Renormalization Constants in Quantum Electrodynamics*

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The electron self-energy and wave-function renormalization constant are calculated with the use of the technique of dispersion relations. In order to do this, exact representations of these renormalization constants are obtained in terms of matrix elements of Heisenberg field operators. The matrix element $\langle e\gamma|\bar{\psi}|0\rangle$ is studied in detail and its contribution to the renormalization constants determined. In retaining only the contributions due to $\langle e|\bar{\psi}|0\rangle$ and $\langle e\gamma|\bar{\psi}|0\rangle$ an infinity of other matrix elements is ignored. A test of the validity of this approximation is discussed. Within the limits of the approximations, the self-energy and wave-function renormalization constants are shown to have a logarithmic, ultraviolet divergence. The wave-function renormalization constant also has an infrared divergence.

I. INTRODUCTION

THE technique of relativistic dispersion relations[1] has been applied with success in the study of many observable processes.[2] It is also of interest to attempt to calculate unphysical quantities, such as renormalization constants, with the use of dispersion theory. It was shown in a previous paper that this method leads to correct results for the renormalization constants in the Lee model.[3] This paper concerns an attempt to extend the analysis to the case of quantum electrodynamics.

In the case of the Lee model it was shown that various matrix elements relevant to the calculation of the V-particle self-energy were equal to products of two types of factors: One was what one obtains from perturbation theory, and the other a cutoff function. If similar results are obtained in quantum electrodynamics, one might find that the electron self-energy and wave-function renormalization constant are finite, at least in some approximation.

In order to pursue this point, we demonstrate that these constants can be expressed exactly in the Heisenberg representation in terms of matrix elements of the form $\langle n|\bar{\psi}|0\rangle$ where the index n enumerates a complete set of interacting states. We approximate these expressions by retaining the contributions due to the one- and two-particle states only. Then, for the calculation of the matrix element $\langle e\gamma|\bar{\psi}|0\rangle$, where $\langle e\gamma|$ is the electron-photon scattering state, we use the technique of dispersion relations. That is, with the use of the reduction formula of Lehmann, Symanzik, and Zimmermann[4] and the analytic properties of $\langle e\gamma|\bar{\psi}|0\rangle$ considered as a function of the total energy of the $\langle e\gamma|$ state, we obtain an integral equation for $\langle e\gamma|\bar{\psi}|0\rangle$. We are able to solve this integral equation because we retain only the contributions to $\mathrm{Im}\langle e\gamma|\bar{\psi}|0\rangle$ due to the physical electron and electron-photon scattering states. This kind of approximation, namely, the retention of only a few intermediate states in the expansion of the absorptive part of the matrix element,[5] has been used many times before. Unfortunately, as is shown below, this approximation destroys the gauge invariance which the full theory possesses.[6] In this section devoted to conclusions we discuss a quantitative measure of the validity of this assumption.

It is found that the electron self-energy and wave-function renormalization constant contain ultraviolet logarithmic divergences. The wave-function renormalization constant also has an infrared divergence.

II. MASS AND WAVE-FUNCTION RENORMALIZATION CONSTANTS IN THE HEISENBERG REPRESENTATION

In this section we write down representations of the electron mass and wave-function renormalization constants in terms of an infinite sum of products of matrix elements.

The self-energy may be written as follows[7]:

$$\delta m=\int_0^\infty [(m-\kappa)\rho_1(\kappa^2)+\rho_2(\kappa^2)]d\kappa^2 \Big/ \int_0^\infty \rho_1(\kappa^2)d\kappa^2, \quad (1)$$

where the spectral function ρ_1 and ρ_2 are given by:

$$\rho_1(\kappa^2)=-\frac{(2\pi)^3}{4\kappa^2}\mathrm{Tr}\,i\kappa\cdot\gamma\sum_{p_n=\kappa}\langle 0|\psi|n\rangle\langle n|\bar{\psi}|0\rangle = \sum_{p_n=\kappa}\rho_1^{(n)}(\kappa^2), \quad (2)$$

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[5] R. Blankenbecler, M. Goldberger, and F. Halpern, Nuclear Phys. **12**, 629 (1959).

[6] R. G. Sachs has pointed out that this deficiency attends the use of this type of approximation in other applications as well. (Private communication from G. Feldman.)

[7] H. Lehmann has proven these relations for the $\pi-N$ system, Nuovo cimento **11**, 343 (1954). The extension of his results to quantum electrodynamics is straightforward and has been carried out in detail by the author in his doctoral thesis, Johns Hopkins University (unpublished). It should be noted that Eq. (1) above is valid only in a gauge in which $\langle 0|A_\mu(x)|0\rangle=0$. The Gupta-Bleuler gauge is one such gauge.

and

$$\rho_2(\kappa^2)=\frac{(2\pi)^3}{4}\,\mathrm{Tr}\sum_{p_n=\kappa}\langle 0|\psi|n\rangle\langle n|\bar{\psi}|0\rangle=\sum_{p_n=\kappa}\rho_2^{(n)}(\kappa^2). \quad (3)$$

We use the caret notation to indicate the scalar product of a vector with a gamma matrix $\hat{K}=K_\mu\gamma_\mu$. All the quantities entering these expressions are renormalized. The ψ is the Heisenberg field operator of the electron. The index n enumerates a complete set of Heisenberg state vectors. We choose these to be "out" states, for convenience. Any other complete set would be adequate.

From Eq. (2) and Eq. (3) we see that matrix elements of the form $\langle n|\bar{\psi}|0\rangle$ are the ones of interest in the discussion of δm. The matrix elements $\langle 0|\psi|n\rangle$ can be obtained readily in terms of $\langle n|\bar{\psi}|0\rangle$.

Finally, we note that the wave-function renormalization constant is given as follows:

$$Z_2^{-1}=\int_0^\infty \rho_1(\kappa^2)d\kappa^2. \quad (4)$$

III. CALCULATION OF THE SPECTRAL FUNCTIONS

We retain only two intermediate states in the sums in Eq. (2) and Eq. (3). The validity of this approximation will be discussed later. For the present we merely note that this approximation along with a subsequent perturbation theory calculation of the matrix elements involved reproduces the usual results of the lowest order perturbation theory calculation of δm.

The contributions to the ρ_i from the intermediate state of a single electron $\rho_i^{(1)}$ are simply

$$\rho_1^{(1)}=\delta(\kappa^2-m^2), \quad (5)$$

and

$$\rho_2^{(1)}(\kappa^2)=0, \quad (6)$$

where the superscript refers to the single-electron intermediate state and we have used the fact that

$$\langle e|\bar{\psi}|0\rangle=(m/p_0)^{\frac{1}{2}}\bar{u}(p)e^{-ipx}, \quad (7)$$

for our choice of normalization.

The contributions to the ρ_i due to the interacting electron-photon state $\rho_i^{(2)}$, are obtained as follows. From Eqs. (2) and (3), it follows that

$$\rho_i^{(2)}\sim\langle 0|\psi|e\gamma\rangle\langle e\gamma|\bar{\psi}|0\rangle. \quad (8)$$

Therefore, we need to calculate the matrix element $L\equiv\langle e\gamma|\bar{\psi}|0\rangle$ in order to find the $\rho_i^{(2)}$. To do this we use dispersion relation techniques. We introduce six form factors, $F_i(\xi)$, in connection with L in such a way that

$$L=\left(\frac{m}{p_0\omega}\right)^{\frac{1}{2}}\bar{u}(p)\{F_1(\xi)[\hat{k}\epsilon\cdot\gamma-k\cdot\epsilon] +F_2(\xi)i\hat{k}p\cdot\epsilon+F_3(\xi)i\hat{k}k\cdot\epsilon+F_4(\xi)k\cdot\epsilon +F_5(\xi)2p\cdot\epsilon+F_6(\xi)i\epsilon\cdot\gamma k\cdot p\}, \quad (9)$$

where p and k are the electron and photon momenta, respectively, and

$$\xi=(p+k)^2. \quad (10)$$

Note that $\langle e\gamma|\bar{\psi}|0\rangle$ is not gauge invariant, hence the presence of F_5 in Eq. (9).

In the next section we shall demonstrate that the F_i obey dispersion relations of the type

$$F_i(\xi)=\frac{1}{\pi}\int_\infty^\infty\frac{\mathrm{Im}F_i(-\xi')}{\xi'+\xi-i\epsilon}d\xi'. \quad (11)$$

It is necessary to assume that the F_i satisfy unsubtracted dispersion relations. In fact, we shall show that our F_i do so.

IV. CALCULATION OF THE FORM FACTORS F_i

In the first part of this section we use the reduction formula of Lehmann, Symanzik, and Zimmermann[4] to obtain an integral representation of $\mathrm{Im}\langle e\gamma|\bar{\psi}|0\rangle$. Upon combining the definition of the F_i in Eq. (9), the integral representation of $\mathrm{Im}\langle e\gamma|\bar{\psi}|0\rangle$, and the dispersion relations for the F_i, we obtain a set of coupled integral equations for the F_i. We solve the integral equations for the $F_i(\xi)$ of particular interest.

A. Integral Representation for L

Using the reduction formula of Lehmann, Symanzik, and Zimmermann, we write the matrix element L,

$$L=-\frac{i}{(2\omega)^{\frac{1}{2}}}\int dx\,e^{-ikx}\epsilon_\mu\Box\langle e|TA_\mu(x)\bar{\psi}|0\rangle, \quad (12)$$

where we employ the convention that an omitted argument means the function is to be evaluated at the zero of its argument, and T denotes the time-ordered product of $A_\mu(x)$ and $\bar{\psi}$. With the use of the equal-time commutation relations

$$[A_\mu(x),\bar{\psi}]_{x_0=0}=[\dot{A}_\mu(x),\bar{\psi}]_{x_0=0}=0, \quad (13)$$

and Eq. (12), we obtain L:

$$L=\frac{i}{(2\omega)^{\frac{1}{2}}}\int dx\,e^{-ikx}\epsilon_\mu\langle e|TJ_\mu(x)\bar{\psi}|0\rangle, \quad (14)$$

where $J_\mu(x)$ is the electromagnetic current,

$$J_\mu(x)=-\Box A_\mu(x). \quad (15)$$

A convenient form of the T product is

$$TA_\mu(x)\bar{\psi}=[A_\mu(x),\bar{\psi}]\theta(x)+\bar{\psi}A_\mu(x), \quad (16)$$

where $\theta(x)$ is the unit step function,

$$\theta(x)=\tfrac{1}{2}[1+\epsilon(x_0)]. \quad (17)$$

The term $\bar{\psi}J_\mu(x)$ does not contribute to the integral in Eq. (14). Thus we can write L as the Fourier transform

of a retarded commutator,

$$L=\frac{i}{(2\omega)^{\frac{1}{2}}}\int dx\, e^{-ikx}\epsilon_\mu\langle e|[J_\mu(x),\bar{\psi}]\theta(x)|0\rangle. \quad (18)$$

B. Proof of the Dispersion Relations for F_i

We now show that L, or, more precisely, the F_i, satisfy dispersion relations. This is most easily seen by choosing a coordinate system in which the electron is at rest. Because $|\mathbf{k}|=\omega$ we may write L

$$L=\frac{i}{(2\omega)^{\frac{1}{2}}}\int dx\, \exp[i\omega(x_0-\mathbf{n}\cdot\mathbf{x})] \times\epsilon_\mu\langle e|[J_\mu(x),\bar{\psi}]|0\rangle\theta(x), \quad (19)$$

where

$$\mathbf{n}=\mathbf{k}/\omega. \quad (20)$$

By virtue of the assumption of microscopic causality,[8]

$$[J_\mu(x),\bar{\psi}]=0, \quad (21)$$

for spacelike x_μ (i.e., in our metric, for $x^2>0$), and we are concerned with times such that $|x_0|>\mathbf{n}\cdot\mathbf{x}$, and because x_0 is restricted to positive values by the $\theta(x)$, we see that $L(\omega)$ defines a function which is analytic in the upper half of the complex ω plane. We can therefore write a dispersion relation for L, or better, for the functions F_i, as follows:

$$F_i(-m^2-2m\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}F_i(-m^2-2m\omega')}{\omega'-\omega-i\epsilon}d\omega'. \quad (22)$$

This may be written in terms of the invariant $\xi=(p+k)^2$ as

$$F_i(\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}F_i(-\xi')d\xi'}{\xi'+\xi-i\epsilon}, \quad (23)$$

in order that this integral exist, it is sufficient that $\mathrm{Im}F_i(-\xi')=O[(\xi')^{-1-\epsilon}]$, $\epsilon>0$, as $|\xi'|\rightarrow\infty$. If the functions F_i were such that this integral did not exist, we might have to write a subtracted dispersion relation. Clearly, however, when calculating the renormalization constants of the electron, we cannot claim knowledge of these functions at any point. It is therefore crucial that our functions satisfy the unsubtracted dispersion relations of Eq. (23). In fact, the F_i which we eventually calculate do satisfy the convergence requirements demanded of them by Eq. (23).

C. Consequences of the Conserved Current

We shall now invoke current conservation, that is, $\partial_\mu J_\mu=0$, in order to show that not all the F_i are independent. Consider the following integral I:

$$I=\int dx\, e^{-ikx}\theta(x)\partial_\mu\langle e|[J_\mu(x),\bar{\psi}]|0\rangle. \quad (24)$$

Evidently $I=0$ by current conservation. Integrating I by parts, we find

$$I=-\int dx\langle e|[J_\mu(x),\bar{\psi}]|0\rangle\partial_\mu(e^{-ikx}\theta(x)), \quad (25)$$

where a vanishing surface term of the form

$$e^{-ikx}\theta(x)\langle e|[J_\mu(x),\bar{\psi}]|0\rangle|_{B(VT)} \quad (26)$$

has been omitted. $B(VT)$ is the boundary of space-time.[9]

It follows from Eq. (25) that

$$k_\mu\int dx\, e^{-ikx}\theta(x)\langle e|[J_\mu(x),\bar{\psi}]|0\rangle =-\int dx\, e^{-ikx}\delta(x_0)\langle e|[J_4(x),\bar{\psi}]|0\rangle. \quad (27)$$

From the equal time commutation relation

$$[J_4(x),\bar{\psi}]_{x_0=0}=-ie\bar{\psi}(x)\delta^3(\mathbf{x}), \quad (28)$$

and Eq. (27), we find

$$ik_\mu\int dx\, e^{-ikx}\theta(x)\langle e|[J_\mu(x),\bar{\psi}]|0\rangle=-e\bar{u}(p)(m/p_0)^{\frac{1}{2}}. \quad (29)$$

Comparing (29) with (18) and (9), and upon replacing ϵ_μ by k_μ in (18), we obtain

$$F_6(\xi)=-e/(\xi+m^2), \quad (30)$$

and

$$F_2(\xi)=-F_6(\xi). \quad (31)$$

These are exact results; they do not depend on perturbation theory. In view of what is proved below, Eq. (30) has a special significance, which we shall discuss in the concluding section.

We now proceed to calculate $\mathrm{Im}L$ approximately, in order that we may determine the $\mathrm{Im}F_i$. Having obtained the $\mathrm{Im}F_i$, we shall calculate F_i with the aid of Eq. (23).

D. Contribution to ImL Due to the Pole Term

Expanding the commutator in Eq. (18), we write for L

$$L=\frac{i}{(2\omega)^{\frac{1}{2}}}\int dx\, e^{-ikx}\epsilon_\mu\{\langle e|J_\mu(x)\bar{\psi}|0\rangle -\langle e|\bar{\psi}J_\mu(x)|0\rangle\}\theta(x). \quad (32)$$

[8] For a discussion of the principle of microscopic causality, see references 1, 2, and also N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, New York, 1959), p. 198 ff.

[9] At this point we note that the Lehmann, Zimmermann, and Symanzik formalism consistently employs the wave packet notion to ensure the vanishing of similar surface terms. We follow their methods, and though we write exp(ikx) in expressions like (26), it should be understood that a wave packet is meant.

Introducing a sum over a complete set of intermediate positive-energy states, we obtain

$$L=\frac{i}{(2\omega)^{\frac{1}{2}}}\int dx\, e^{-ikx}\epsilon_\mu \sum_n\{\langle e|J_\mu|n\rangle\langle n|\bar{\psi}|0\rangle - \langle e|\bar{\psi}|n\rangle\langle n|J_\mu(x)|0\rangle\}\theta(x). \quad (33)$$

Using translational invariance we are able to rewrite (33) as follows:

$$L=\frac{i}{(2\omega)^{\frac{1}{2}}}\int dx\, e^{-ikx}\epsilon_\mu \sum_n\{\langle e|J_\mu|n\rangle\langle n|\bar{\psi}|0\rangle e^{-i(p-p_n)x} - \langle e|\bar{\psi}|n\rangle\langle n|J_\mu|0\rangle e^{-ip_nx}\}\theta(x). \quad (34)$$

Using Eq. (17) in Eq. (34) and the time reversal invariance of the theory, it can be shown[10] that

$$\mathrm{Im}L=\frac{(2\pi)^4}{2(2\omega)^{\frac{1}{2}}}\sum_n \epsilon_\mu\langle e|J_\mu|n\rangle\langle n|\bar{\psi}|0\rangle\delta^4(k+p-p_n), \quad (35)$$

where we have explicitly used the assumption that the intermediate states have positive energy.

In the following we shall consider only the contributions to the sum in Eq. (35) corresponding to the single-electron state $|e\rangle$ and the electron-photon scattering state $|e\gamma\rangle$.

Consider the contribution to ImL due to the single-electron state $|e\rangle$, Im$L^{(1)}$.

$$\mathrm{Im}L^{(1)}=\frac{(2\pi)^4}{2(2\omega)^{\frac{1}{2}}}\sum_{\text{spin}}\epsilon_\mu\int\frac{d^3p'}{(2\pi)^3}\times\langle e|J_\mu|e'\rangle\langle e'|\bar{\psi}|0\rangle\delta^4(p+k-p')$$
$$=\frac{\pi}{(2\omega)^{\frac{1}{2}}}\epsilon_\mu\sum_{\text{spin}}\langle e|J_\mu|e'\rangle\langle e'|\bar{\psi}|0\rangle\delta(p_0+\omega-p_0'), \quad (36)$$

with the condition

$$\mathbf{p}'=\mathbf{p}+\mathbf{k}. \quad (37)$$

We now make use of the usual representation[11] of $\langle e|J_\mu|e'\rangle$,

$$\langle e|J_\mu|e'\rangle=-i\left(\frac{m^2}{p_0p_0'}\right)^{\frac{1}{2}}\bar{u}(p)[g_1(q^2)\gamma_\mu + g_2(q^2)i\sigma_{\mu\nu}q_\nu]u(p'), \quad (38)$$

where $q_\nu=p_\nu-p_\nu'$, and

$$g_1(q^2)\gamma_\mu|_{q^2=0}=e\gamma_\mu,\quad g_2(q^2)i\sigma_{\mu\nu}q_\nu|_{q^2=0}=0. \quad (39)$$

By virtue of the δ function in Eq. (36), we need know only the $g_i(0)$. Using Eqs. (38) and (39) in Eq. (36), we find for Im$L^{(1)}$

$$\mathrm{Im}L^{(1)}=-e\pi\left(\frac{m}{2\omega p_0}\right)^{\frac{1}{2}}i\epsilon_\mu\bar{u}(p)\gamma_\mu\times\left(\frac{m-i\hat{p}'}{2p_0'}\right)\delta(p_0'-p_0-\omega). \quad (40)$$

It follows from Eqs. (40) and (9) that the pole term contributions to the F_i are

$$\mathrm{Im}F_1^{(1)}(\xi)=\mathrm{Im}F_4^{(1)}(\xi)=\mathrm{Im}F_5^{(1)}(\xi)=-e\pi\delta(\xi+m^2), \quad (41)$$

and

$$\mathrm{Im}F_2^{(1)}=\mathrm{Im}F_3^{(1)}=\mathrm{Im}F_6^{(1)}=0. \quad (42)$$

These expressions, when used in connection with the dispersion relation Eq. (23), reproduce lowest order perturbation theory.

E. Contribution to ImL Due to the Interacting Electron-Photon State

We now consider the contribution to Eq. (35) due to the $|e\gamma\rangle$ intermediate state, Im$L^{(2)}$. With the use of the reduction formula it can be shown that

$$\mathrm{Im}L^{(2)}=\frac{i}{2}\sum_{\text{spin pol}}\int\int\frac{d^3p'd^3k'}{(2\pi)^6}\langle e\gamma|e'\gamma'\rangle\langle e'\gamma'|\bar{\psi}|0\rangle, \quad (43)$$

where $\langle e\gamma|e'\gamma'\rangle$ is the Compton scattering amplitude. Note, however, that $k\cdot\epsilon\neq0$ and $k'\cdot\epsilon'\neq0$. In order to proceed we need information about this scattering amplitude.

For lack of anything better, we shall use the first Born approximation for the Compton scattering amplitude. Using usual perturbation theory[12] we find that

$$\langle e\gamma|e'\gamma'\rangle=\frac{ie^2(2\pi)^4}{2(p_0p_0'\omega\omega')^{\frac{1}{2}}}\bar{u}(p)\left\{\epsilon\cdot\gamma\frac{i(\hat{p}+\hat{k})-m}{-2\kappa}\epsilon'\cdot\gamma + \epsilon'\cdot\gamma\frac{i(\hat{p}'-\hat{k})-m}{2\kappa'}\epsilon\cdot\gamma\right\}u(p')\delta^4(p+k-p'-k'), \quad (44)$$

where

$$\kappa=-p\cdot k=-p'\cdot k', \quad (45)$$

and

$$\kappa'=-p'\cdot k=-p\cdot k'. \quad (46)$$

Writing the quantity $\langle e'\gamma'|\bar{\psi}|0\rangle$ in terms of form factors F_i, as before, in Eq. (9) and using Eq. (44) in Eq. (43), we obtain for Im$L^{(2)}$

$$\mathrm{Im}L^{(2)}=-\frac{e^2}{4(2\pi)^2}\left(\frac{m}{2p_0\omega}\right)^{\frac{1}{2}}\sum_{\text{spin pol}}\int\int d^3p'd^3k'\frac{m}{p_0'\omega'}$$
$$\times\left\{\bar{u}(p)\left[\epsilon\cdot\gamma\frac{i(\hat{p}+\hat{k})-m}{-2\kappa}\epsilon'\cdot\gamma+\epsilon'\cdot\gamma\frac{i(\hat{p}'-\hat{k})-m}{2\kappa'}\epsilon\cdot\gamma\right]\right.$$
$$\times\bar{u}(p')u(p')[\mathrm{Re}F_1 i\sigma_{\mu\nu}\epsilon_\mu'k_\nu'+\mathrm{Re}F_2 i\hat{k}'p'\cdot\epsilon'$$
$$+\mathrm{Re}F_3 i\hat{k}'k'\cdot\epsilon'+\mathrm{Re}F_4 k'\cdot\epsilon'+\mathrm{Re}F_5 2p'\cdot\epsilon'$$
$$\left.+\mathrm{Re}F_6 i\epsilon'\cdot\gamma k'\cdot p']\delta^4(p+k-p'-k')\right\}. \quad (47)$$

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[10] For the details of this proof see the author's doctoral thesis, Johns Hopkins University (unpublished).

[11] G. Salzman, Phys. Rev. 99, 973 (1953).

[12] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955).

Recall that we intend to use the final $\mathrm{Im}F_i$ in the dispersion relations Eq. (23) in order to calculate the F_i themselves. With these new F_i we shall calculate the self-energy. Since perturbation theory gives correct results in quantum electrodynamics for processes taking place at low energies, our calculated F_i must agree with the results of perturbation theory for small $|\xi|$. Of greatest importance to us, however, is the high-energy behavior of the F_i. We hope that the F_i which we calculate will vanish more rapidly as $-\xi\to\infty$, than do the results of perturbation theory, for it is only in this case that we can hope to get finite renormalization constants.

The integrals in Eq. (47) may be evaluated in a straightforward way in the barycentric system of the $|e\gamma\rangle$ state ($\mathbf{p}=0$). The results of the integration are too lengthy to reproduce completely here and thus we shall simply state their salient features in the following.

F. Application of the Dispersion Relations

Adding the $\mathrm{Im}F_i^{(2)}$ to the contribution due to the pole term $\mathrm{Im}F_i^{(1)}$, we find in our approximation that the F_i satisfy the following equations.

$$F_i(\xi)=f_i(\xi)+\frac{1}{\pi}\int_{m^2}^{\infty}\left[\sum_{j=1}^{6}h_{ij}(-\xi')\,\mathrm{Re}F_j(-\xi')\Big/(\xi'+\xi-i\epsilon)\right]d\xi', \tag{48}$$

where the $f_i(\xi)$ are given as follows:

$$f_1=f_4=f_5=-e/(\xi+m^2), \tag{49}$$

$$f_2=f_3=f_6=0. \tag{50}$$

and the $h_{ij}(-\xi)$ are known functions.

We now summarize a few important properties of the h_{ij}. The most significant property is that

$$h_{i4}(\xi)\equiv 0 \tag{51}$$

for all i. Especially note that $h_{44}=0$ and therefore, according to Eq. (48)

$$F_4(\xi)=-\frac{e}{\xi+m^2}+\frac{1}{\pi}\int_{m^2}^{\infty}\left[\sum_{j=1}^{6}h_{4j}\,\mathrm{Re}F_j(-\xi')\Big/(\xi'+\xi-i\epsilon)\right]d\xi', \tag{52}$$

where the right-hand side of Eq. (52) does not depend on an integral over $\mathrm{Re}F_4$. That is, Eq. (52) is *not* an integral equation for F_4. It would seem that the right-hand side of (52) does depend on $\mathrm{Re}F_4$ through $\mathrm{Re}F_1$, $\mathrm{Re}F_2$, etc. However, the fact that $h_{i4}=0$ for all i makes certain that this is not the case. These facts follow as a direct consequence of Eq. (47) when the sum over the polarization of ϵ' has been completed.

On the other hand, consider F_1. We obtain from Eq. (48) the following integral equation for F_1

$$F_1(\xi)=-\frac{e}{\xi+m^2}+\frac{1}{\pi}\int_{m^2}^{\infty}\left[\sum_{j=1}^{6}h_{1j}(-\xi')\,\mathrm{Re}F_j(-\xi')\Big/(\xi'+\xi-i\epsilon)\right]d\xi', \tag{53}$$

where $h_{14}=0$ and $h_{11}\neq 0$.

There is also an integral equation for F_5, if we desire to use it. Since we know $F_5(\xi)$ exactly [see Eq. (30)], we do not need to solve the equation. Actually, we have a rare opportunity of seeing how good is the approximation of retaining only two intermediate states in the sum in (35) by calculating F_5 from the integral equation and comparing the result with the exact result Eq. (30). We shall return to this point later.

We are primarily concerned with determining the high-energy behavior of the F_i. In view of this and the fact that the integrals in the dispersion relations Eq. (23) get their major contribution when $\xi'=-\xi$ we retain only the leading terms in the h_{5j} as $-\xi'\to\infty$. In this way we obtain the following integral equation for F_5

$$F_5(\xi)=-\frac{e}{\xi+m^2}+\frac{1}{\pi}\int_{m^2}^{\infty}\frac{(\alpha/8\pi)\,\mathrm{Re}F_5(-\xi')}{\xi'+\xi-i\epsilon}d\xi',\quad -\xi\gg m^2. \tag{54}$$

This is an integral equation of the Carleman type.[13] Its solution is

$$F_5(\xi)=-\frac{e}{\xi+m^2}\left(\frac{m^2}{\xi+m^2}\right)^{\arctan(\alpha/8\pi)}, \tag{55}$$

where α is the fine structure constant $\alpha=e^2/4\pi$.

We now turn to Eq. (53), the integral equation for F_1. We employ the exact result for F_5 in the right-hand side of (53). In this way we obtain the equation

$$F_1(\xi)=-\frac{e}{\xi+m^2}+\frac{e^3}{8\pi^2}\int_{m^2}^{\infty}\frac{[1-\ln(\xi'/m^2)]\,\mathrm{Re}F_1(-\xi')}{\xi'+\xi-i\epsilon}d\xi',\quad -\xi\gg m^2. \tag{56}$$

The solution of Eq. (56) in the limit of large $|\xi|$ is

$$F_1(\xi)=-\frac{e}{\xi+m^2}\left(\frac{m^2}{\xi+m^2}\right)\exp[\tan^{-1}(\alpha/8\pi)]. \tag{57}$$

[13] F. G. Tricomi, *Integral Equations* (Interscience Publishers, New York, 1957), p. 185. Cf. also, R. Omnes, Nuovo cimento 8, 316 (1958).

Thus, the pole term has been damped by the inclusion of the $|e\gamma\rangle$ intermediate state.[14]

We now consider Eq. (52) for F_4. As was remarked before, this is not an integral equation. If we use our calculated F_1 and exact F_5 on the right-hand side of (52), we can simply integrate to obtain the result that

$$F_4(\xi)=-e/(\xi+m^2)+A(\xi), \tag{58}$$

where $A(\xi)$ is such that

$$\lim_{|\xi|\to\infty} |\xi| A(\xi)=0. \tag{59}$$

We have omitted F_2 and F_3 from the integral on the right-hand side of Eq. (52). Including these terms does not change the result that $F_4=O(\xi^{-1})$.

We summarize the results of the last part of this section as follows. We found that our dispersion theory approach gives a value for $F_5=-[e/(\xi+m^2)][m^2/(\xi+m^2)]^{\alpha/8\pi}$ which disagrees with the exact result $-e/(\xi+m^2)$. Also, the solution of the integral equation for F_1 approaches zero as $|\xi|\to\infty$, faster than the result of perturbation theory indicates. Finally, and most important, we saw that $F_4=O(\xi^{-1})$ for $|\xi|\gg m^2$.

It is worth stressing that the result that Eq. (52) determining F_4 is not an integral equation does not depend on the high-energy limit argument we made in connection with the solution of Eq. (48) for F_5. Rather, it is a rigorous result of the complete system of equations (48) that $F_4=O(\xi^{-1})$ as $-\xi\to\infty$.

V. CALCULATION OF THE RENORMALIZATION CONSTANTS δm AND Z_2

In this section we determine the self-energy and wave-function renormalization constant for the electron in terms of the $F_i(\xi)$.

Let us first observe that from Eqs. (1), (2), and (3) it follows that

$$\delta m=Z_2\int_0^\infty [(m-\kappa)\rho_1(\kappa^2)+\rho_2(\kappa^2)]d\kappa^2$$

$$=Z_2\frac{(2\pi)^3}{4}\int_0^\infty\left\{-\frac{m}{\kappa^2}\operatorname{Tr} i\kappa\cdot\gamma\sum_{p_n=\kappa}\langle 0|\psi|n\rangle\langle n|\bar\psi|0\rangle -\operatorname{Tr}\sum_{p_n=\kappa}\langle 0|\psi|n\rangle\langle n|\bar\psi|0\rangle\right\}. \tag{61}$$

From Eqs. (5) and (6) we see that the single-electron state $|e\rangle$ contributes nothing to the sums in Eq. (61).

In the following we neglect F_2 and F_3 relative to F_1, F_4, and F_5. That is, we do not consider the effect of the F_2 and F_3 on the self-energy. If we retained the terms involving these functions in the expression Eq. (61) for the self-energy, their contributions would be proportional to e^4 at least. The contributions to Eq. (60) from F_1, F_4, and F_5, on the other hand, are proportional to e^2.

From the definitions of the F_i, Eq. (9), we see that

$$\operatorname{Tr}\sum_{p+k=\kappa}\langle 0|\psi|e\gamma\rangle\langle e\gamma|\bar\psi|0\rangle$$

$$=\frac{1}{(2\pi)^6}\int\int\frac{d^3p d^3k}{p_0\omega}\{F_4^*F_5 2mp\cdot k +F_5^*F_4 2mp\cdot k-|F_5|^2 4m^3\}\delta^4(p+k-\kappa), \tag{62}$$

and

$$\operatorname{Tr}\sum_{p+k=\kappa}\frac{m}{\kappa^2}i\kappa\cdot\gamma\langle 0|\psi|e\gamma\rangle\langle e\gamma|\bar\psi|0\rangle=\frac{m}{\kappa^2}\frac{1}{(2\pi)^6}\int\int\frac{d^3pd^3k}{p_0\omega}$$

$$\times\{-|F_1|^2 4(p\cdot k)^2-4\operatorname{Re}(F_1F_5^*)(k\cdot p)^2 +4\operatorname{Re}(F_4F_5^*)p\cdot k(p\cdot k-m^2) -|F_5|^2 4m^2(k\cdot p-m^2)\}\delta^4(p+k-\kappa). \tag{63}$$

Upon performing the integrations in Eqs. (62) and (63) and combining the results with Eq. (61), we find for the self-energy

$$\delta m=Z_2\frac{m}{16\pi^2}\int\{-|F_1(-a)|^2(m^2-a)^3 -\operatorname{Re}[F_1(-a)F_5^*(-a)](m^2-a)^3 -\operatorname{Re}[F_4(-a)F_5^*(-a)](m^2-a)^3 +|F_5(-a)|^2 2m^2(m^2-a)^2\}\frac{da}{a^2}. \tag{64}$$

This equation, along with Eqs. (30), (57), and (58), forms the basis of our quantitative discussion of the self-energy.

We now consider how the pole terms alone affect the self-energy if they are not damped as in (57). That is, we see now what our method of calculation gives when we go to the limit of no interaction in the $|e\gamma\rangle$ state, which corresponds to the transition to lowest order perturbation theory. In this limit

$$F_1(-a)=F_4(-a)=F_5(-a)=-e/(m^2-a), \tag{65}$$

and we find for the self-energy

$$\delta m=Z_2\frac{\alpha m}{4\pi}\int_2^\infty\left(\frac{3}{a}-\frac{m^2}{a^2}\right)da \tag{66}$$

$$=Z_2\frac{\alpha m}{4\pi}\left\{3\int_{m^2}^\infty\frac{da}{a}-1\right\}. \tag{67}$$

Introducing a cutoff at the high momentum limit $(a=M^2)$, we write

$$\delta m=Z_2(\alpha m/2\pi)\{3\ln(M/m)-\tfrac{1}{2}\}. \tag{68}$$

[14] Bogoliubov has obtained a similar expression for electron Green's function in the low-energy limit using the Bloch-Nordsieck model. Bogoliubov and Shirkov, reference 8.

To make a comparison with the lowest order perturbation theory, we should set $Z_2=1$. Note that our answer differs by an additive constant from that which one obtains using perturbation theory in the usual way. The latter yields[10]

$$\delta m=(\alpha m/2\pi)\{3\ln(M/m)+\tfrac{1}{4}\}. \tag{69}$$

The difference arises because of the way the cutoff has been inserted. We observe that the dominant (infinite) term is the same in the results of both our analysis and that of usual perturbation theory.

Let us consider how the F_i which we have calculated affect the self-energy when they are used in Eq. (64). According to (57)

$$\int_{m^2}^{\infty}|F_1|^2(m^2-a)^3\frac{da}{a^2}<\infty. \tag{70}$$

Also, because of (30),

$$\int_{m^2}^{\infty}\mathrm{Re}(F_1F_5^*)(m^2-a)^3\frac{da}{a^2}<\infty. \tag{71}$$

According to Eq. (30)

$$\int_{m^2}^{\infty}|F_5|^2(m^2-a)^2\frac{da}{a^2}<\infty. \tag{72}$$

However, since $F_4=O(\xi^{-1})$ for large $|\xi|$ according to Eq. (58), we see that

$$\int_{m^2}^{\infty}\mathrm{Re}(F_4F_5^*)(m^2-a)^3\frac{da}{a^2}\text{ diverges.} \tag{73}$$

Introducing a cutoff as before, we see that

$$\delta m=(\alpha m/2\pi)\ln(M/m)+\text{a finite constant.} \tag{74}$$

Our calculation of the F_i leads to a self-energy which is only slightly different from Eq. (68).

In calculating the Z_2^{-1} we use Eq. (2) to find that

$$\begin{aligned}\rho_1^{(2)}(\kappa^2)=\frac{m}{\kappa^2}\frac{16\pi}{(2\pi)^3}\left(\frac{m^2-\kappa^2}{2\kappa^2}\right)\Bigg\{&-|F_1|^2\left(\frac{m^2-\kappa^2}{2}\right)^2\\&-\mathrm{Re}(F_1F_5^*)\left(\frac{m^2-\kappa^2}{2}\right)^2\\&-\mathrm{Re}(F_4F_5^*)\left(\frac{m^2-\kappa^2}{2}\right)\left(\frac{m^2+\kappa^2}{2}\right)\\&+|F_5|^2m^2\left(\frac{m^2+\kappa^2}{2}\right)\Bigg\}.\end{aligned} \tag{75}$$

Using Eqs. (57), (58), and (30) in (75), we obtain for Z_2^{-1}

$$\begin{aligned}Z_2^{-1}=1+\frac{1}{16\pi^2}\int_{m^2}^{\infty}\frac{da}{a^2}\{&(a-m^2)^3|F_1|^2\\&+(a-m^2)^3\,\mathrm{Re}(F_1F_5^*)-\mathrm{Re}(F_4F_5^*)(m^2+a)(m^2-a)^2\\&+2m^2|F_5|^2(m^2-a)(m^2+a)\}.\end{aligned} \tag{76}$$

Following the same procedure as in the discussion of δm, we find that

$$Z_2^{-1}=1-\frac{\alpha}{2\pi}\ln(M/m)-\frac{\alpha}{\pi}\ln\lambda+C, \tag{77}$$

where C is a finite constant having no special significance. As before, M is the high-momentum cutoff. The constant λ is a low-momentum cutoff, the use of which serves to avoid an infrared catastrophe. We wish to make the following observations about Eq. (77). The ultraviolet divergence in Eq. (77) is opposite in sign, but similar in origin to the ultraviolet divergence in the perturbation calculation of Z_2^{-1}. The infrared divergence is of exactly the same form in Eq. (77) as it is in perturbation theory. There is one significant difference, however. Since our calculation of F_5 is exact, the statement that Z_2^{-1} contains an infrared divergence is also exact; it is independent of both perturbation theory and our specific approximations concerning the intermediate-state sum.

VI. CONCLUSIONS

We have shown that the use of the dispersion relation technique as applied to quantum electrodynamics does not eliminate the divergences which occur in the perturbation theory results for the electron self-energy and wave-function renormalization constant. In arriving at this result, we have made use of two approximations: (A) We retained only two terms in the sum over intermediate states in Eq. (35); (B) we used the first Born approximation for the matrix element $\langle e\gamma|e'\gamma'\rangle$. We shall now discuss these two approximations.

First of all, we retain only two intermediate states in Eq. (35), because we cannot calculate the matrix elements involving an n-particle state where $n>2$. However, we have the benefit of the experience in meson theory where similar approximations have led to reasonable agreement with data from low-energy experiments. Also, one might argue that n-particle intermediate states contribute to the amplitude terms of the order of e^n when calculated in perturbation theory, and although the expansion in intermediate states is not an expansion in e, we should still make use of such information to give us the order of magnitude of such contributions. These statements tend to make plausible the validity of approximation A, but they stand in need of support of a more rigorous nature. The remarks in the text about $F_5(\xi)$ are of such a nature. From Eqs. (30) and (55) we see that the

approximation A has led to an $F_5(\xi)$ in *apparently* good agreement with the exact F_5. On the other hand, the lowest order perturbation theory approximation reproduces F_5 exactly. Yet one would rather not say that therefore perturbation theory is more accurate than dispersion theory plus approximation A. It is at least satisfying that approximation A does not lead to an answer in gross disagreement with the exact result. However, note that although the disagreement between Eqs. (30) and (55) is slight, it is precisely such a slight difference, which, if it occurred in the case of F_4, could have led to a finite result for both δm and Z_2^{-1}. The main conclusion to be drawn from this may be that with the use of approximation A, the dispersion relation technique is unreliable in the calculation of high-energy effects.

Within the framework of our approximations

$$F_5(\xi)=-\frac{e}{\xi+m^2}\left(\frac{m^2}{\xi+m^2}\right)^{\alpha/8\pi}\quad\text{for}\quad |\xi|\gg m^2.$$

Yet it is an exact consequence of the gauge invariance of electrodynamics that $F_5=-e/(\xi+m^2)$. Therefore, our method of choosing relevant intermediate states in Eq. (33) has destroyed this symmetry of the full theory. Unfortunately, we do not see what finite set of intermediate states would be sufficient to maintain the gauge invariance of the theory in the approximation.

If we were optimistic about these difficulties we should note that the δm is infinite because we have used approximation B. It is easy to convince oneself that a slight modification of the Compton scattering amplitude at high energies would be sufficient to produce an integral equation for F_4 and thus a possible damping of the pole term in Eq. (52) as $-\xi\rightarrow\infty$.

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$K_2{}^0$ and the Equivalence Principle*†

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It is shown that the existence of the long-lived neutral K meson, and the absence of its decay into two pions, establishes that the gravitational masses of the K^0 and $\bar{K}^0$ are equal to a few parts in 10^{-10} of the K inertial mass. This is of interest since the $\bar{K}^0$ is the antiparticle of the K^0, and is not identical with the K^0. The gravitational mass of such a nonidentical antiparticle has never been directly measured.

Also, the $\bar{K}^0$ has opposite strangeness to the K^0. Thus the argument rules out any linear dependence of the gravitational mass on the strangeness quantum number, a point on which all previous experiments say nothing.

These observations are in accord with, and serve as a confirmation of, the equivalence principle of Einstein.

SINCE the discovery of antinucleons, the interesting possibility that antimatter may have gravitational mass opposite in sign to its inertial mass has been widely discussed.[1-3] Although such a possibility would necessarily involve major modifications in present theoretical ideas,[2] it is generally regarded as something to be settled by experiment.

Schiff[4] has recently put forth considerable evidence against the antigravity idea, by showing that negative gravitational mass of the positrons in the virtual pairs of the Coulomb field of the nucleus would very likely (i.e., barring fortuitous cancellations) produce an observable effect in the Eötvös experiment.[5] The argument is necessarily somewhat indirect, since the antiparticles are virtual rather than real. In any case, it is useful to extend the proof to other types of particles.

We consider here the effect of gravity on the $K_2{}^0$, and show that it affords a direct measurement of the difference between the gravitational mass of a particular particle, the K^0, and the gravitational mass of its antiparticle, the $\bar{K}^0$. We conclude that this difference is zero, to an accuracy of a few $\times 10^{-10} M_K$. This is in disagreement with the antigravity hypothesis, and instead affords an extremely precise check, in a new context, of the equivalence principle of Einstein.

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