+x^{-1}J_1(x)(\nu/\kappa)]$$

$$\cong\frac{ik\kappa}{(1+\kappa)^{\frac{1}{2}}}J_0(x).\quad(27)$$

We note that for the mode identified by the root x_i of (16),

$$\mathcal{G}\propto1/x_i^2J_1(x_i).\quad(28)$$

For the first four roots, the relative values of $|\mathcal{G}|$ are 1, 0.3, 0.14, and 0.10. Thus the first root dominates the excitation, but other modes should perhaps be observable. We note from (27) that $m_\rho\approx0$ at the surface, so that modes of this form will be scattered very little by surface imperfections. There are similar modes with low surface magnetization in a flat plate with $\mathbf{k}\|\mathbf{H}_0$ and in the plane of the plate.

For low k the velocity of the magnetostatic waves may exceed appreciably the maximum magnon velocity for frequencies in the microwave range—at K band the magnon velocity is about 2×10^5 cm/sec in YIG if all the energy is exchange energy. We have above made the estimate of 2.5×10^7 cm/sec for the magnetostatic mode with $kR=10$, and $R=0.1$ cm. Our derivation involves approximations which break down at low values of kR, but the exact solutions[5] for prolate spheroids with high axial ratios suggest that the limiting velocities of low-order magnetostatic modes may be of the order of

$$|v_g|\approx|\Delta\omega/\Delta k|\approx\gamma M_sL,\quad(29)$$

where L is the length of the specimen. For $L=1$ cm, $|v_g|\approx4\times10^8$ cm/sec. In this situation it is essential to take account of electromagnetic propagation effects, but we see that very rapid transmission of magnetostatic pulses may be expected to occur. This conclusion is in agreement with some preliminary experiments[8] on magnetic pulse propagation in a ferrite cylinder.

The mean-free path Λ of a wave is given by

$$\Lambda\approx v_g\tau,\quad(30)$$

[8] P. C. Fletcher (unpublished).

so that if $\tau=10^{-7}$ sec, the amplitudes of waves having $v_g=10^9$, 10^7, and 10^5 cm/sec will be reduced by e^{-1} in distances of 100, 1, and 0.01 cm, respectively. Thus the low k magnetostatic waves propagate best.

ACKNOWLEDGMENTS

We wish to thank R. L. White and F. Keffer for helpful discussions.

APPENDIX A: MAGNETOSTATIC WAVES IN ZERO APPLIED FIELD

We consider the same infinite circular cylinder as above, but let the static magnetic field approach zero while still maintaining the saturated state of the specimen. Then $\kappa=0$, and we have (for $n=1$)

$$\psi_{\rm in}=A_{\rm in}J_1(ik\rho)e^{ikz}e^{i\varphi}.\quad(A.1)$$

Now $J_1(ix)=I_1(x)$, which for $x\gg1$ approaches $(2\pi x)^{-\frac{1}{2}}e^x$, so that $J_1'(ix)\cong-iI_1'(x)$ and the charac-

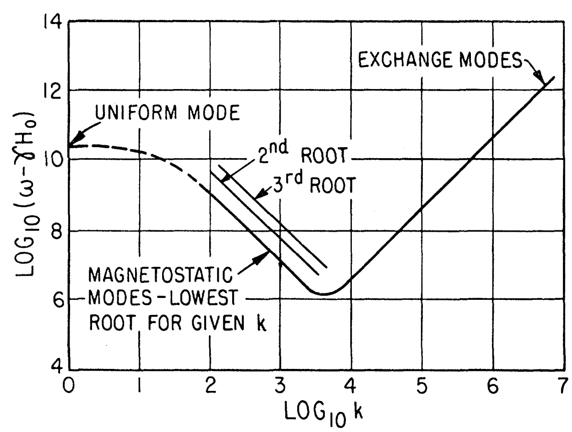

FIG. 1. Plot of $\log_{10}(\omega-\omega_0)$ vs $\log_{10}k$ for z-directed spin waves, $n=1$, in a circular cylinder of radius 0.1 cm, with static field and cylinder axis parallel to the z axis. Material constants as for YIG at 0°K; here $\omega_0=\gamma H_0$; ω, ω_0 in sec^{-1}; k in cm^{-1}. It is assumed that $(\omega-\omega_0)/\omega\ll1$.

teristic equation, Eq. (12) becomes

$$\nu=-2kR,\quad(A.2)$$

or

$$\omega=2\pi\gamma M_s/kR.\quad(A.3)$$

This is rather different in form from the earlier result applicable to the region $\nu\cong\kappa$.

The present result is not hard to understand qualitatively, because the magnetostatic energy of a cylinder divided into disks of thickness k^{-1} magnetized alternately in opposite directions (and normal to the axis) is of the order of M_s^2/kR per unit volume, for $kR\gg1$. The distribution of magnetization is actually peaked at the surface, whereas in the earlier problem the distribution had a node at the surface.