Energy-Momentum Tensor for Plane Waves*

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A general form is established for the energy momentum tensor for plane waves propagating in a homogeneous medium, the field equations of which are derivable from a quadratic Lagrangian function. Energy density and momentum density are proportional to frequency and the wave vector, the coefficient of proportionality being "action density." Energy flow and momentum flow are related to energy density and momentum density by the group velocity. The relation between momentum density and the wave vector is valid even in a nonlinear system. For a wave packet, one finds that the total energy is related to frequency and the total momentum to the wave vector by the total action of the packet, in close analogy with corresponding relations of quantum mechanics.

IN a recent article in this journal, Post[1] has established a simple formula [Eq. (1) of that article] for the energy-momentum tensor of plane acoustic waves. This formula is, as it stands, misleading: The Lagrangian density, which appears in the formula, is nonzero only if the field equations are nonlinear, whereas the assumption of linearity is employed frequently in Post's derivation.

When corrected, the formula given by Post is valid for *any* propagating medium of which the field equations are derivable from a quadratic Lagrangian density, as has already been noted and used elsewhere.[2] We shall derive this formula and point out where the assumption of linearity enters, showing that the momentum components of the formula are valid even in a nonlinear medium. It will be seen that introduction of the action density enhances the beauty and significance of the formula.

If we write (x^μ) or (x^0,x^r) for the coordinates of time $(x^0=t)$ and space $(x^r=x^1, x^2,x^3)$, $\phi_\alpha(x)$ $(\alpha=1, 2,\cdots)$ for the dynamical variables, and $\phi_{\alpha;\mu}$ for $d\phi_\alpha/dx^\mu$, the action principle for a uniform stationary medium,

$$\delta\int \mathfrak{L}(\phi_{\alpha;\mu},\phi_\beta)d^4x=0, \tag{1}$$

leads to the following field equations,

$$\pi^{\alpha\mu}{}_{;\mu}\equiv(d/dx^\mu)(\partial\mathfrak{L}/\partial\phi_{\alpha;\mu})=\partial\mathfrak{L}/\partial\phi_\alpha. \tag{2}$$

The canonical energy-momentum tensor

$$T_\mu{}^\nu=\pi^{\alpha\nu}\phi_{\alpha;\mu}-\mathfrak{L}\delta_\mu{}^\nu \tag{3}$$

satisfies the conservation equation

$$T_\mu{}^\nu{}_{;\nu}=0. \tag{4}$$

$T_0{}^0=$ energy density; $T_0{}^r=$ energy flux vector; $T_r{}^0=$ momentum density; $T_r{}^s=$ momentum flux tensor.

We now consider a plane-wave solution of (2) for which all quantities are expressible as functions (in general nonsinusoidal) of θ of period 2π, where

$$\theta=k_\mu x^\mu+\kappa\equiv\omega t+k_r x^r+\kappa. \tag{5}$$

The phase parameter κ enables us to ascribe an action density $2\pi J^0$ to the wave function, where

$$J^0=(1/2\pi)\oint d\kappa\,\pi^{\alpha0}\partial\phi_\alpha/\partial\kappa\equiv\langle\pi^{\alpha0}(\partial\phi_\alpha/\partial\kappa)\rangle_{\text{av}}, \tag{6}$$

the angular brackets indicating an average taken over space (or time). We see immediately from (3), (5), and (6) that the average momentum density is given by

$$\langle T_r{}^0\rangle_{\text{av}}=J^0k_r. \tag{7}$$

The corresponding energy-frequency relation is less simple. (3), (5), and (6) now give

$$T_0{}^0=J^0\omega-\mathfrak{L}. \tag{8}$$

We see from (2) that

$$\delta\langle\mathfrak{L}\rangle_{\text{av}}=(d/dt)\langle\pi^{\alpha0}\delta\phi_\alpha\rangle_{\text{av}}. \tag{9}$$

$\delta\phi_\alpha$ comprises two parts: $\delta_f\phi_\alpha$, a change of functional form, and a part due to the dependence of ω on amplitude:

$$\delta\phi_\alpha=\delta_f\phi_\alpha+\delta\omega\, t\partial\phi_\alpha/\partial\kappa. \tag{10}$$

Since $\delta_f\phi_\alpha$ and $\partial\phi_\alpha/\partial\kappa$ are still of frequency ω, (9) and (10) give

$$\delta\langle\mathfrak{L}\rangle_{\text{av}}=J^0\delta\omega, \tag{11}$$

so that (8) gives

$$\delta\langle T_0{}^0\rangle_{\text{av}}=\omega\delta J^0. \tag{12}$$

Hence in nonlinear theory

$$\langle T_0{}^0\rangle_{\text{av}}=\int_0^{J^0}\omega(\chi)d\chi. \tag{13}$$

In *linear* theory, ω is independent of J^0. It then follows from (11) that $\langle\mathfrak{L}\rangle_{\text{av}}=0$ if $\langle\mathfrak{L}\rangle_{\text{av}}=0$ in the unperturbed medium, and from (13) that

$$\langle T_0{}^0\rangle_{\text{av}}=J^0\omega. \tag{14}$$

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[1] E. J. Post, Phys. Rev. **118**, 1113 (1960).
[2] P. A. Sturrock, J. Appl. Phys. **31**, 2052 (1960).

In linear theory (assuming the system has not a nondenumerably infinite number of degrees of freedom) we may go further by setting up a wave packet[3] of the form

$$\phi_\alpha(x^r,t)=\Phi_\alpha(k_r)A(x^r-u^r t)\exp(i\theta)+\text{c.c.}, \tag{15}$$

where $u^r=-\partial\omega/\partial k_r$ is the group velocity, and the amplitude $A(x^r)$ may be regarded as arbitrary provided it varies sufficiently slowly. For such wave packets, the "local" averages of quadratic functions of the field variables are expressible in the following form:

$$\langle T_0{}^0(x^r,t)\rangle_{\text{av}}=|A|^2\tilde{T}_0{}^0(k_r), \quad \text{etc.} \tag{16}$$

We now obtain from (7) and (14) relations analogous to familiar relations of quantum mechanics,

$$P_r=Jk_r, \quad E=J\omega, \tag{17}$$

where P_r, E, and J are integrals over the packet of $T_r{}^0$, $T_0{}^0$, and J^0. We may also deduce from (4) and (16) that

$$\langle T_0{}^r\rangle_{\text{av}}=\langle T_0{}^0\rangle_{\text{av}}u^r, \quad \langle T_r{}^s\rangle_{\text{av}}=\langle T_r{}^0\rangle_{\text{av}}u^s, \tag{18}$$

thus demonstrating the equivalence of the group velocity, "energy velocity," and "momentum velocity." Hence, on compiling (7), (14), and (18), we obtain the following formula for the energy-momentum tensor of a plane wave in a linear system:

$$\begin{aligned}\langle T_0{}^0\rangle_{\text{av}}&=J^0\omega, & \langle T_0{}^r\rangle_{\text{av}}&=J^0\omega u^r,\\ \langle T_r{}^0\rangle_{\text{av}}&=J^0k_r, & \langle T_r{}^s\rangle_{\text{av}}&=J^0k_r u^s.\end{aligned} \tag{19}$$

[3] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 332.

Permissible transformations of the field variables are conventionally restricted to the form[4]

$$\Theta_\mu{}^\nu=T_\mu{}^\nu+f_\mu{}^{\nu\sigma}{}_{;\sigma}, \tag{20}$$

where $f_\mu{}^{\nu\sigma}$ is a function of the field variables antisymmetric in ν,σ; this term does not contribute to $\langle\Theta_\mu{}^\nu\rangle_{\text{av}}$, so that the form (19) is preserved.

The equation of conservation of angular momentum, applied to a classical (spinless) system, leads to the requirement that the energy-momentum tensor be symmetric,[4] but we see from (19) that in an anisotropic medium the energy-momentum tensor of a linear system is asymmetric. Of possible resolutions of this paradox, the following should not be overlooked: If the system is strictly nonlinear so that the Lagrangian function is not purely quadratic and contains in particular a cubic contribution, and if the unperturbed state is not stress-free, one may find that the energy-momentum tensor formed from the quadratic part of the Lagrangian function is not identical with the quadratic part of the energy-momentum tensor formed from the exact Lagrangian function. Of these two tensors, it is the former which is given by (19) but the latter to which physical arguments concerning symmetry apply. This explains, in particular, the asymmetry of the energy-momentum tensor derived elsewhere[5] for small-amplitude disturbances of an electrodynamic system.

[4] L. Landau and E. Lifschitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1951), p. 82.

[5] P. A. Sturrock, Ann. Phys. 4, 306 (1958).