A second experimental difficulty is that scattered particle currents are extremely small and difficult to measure at large angles, especially in the light ion-atom combinations. There is much scatter in the data and this might possibly explain the diverse slopes and discontinuities of all the combinations of Fig. 5.

b. Extrapolation Errors

In Sec. 3(a) above an extrapolation was necessary because of the lack of differential cross section data beyond the largest angle measured. Besides the reasonable and smooth extrapolation used to extend the curve of Fig. 1(b), two extreme and less reasonable extrapolations were investigated.

(1) It was assumed that the differential cross section did not decrease below the last measured value and was constant out to 180°. Since $\sigma(\theta)$ is a decreasing function of angle, this grossly overestimates its value. When this assumption is followed through it causes the left end of the $rV(r)$ curves to turn up as shown for the 25-kev Ar^+ on Ar case in the curve labeled A in Fig. 4(a).

(2) The second assumption is that $\sigma(\theta)$ drops exponentially with angle as shown by the straight line labeled B on Fig. 1(a) joining smoothly to the data. Since the measured data have an upward curvature on this semi-log plot this assumption is somewhat of an underestimation of $\sigma(\theta)$ and the effect is to slightly lower the left end of the $rV(r)$ curves. This is shown for the 25-kev Ar^+ on Ar case by the curve labeled B in Fig. 4(a).

The other extrapolation, made necessary as discussed in Sec. 3(b) by the lack of data between 0° and 1°, was investigated in a similar manner, taking extreme assumptions and determining their effect on the final $rV(r)$ curves. These assumptions had only very small effects on the right end of these curves.

PHYSICAL REVIEW VOLUME 120, NUMBER 6 DECEMBER 15, 1960

Coulomb Energies of Closed-Shell Nuclei from Shell-Model Wave Functions*

N. V. V. J. SWAMY AND V. K. KEMBHAVI
Karnatak University, Dharwar, India

AND

D. G. GALGALI
Institute of Science, Bombay, India

(Received June 1, 1960; revised manuscript received August 11, 1960)

Using harmonic oscillator wave functions, the Slater integrals have been evaluated and expressed in the form of summation formulas. The Coulomb energies of seven closed-shell nuclei are estimated using these integrals. These estimates are compared with those based on a statistical model and a trapezoidal model. The influence of exchange energy on the Coulomb radius of a nucleus is shown to be sensitive to the model used. Although the $Z^{4/3}$ variation of exchange energy appears to be a suitable characterization, present estimates require a larger multiplying constant in the usually accepted expression for the exchange energy. The direct and "net" energies, computed from the three models, show very good agreement.

A STUDY has been made by Swamy and Green[1] of the Coulomb exchange energies of light nuclei, wherein it was noticed that estimates with shell-model wave functions do not agree with those of Bethe and Bacher[2] based on the statistical model of the nucleus. This study of Coulomb energies is now extended to heavier nuclei up to $Z=70$. In order, however, to facilitate a close comparison of the direct energies with the results of a uniform spherical distribution of charge, the present calculations are restricted to closed-shell nuclei. The Slater integrals[3] have been evaluated using harmonic oscillator wave functions and the results are given in the Appendix. The direct and exchange energies are linear functions of the oscillator parameter α. These, inclusive of self-energies, have been calculated for seven nuclei and are shown in Table I.

If a phase-shift analysis were made with the oscillator model to fit electron scattering, an exact experimental oscillator constant could have been available which would uniquely determine the nuclear radius. In our qualitative study we have, however, chosen the oscillator parameter to satisfy the "equivalent uniform radius" criterion

$$[5/3\,\langle r^2\rangle]^{\frac{1}{2}}=r_0A^{\frac{1}{3}}=R, \tag{1}$$

where $\langle r^2\rangle^{\frac{1}{2}}$ is the rms radius of the charge distribution computed with harmonic oscillator wave functions, and r_0 is the familiar radius constant. In the case of ${}_{20}Ca^{40}$ we have assumed r_0 to be equal to 1.22 in order to secure

* This work was supported by the Department of Atomic Energy, Government of India.

[1] N. V. V. J. Swamy and A. E. S. Green, Phys. Rev. **112**, 1719 (1958).

[2] H. A. Bethe and R. F. Bacher, Revs. Modern Phys. **8**, 162 (1936).

[3] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1951), p. 176.

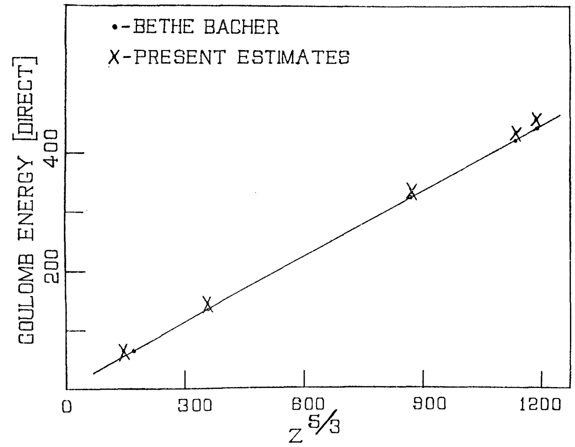

FIG. 1. Coulomb energy (direct) in units of $e^2/1$ fermi vs $Z^{5/3}$. The estimates of the trapezoidal model (Cameron) are too close to ours to be shown in this figure.

agreement with the "net" Coulomb energy (direct $-|\text{exchange}|$) calculated on the basis of electron scattering data.[4] The results obtained, using arbitrary values of r_0 for the other nuclei, are shown in Table II and the direct and exchange energies are plotted in Figs. 1 and 2, respectively. For the purpose of comparison with the present oscillator model, estimates based on the statistical model and a trapezoidal model[5] are also shown in the tables and figures. The comparisons are, of course, independent of the choice of the oscillator constant since a change in this causes only a scale magnification of all the comparative estimates.

DISCUSSION

The Coulomb exchange energy, arising from correlations in the positions of protons required by the Pauli principle, is a matter of considerable interest. In light nuclei it rather strongly affects a fit of nuclear masses to a semiempirical mass formula.[6] As a first estimate of this energy we have the supermultiplet calculations of Feenberg and Phillips[7] who have shown that the exchange energy turns out to be about 6–18% of the direct energy in the ground-state configurations of light nuclei. Later, the influence of the exchange energy on the Coulomb radii of such nuclei has been studied in detail.[8] In the case of mirror nuclei, the difference in the binding energies of the isobars is given by the difference in Coulomb energies of the two nuclei on the assumption of charge independence of nuclear forces. Since this binding energy difference is obtained experimentally, a theoretical estimate of the Coulomb energy difference fixes the radius of the charge distribution inside such nuclei. Using a statistical model Bethe and Bacher[2] had given the following expression for the Coulomb exchange energy of a spherical nucleus of radius R:

$$0.46(e^2/R)Z^{4/3}. \qquad (2)$$

Inclusion of this exchange energy in the Coulomb energy difference of a mirror pair decreases the radius, computed without it, by a Z-dependent factor[9] $(1-0.51Z^{-2/3})$. The trend of decrease of mirror nuclear radii with increasing mass number has been investigated by Carlson and Talmi[8] who have included pairing effects in their estimates based on the harmonic oscillator model. They find that r_0 falls off gradually from 1.34 fermis at $A=13$ to about 1.23 at $A=25-29$. According to their calculations the exchange energy reduces, for instance, the radius of the $O^{17}-F^{17}$ pair by about 13%. It is important to note, however, that the extent of influence of the exchange energy on the nuclear radius is sensitive to the model used. Thus for the same mirror pair, the percentage reduction computed with a square well model[10]

TABLE I. Coulomb energies of nuclei in units of $e^2\alpha$.

Nucleus	Direct energy	Exchange energy	Percentage decrease in Coulomb radius as a result of inclusion of exchange energy: Present estimates	Statistical model	Trapezoidal model
$_{18}A^{40}$	94.25	12.64	15.5	12.6	8.5
$_{20}Ca^{40}$	114.97	14.21	14.1	11.6	7.8
$_{34}Se^{80}$	294.56	24.70	9.2	7.9	5.3
$_{40}Zr^{90}$	399.73	29.59	8.0	7.0	4.7
$_{58}Ce^{140}$	774.88	44.05	6.0	5.4	3.6
$_{68}Er^{168}$	1043.34	52.98	5.3	4.8	3.3
$_{70}Yb^{174}$	1105.94	54.98	5.2	4.7	3.2

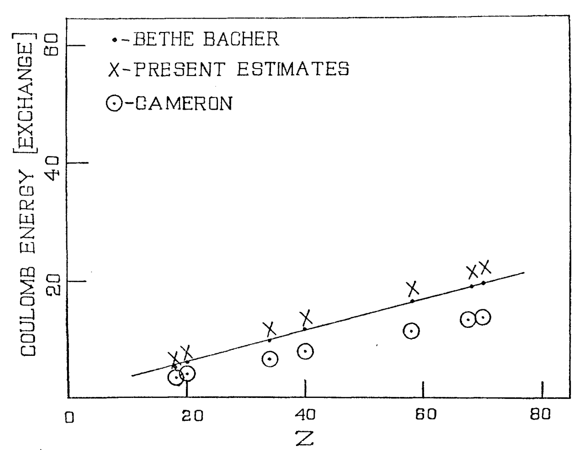

FIG. 2. Coulomb energy (exchange) in units of $e^2/1$ fermi vs Z.

[4] B. Hahn, D. Ravenhall, and R. Hofstadter, Phys. Rev. **101**, 1131 (1956).
[5] A. G. W. Cameron, Can. J. Phys. **35**, 1021 (1957).
[6] A. E. S. Green, Revs. Modern Phys. **30**, 569 (1958).
[7] E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937).
[8] L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953); B. G. Jancovici, Phys. Rev. **95**, 389 (1954); D. C. Peaslee, Phys. Rev. **95**, 717 (1954); B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954); P. C. Sood and A. E. S. Green, Nuclear Phys. **5**, 274 (1958).
[9] L. N. Cooper, reference 8.
[10] B. G. Jancovici, reference 8.

TABLE II. Net Coulomb energies (direct−|exchange|) in units of $e^2/1$ fermi.

Nucleus	r_0	Present estimates	Statistical model	Trapezoidal model
$_{18}A^{40}$	1.25	42.29	40.39	40.51
$_{20}Ca^{40}$	1.22	54.16	51.53	50.65
$_{34}Se^{80}$	1.21	127.10	123.31	125.03
$_{40}Zr^{90}$	1.21	170.65	165.46	168.90
$_{58}Ce^{140}$	1.20	313.74	307.36	316.53
$_{68}Er^{168}$	1.20	408.27	400.62	413.37
$_{70}Yb^{174}$	1.20	429.62	419.04	434.28

is 6.4. The correlations in the positions of protons are thus dependent on the type of well in which they move. This point is further confirmed by our Table I, where is shown the effect of exchange energy on the radii of *heavier* nuclei. Further, as is evident from Fig. 2, while the $Z^{4/3}$ variation appears to be a suitable characterization of the exchange energy, the present estimates pertaining to heavy nuclei require a larger multiplying constant in Eq. (2) than that required by the other two models. As was noticed earlier,[1] this "constant" happens to be a mildly Z-dependent quantity.

It is interesting to note, however, that the direct energies do not show any serious disagreement among the three models, as also the corresponding "net" energies. Table II shows the trapezoidal model estimates as intermediate between the other two models. The diffuse surface of the charge distribution corresponding to the harmonic oscillator model may be more realistic than the linear falloff prescribed by the trapezoidal model.

ACKNOWLEDGMENTS

We are grateful to Shri R. J. Kulkarni and Shri V. V. Itagi for checking some of the calculations. One of us (V.K.K.) would like to express his indebtedness to the Department of Atomic Energy, Government of India, for financial support while carrying out this work.

APPENDIX

The radial wave functions of protons bound in a harmonic oscillator well are

$$R_{vl}(r)=N_{vl}(\alpha r)^{l+1}\exp\left[-\tfrac{1}{2}(\alpha r)^2\right]\times {}_1F_1\left(-v,\, l+\tfrac{3}{2};\, (\alpha r)^2\right). \quad \text{(I)}$$

When the above functions are used, the Slater integrals can be evaluated analytically and the results are

$$\begin{aligned}F^{\lambda}(v_1l_1,v_2l_2) &= \frac{e^2}{4\alpha}(N_{v_1l_1}N_{v_2l_2})^2\\
&\times\Bigg\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(L_{12}+m+n+\frac{5}{2})}{2^{L_{12}+m+n+\frac{5}{2}}[l_2+n+(\lambda+3)/2]}\\
&\times A(v_1v_1l_1l_1m)A(v_2v_2l_2l_2n)\\
&\times {}_2F_1(L_{12}+m+n+\tfrac{5}{2},\, 1,\, l_2+n+(\lambda+5)/2;\, \tfrac{1}{2})\\
&+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(L_{12}+m+n+\frac{5}{2})}{2^{L_{12}+m+n+\frac{5}{2}}[l_1+m+(\lambda+3)/2]}\\
&\times A(v_1v_1l_1l_1m)A(v_2v_2l_2l_2n)\\
&\times {}_2F_1(L_{12}+m+n+\tfrac{5}{2},1,l_1+m+(\lambda+5)/2;\tfrac{1}{2})\Bigg\}, \quad \text{(II)}\end{aligned}$$

and

$$\begin{aligned}G^{\lambda}(v_1l_1,v_2l_2) &= \frac{e^2}{\alpha}(N_{v_1l_1}N_{v_2l_2})^2\\
&\times\Bigg\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(L_{12}+m+n+\frac{5}{2})A(v_1v_2l_1l_2m)A(v_1v_2l_1l_2n)}{2^{L_{12}+m+n+\frac{5}{2}}(L_{12}+\lambda+2n+3)}\\
&\times {}_2F_1(L_{12}+m+n+\tfrac{5}{2},1,\tfrac{1}{2}(L_{12}+2n+\lambda+5);\tfrac{1}{2})\Bigg\}. \quad \text{(III)}\end{aligned}$$

Here we have used the following abbreviations:

$$\alpha=(\mu\omega/\hbar)^{\frac{1}{2}},$$

$$N_{vl}\equiv\left[\frac{2\alpha(v+l+\frac{1}{2})!}{v!(l+\frac{1}{2})!^2}\right]^{\frac{1}{2}},$$

$$A(v_iv_jl_il_jc)\equiv\frac{(-v_i)_c}{(a_i)_cc!}\times {}_3F_2(-v_j,\, -(l_i+c+\tfrac{1}{2}),\, -c,\, l_j+\tfrac{3}{2},\, 1+v_i-c;\, -1),$$

$$(x)_y\equiv\Gamma(x+y)/\Gamma(x),$$

$$L_{12}\equiv l_1+l_2,$$

$$A_i\equiv l_i+\tfrac{3}{2}.$$