Total Compton Cross Section for Arbitrary Spin

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The magnitude of the total Compton cross section for arbitrary spin is calculated in lowest order by using the absorptive part of the forward scattering amplitude and the unitarity property of the S matrix. To find the general form (for any spin) of the forward amplitude a universal electromagnetic interaction, implying the value 2 for the gyromagnetic ratio, is postulated. The deduced cross section contains as special cases the known cross sections for spinless and Dirac particles. The average of the cross section over all the possible polarization states of the particle is also performed. In the high-energy region the cross section never increases faster than linearly with the energy of the photon. If a development in powers of the photon energy is made, the first and second coefficients are found to be spin independent. The coefficient of the square of the energy is proportional to the square of the modulus of the spin vector.

1. INTRODUCTION

It should certainly be interesting to be able to see how some electromagnetic properties of the elementary particles vary with the spin. Unfortunately, the complications and the unrenormalizability of the Fierz-Pauli theory,[1] even after the achievement of some simplifications,[2] act as a deterrent against any general calculation and, apart from the static limit, very few results have so far been obtained for spin greater than $\frac{1}{2}$. We will here abandon altogether the above-mentioned theory. Instead, we will try a more naïve approach.

The link between the particle and the electromagnetic field must be provided by a tensor capable of describing the four-current interaction and the magnetic moment interaction. The most natural tensor with that property appears to be the transformation matrix $T_{\mu\nu}{}^{ab}$ defined by

$$\Psi^{a\prime}=\tfrac{1}{4}T_{\mu\nu}{}^{ab}a^{\mu\nu}\Psi_b,$$

where Ψ^a is the wave function and $a^{\mu\nu}$ are the coefficients of an infinitesimal Lorentz transformation: $x^{\mu\prime}=a^{\mu\nu}x_\nu$.

We have[3]

$$T_{\mu\nu}=g_{\mu\nu}+2is_{\mu\nu}, \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor and $s_{\mu\nu}$ is the spin matrix.

Using $T_{\mu\nu}$ the Klein-Gordon equation can be written

$$(T_{\mu\nu}p^\mu p^\nu+m^2)\Psi=(p^2+m^2)\Psi=0.$$

Now, the well-known gauge-invariant substitution yields

$$[T_{\mu\nu}(p^\mu-eA^\mu)(p^\nu-eA^\nu)+m^2]\Psi=0,$$

i.e.,

$$(p^2+m^2)\Psi=I\Psi, \tag{2}$$

$$I=T_{\mu\nu}(ep^\mu A^\nu+eA^\mu p^\nu-e^2A^\mu A^\nu). \tag{3}$$

This interaction has been investigated with some detail elsewhere[4] and is the one we intend to use in the present paper. The number 2 appearing in (1) as a coefficient of the spin matrix turns out to be the gyromagnetic factor.[5] This should be compared with the value given by the Fierz-Pauli theory which is the inverse of the spin.[6]

The usual perturbation treatment of the equation of motion can be carried out, but we will need only the part of the theory that is related to the determination of the forward amplitude. For this reason we think it better to introduce as a postulate the part that is essential for the problem we have in hand. So, let us start with the following *hypothesis: The forward scattering amplitude for particles of arbitrary spin can be calculated by means of the Feynman techniques*[7] *using the simple (universal) propagator* $(p^2+m^2)^{-1}$ *and the universal electromagnetic interaction (3).*

Of course, it is necessary to say something more in favor of the adoption of such an assumption. For this purpose let us point out that: (1) the procedure is evidently gauge invariant; (2) in the usual theory for spinless particles the postulate is true; (3) for spin $\frac{1}{2}$ the results obtained with the use of the postulate are coincident with those obtained with the Dirac theory.[4,8]

We think that these three points, together with the generality and simplicity of the procedure, justify the use of the postulate from a theoretical point of view.

2. MATRIX ELEMENTS

From the unitarity of the S matrix it follows[9] that

$$\sigma(\omega)=(4\pi/\omega)\,\mathrm{Im}\,a(\omega), \tag{4}$$

where $a(\omega)$ is the forward scattering amplitude and ω is

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[1] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).
[2] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941); P. A. Moldauer and K. M. Case, Phys. Rev. **102**, 279 (1956); C. Fronsdal, Suppl. Nuovo cimento **9**, 416 (1958).
[3] E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (Blackie and Son Limited, London and Glasgow, 1953), Chap. 1.
[4] C. G. Bollini, Nuovo cimento **14**, 560 (1959).
[5] Reference 4, Sec. 7.
[6] C. Fronsdal, reference 2, Sec. 8.
[7] R. P. Feynman, Phys. Rev. **76**, 769 (1949).
[8] This will also be shown in Secs. 5 and 6 for the total cross section.
[9] See, for instance, J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addinson-Wesley Publishing Company, Inc., Reading, Massachusetts 1955), Appendix A7.

the energy of the photon. To calculate $\sigma(\omega)$ in the lowest order we shall have to calculate the forward amplitude in fourth order. [In second order $a(\omega)=-e^2/4\pi m=-\alpha/m$, whatever the spin.]

There are nine fourth order Feynman diagrams contributing to $\sigma(\omega)$. They are shown in Fig. 1.

The rules for obtaining the forward amplitude from those diagrams are the following.[10] One must write down a factor $p_\mu+p_\mu'+2is_{\mu\nu}q^\nu$ for each single corner where p_μ (p_μ') is the momentum of the incoming (outgoing) particle and q_μ the momentum of the outgoing photon. There is also a factor $-2g_{\mu\nu}$ for each double corner. Each internal particle line contributes with a factor $(p^2+m^2)^{-1}$ and the internal photon line with the factor $g_{\mu\nu}/q^2$. One must integrate over the four components of the momentum of the virtual photon. There is a polarization vector ϵ_μ for each external photon line. For the external incoming particle there is a polarization tensor or tensor-spinor u.[11] The external outgoing particle is represented by the conjugated entity $\bar{u}$, $\bar{u}u=1$. Finally, there is an over-all constant coming from the perturbation expansion and the normalization of the wave functions and propagators. It is $mr_0^2/8\pi^3$, where r_0 is the classical particle radius.

3. TECHNIQUES OF CALCULATION

We will now describe the method of calculation we have used. The labor may be divided into several parts. As a first step, the complete integral corresponding to each diagram is written down according to the rules stated in Sec. 2. In the second step, all the denominators are combined into a single one by introducing the Feynman parameters. After that the denominator have the form $(x-i\xi)^n$, where x is real and $i\xi$ is the infinitesimal imaginary constant defining the Feynman path of integration. Because we are only interested in the absorptive part of the scattering amplitude we need only the imaginary part of $(x-i\xi)^{-n}$. For $n=1$, $\mathrm{Im}(x-i\xi)^{-1}=\pi\delta(x)$, so that, by taking the nth derivative, we obtain

$$\mathrm{Im}(x-i\xi)^{-n-1}=(-1)^n\pi\delta^{(n)}(x)/n!. \tag{5}$$

After the use of (5), no denominator is left in the integrand. A change of variable may now be used to reduce the argument of the $\delta^{(n)}$ function to the form q^2-y, where y is q-independent and $q^2=q_\mu q^\mu$. The next step consists of the elimination of the odd powers of q_μ, the rotation of the path of integration so as to obtain an Euclidean metric, and the introduction of spherical coordinates.[12] As a result of the last operation the integration $\int d^4q$ is transformed into $\pi^2\int\lambda d\lambda$, $\lambda=q^2$.[12] The integral over the virtual momentum has the form

$$\int_0^\infty F(\lambda)\delta^{(n)}(\lambda-y)d\lambda. \tag{6}$$

Introducing the Heaviside function $\theta(\lambda)$, (6) may be written

$$\int_{-\infty}^\infty \Theta(\lambda)F(\lambda)\delta^{(n)}(\lambda-y)d\lambda, \tag{7}$$

and the λ integration can easily be performed.

$$\int_{-\infty}^\infty \Theta(\lambda)F(\lambda)\delta^{(n)}(\lambda-y)d\lambda =(-1)^n\left|\frac{d^n}{d\lambda^n}[\Theta(\lambda)F(\lambda)]\right|_{\lambda=y}. \tag{8}$$

The final step, consisting of the remaining integration over the Feynman parameters, can be carried out without difficulties if due care is taken of the presence of $\Theta(y)$ and its derivatives in the integrand.

In Appendix I we will give as an example the calculation of the contribution of diagrams II′ and II″ of Fig. 1.

4. RESULTS OF THE CALCULATION

The following gauge and notation has been used: $\epsilon_\mu\epsilon^\mu=\epsilon\cdot\epsilon=1$, $\epsilon\cdot k=0$, $\epsilon\cdot p=0$, $k^2=0$, $p^2=-m^2$, $p\cdot k=-m\omega$, $\gamma=\omega/m$, $\rho=(1+2\gamma)^{-1}$, $\lambda=-(\ln\rho)/2\gamma$, $K_\mu=s_{\mu\nu}k^\nu/\omega$, $K_\epsilon=K\cdot\epsilon$, $P_\mu=s_{\mu\nu}p^\nu/m$, $P_\epsilon=P\cdot\epsilon$, $S=(\epsilon\cdot s)\cdot(s\cdot\epsilon)=\epsilon_\mu s^{\mu\nu}s_{\nu\rho}\epsilon^\rho$, $a=2\pi r_0^2$. σ^j is the partial contribution of the jth diagram of Fig. 1 to the total cross section.

$$\sigma=\sum_{j=I}^{VI}\sigma^j,$$

$$\sigma^{I}=2a\rho,$$

$$\sigma^{II}=2a\gamma\rho^2\bar{u}(P_\epsilon K_\epsilon+K_\epsilon P_\epsilon+2\gamma K_\epsilon^2)u,$$

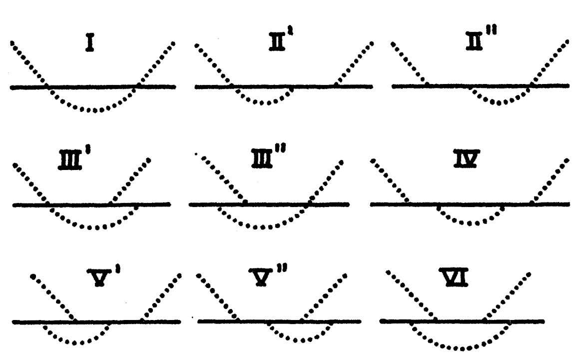

FIG. 1. Fourth order Feynman diagrams with nonzero contribution to the absorptive part of the forward scattering amplitude.

[10] F. Rohrlich, Phys. Rev. **80**, 666 (1950); S. S. Schweber, H. A. Bethe, and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, Illinois, 1956), Vol. I, Sec. 19d. The rules given by these authors correspond to the spin zero case, but the addition of the spin term causes no difficulty.

[11] We are not going to write explicitly the indices of the polarization representative u of the particle. For integer spin, u is a symmetric and traceless tensor of the sth rank, satisfying the Lorentz condition. For half-integer spin $s=n+\frac{1}{2}$, u is a symmetric and traceless tensor of the nth rank plus a first degree spinor (Rarita-Schwinger representation). In addition u satisfies the Dirac equation and is perpendicular to γ_μ.

[12] Reference 9, Appendix A5-2.

$$\sigma^{\rm III}=(a/\gamma)(-1+\gamma\rho+\lambda)(1+4\gamma\bar{u}K_e^2u)+2a(\rho-\lambda)\bar{u}(P_eK_e+K_eP_e)u,$$

$$\sigma^{\rm IV}=-a(1+\rho)\bar{u}K_e^2u+\tfrac{2}{3}a\gamma^2\rho^2\bar{u}K_e[s_{\mu\nu}s^{\mu\nu}+4\rho(P+\gamma K)^2]K_eu,$$

$$\begin{aligned}\sigma^{\rm V}=2a[2+\rho-(1+\gamma)\lambda]\bar{u}K_e^2u
&+(a/3\gamma)(1+2\rho+\gamma\rho^2-3\lambda)\bar{u}(2S-K\cdot P-P\cdot K)u\\
&+(a/3\gamma)(3-\gamma+3\rho+\gamma\rho^2-6\lambda)\bar{u}K^2(1+3\gamma K_e^2)u\\
&+a(-1+\gamma\rho+\lambda)\bar{u}(K_es_{\mu\nu}K_es^{\mu\nu}+s_{\mu\nu}K_es^{\mu\nu}K_e)u\\
&+a(3\rho-\rho^2-2\lambda)\bar{u}(K_eP_\mu K_eP^\mu+P_\mu K_eP^\mu K_e)u\\
&+a(-1-2\rho-\gamma\rho^2+3\lambda)\bar{u}(K_eK_\mu K_eP^\mu\\
&\quad+K_eP_\mu K_eK^\mu+K_\mu K_eP^\mu K_e+P_\mu K_eK^\mu K_e)u,\end{aligned}$$

$$\begin{aligned}\sigma^{\rm VI}=(a/\gamma^2)[2+(1+\gamma)\gamma\rho-(2+3\gamma)\lambda]
&+(a/3\gamma^2)[3+\gamma^2\rho-3(1+\gamma)\lambda]\bar{u}(s_{\mu\nu}s^{\mu\nu}-2S)u\\
&+(a/\gamma^2)[-3-\rho+\tfrac{2}{3}\gamma^2\rho^2+2(2+\gamma)\lambda]\bar{u}P^2u\\
&+(a/\gamma^2)[5-(1-\gamma)\gamma\rho-\tfrac{1}{3}\gamma^2\rho^2\\
&-(5+4\gamma)\lambda]\bar{u}(P\cdot K+K\cdot P)u\\
&+(a/\gamma^2)[-5+\tfrac{1}{3}\gamma^2\rho(2\gamma^2\rho-5)+5(1+\gamma)\lambda]\bar{u}K^2u\\
&+2a[1-(2+\gamma)\lambda]\bar{u}K_e^2u\\
&+2a[-1+(1+\gamma)\lambda]\bar{u}s_{\mu\nu}K_e^2s^{\mu\nu}u\\
&+2a(1+\rho-2\lambda)\bar{u}P_\mu K_e^2P^\mu u\\
&+2a[3+\gamma^2\rho-3(1+\gamma)\lambda]\bar{u}K^2K_e^2u\\
&+2a[-3+\gamma\rho+(3+2\gamma)\lambda]\bar{u}(P_\mu K_e^2K^\mu+K_\mu K_e^2P^\mu)u.\end{aligned}$$

5. AVERAGE OVER THE SPIN STATES

The formulas written down in Sec. 4 are not suitable for a general discussion because they are rather long and explicitly contain the state of polarization of the particle. It is convenient to deduce the average of the cross section over all the possible spin states of the particle. It is easy to realize that all we need is the following two average values:

$$S_{\alpha\beta,\mu\nu}=\mathrm{Av}[\bar{u}s_{\alpha\beta}s_{\mu\nu}u], \tag{9}$$

and

$$T_{\alpha\beta}=\mathrm{Av}[\bar{u}(K_es_{\alpha\rho}K_es_\beta{}^\rho+s_{\alpha\rho}K_es_\beta{}^\rho K_e)u]. \tag{10}$$

They could be found by noting that $u\bar{u}$ may be replaced by the projection operator appropriate to the spin of the particle. Unfortunately the general expression for the projection operator[13] is unsuitable for our purpose. It is here preferable to use only arguments of relativistic covariance.

The tensor $S_{\alpha\beta,\mu\nu}=S_{\mu\nu,\alpha\beta}=-S_{\beta\alpha,\mu\nu}$ can only be formed out of products of the metric tensor $g_{\mu\nu}$ and the impulse vector p_μ. It must then be of the form

$$\begin{aligned}S_{\alpha\beta,\mu\nu}=A(g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\nu}g_{\beta\mu})+(B/m^2)\\
\times(g_{\alpha\mu}p_\beta p_\nu+g_{\beta\nu}p_\alpha p_\mu-g_{\alpha\nu}p_\beta p_\mu-g_{\beta\mu}p_\alpha p_\nu),\end{aligned} \tag{11}$$

$T_{\alpha\beta}$ is a second rank symmetric tensor that depends on $g_{\alpha\beta}$, p_α and also k_α. (The average values are ϵ-independent.)

$$T_{\alpha\beta}=Ck_\alpha k_\beta/\omega^2+D(p_\alpha k_\beta+k_\alpha p_\beta)/m\omega+Ep_\alpha p_\beta/m^2+Fg_{\alpha\beta}. \tag{12}$$

[13] R. E. Behrends and C. Fronsdal, Phys. Rev. **106**, 345 (1957).

In Appendix II it is shown how to get the values of the spin-dependent constants of (11) and (12). The result is

$$\begin{aligned}&A=s(s+1)/3,\quad B=(2s^2-\eta)/6,\quad C=3B^2+2B-3G,\\
&D=-2G,\qquad E=8G,\qquad F=6AB-C-2G,\\
&G=(1/15)[s^2(s-1)^2-\eta(s^2-s+\tfrac{3}{8})],\end{aligned}$$

where $\eta=0$ for $s=$integer and $\eta=\frac{1}{2}$ for $s=$half an odd integer.

Using (11), (12), and the equality $E=-4D$, we obtain for the average of the total cross section over all the possible spin states of the particle

$$\begin{aligned}\sigma(s,\gamma)=(a/\gamma^2)(1+\rho-2\lambda)\\
\times[1+\gamma+2\gamma^2(-A+B+C)-\gamma^3D]\\
+\tfrac{4}{3}a\gamma^2\rho^2[6A+(\rho-3)(B+C)+\gamma D].\end{aligned} \tag{13}$$

The first few particular cases are:

$$\sigma(0,\gamma)=(a/\gamma^2)(1+\gamma)(1+\rho-2\lambda), \tag{14}$$

$$\sigma(\tfrac{1}{2},\gamma)=\sigma(0,\gamma)-a(\rho+\gamma\rho^2-\lambda), \tag{15}$$

$$\sigma(1,\gamma)=\sigma(0,\gamma)+\tfrac{4}{3}a(1+\rho+\tfrac{4}{3}\gamma^2\rho^3-2\lambda), \tag{16}$$

$$\sigma(\tfrac{3}{2},\gamma)=\sigma(0,\gamma)+(5/3)a[2+3\rho+\gamma\rho^2+(8/3)\gamma^2\rho^3-5\lambda]. \tag{17}$$

6. DISCUSSION

The result given in formula (13) shows some expected features which are shared with the cross section of Sec. 4. First of all, the limit as $\gamma\rightarrow 0$ gives the Thomson cross section whatever the spin:

$$\lim_{\gamma\rightarrow 0}\sigma(s,\gamma)=\sigma(s,0)=4a/3=8\pi r_0^2/3.$$

This is actually a consequence of the gauge invariance.[14] Second, $d\sigma(s,0)/d\gamma$ is also spin independent, as it should be.[15] Third, (14) is the usual cross section for spinless particles. Fourth, (15) is the known total Compton cross section for Dirac particles.[16] (14) and (15) could also be obtained directly from the formulas of Sec. 4. (There is no need of an average for these two cases.)

The behavior of (13) in the low-energy region is best expressed by a development of $\sigma(s,\gamma)/\sigma(0,\gamma)$ in powers of γ. We have

$$\sigma(s,\gamma)/\sigma(0,\gamma)=1+4A\gamma^2+\text{higher order terms},$$

i.e.,

$$\sigma(s,\gamma)/\sigma(0,\gamma)=1+\tfrac{4}{3}s(s+1)\gamma^2+\cdots, \tag{18}$$

and the coefficient of γ^2 turns out to be proportional to the square of the modulus of the spin vector.

[14] N. Kroll and M. Ruderman, Phys. Rev. **93**, 233 (1954).

[15] F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954). I am indebted to Professor A. Salam for calling my attention to these papers.

[16] The known total Compton cross section for Dirac particles has been derived from the absorptive part of the forward scattering amplitude by L. M. Brown and R. P. Feynman, Phys. Rev. **85**, 231 (1952).

The behavior in the extreme relativistic region varies with the spin. For $s=0$ the cross section decreases like $1/\gamma$ and for $s=\frac{1}{2}$ it decreases like $\ln\gamma/\gamma$. For $s=1$ it asymptotically approaches the Thomson cross section and for $s=\frac{3}{2}$ it approaches 2.5 times that value. For $s=2$ and greater, the cross section increases linearly with γ. The leading term of (13) in the extreme relativistic region is

$$\sigma_r(s,\gamma)=\tfrac{4}{3}aG(s)\gamma, \tag{19}$$

G being the function of s given in Sec. 5. Although the coefficient of γ in (19) is strongly spin dependent, in no case does the cross section increase faster than linearly. This should be contrasted with the behavior found by Mathews[17] using the Fierz-Pauli theory. He found that for $s=\frac{3}{2}$ the total cross section increases cubically with the energy of the photon.

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APPENDIX I

We shall calculate, as an example, the contribution of diagrams II′ and II″ of Fig. 1 to the total scattering cross section. The gauge and notation are those of Sec. 4.

Using the rules stated in Sec. 2, we obtain

$$a^{\mathrm{II}'}=\left(\frac{mr_0^2}{8\pi}\right)(-2)\bar{u}\times\int d^4q\frac{(-\epsilon\cdot q-2i\epsilon\cdot s\cdot q)2i\epsilon\cdot s\cdot k}{q^2[(p+k-q)^2+m^2][(p+k)^2+m^2]}u,$$

$$a^{\mathrm{II}''}=\left(\frac{mr_0^2}{8\pi}\right)(-2)\bar{u}\times\int d^4q\frac{(-2i\epsilon\cdot s\cdot k)(-\epsilon\cdot q+2i\epsilon\cdot s\cdot q)}{q^2[(p+k-q)^2+m^2][(p+k)^2+m^2]}u.$$

The total amplitude $a^{\mathrm{II}}=a^{\mathrm{II}'}+a^{\mathrm{II}''}$ is

$$a^{\mathrm{II}}=-\frac{mr_0^2}{2p\cdot k\pi}\bar{u}\int d^4q\,D^{-1}(\epsilon\cdot s\cdot q\epsilon\cdot s\cdot k+\epsilon\cdot s\cdot k\epsilon\cdot s\cdot q)u,$$

where $D=q^2[(p+k-q)^2+m^2]$.

Combining the q-dependent denominators by means of the Feynman formula,

$$D^{-1}=\int_0^1 dx[(q-xp-xk)^2+m^2x^2-2m\omega x(1-x)]^{-2},$$

and using (5) to find the imaginary part of $1/D$,

$$\mathrm{Im}(1/D)=-\pi\int_0^1 dx\,\delta'(z),$$

$$z=(q-xp-xk)^2+m^2x^2-2m\omega x(1-x),$$

we can obtain the absorptive part of a^{II}:

$$\mathrm{Im}a^{\mathrm{II}}=-\frac{r_0^2}{2\omega\pi^2}\bar{u}\int_0^1 dx\times\int d^4q(\epsilon\cdot s\cdot q\epsilon\cdot s\cdot k+\epsilon\cdot s\cdot k\epsilon\cdot s\cdot q)\delta'(z).$$

Multiplying by $4\pi/\omega$ [see Eq. (4)] and changing the variable of integration $q\rightarrow q+xp+xk$, we find after elimination of the odd powers of q

$$\sigma^{\mathrm{II}}=-\frac{2r_0^2}{\pi\omega^2}\mathbf{Q}\int_0^1 dx\,x\int d^4q\,\delta'[q^2+m^2x^2-2m\omega x(1-x)],$$

where

$$\mathbf{Q}=\bar{u}(\epsilon\cdot s\cdot p\epsilon\cdot s\cdot k+\epsilon\cdot s\cdot k\epsilon\cdot s\cdot p+2\epsilon\cdot s\cdot k\epsilon\cdot s\cdot k)u$$
$$=m\omega\bar{u}(P_eK_e+K_eP_e+2\gamma K_e^2)u.$$

A further change of variable brings the integral into the form[12]

$$\sigma^{\mathrm{II}}=-\frac{a}{\omega^2}\mathbf{Q}\int_0^1 xdx\times\int_0^\infty \lambda d\lambda\,\delta'[\lambda+m^2x^2-2m\omega x(1-x)],\quad \lambda=q^2.$$

According to (6), (7), and (8), we have

$$\sigma^{\mathrm{II}}=-\frac{a}{\omega^2}\mathbf{Q}\int_0^1 xdx\int_{-\infty}^\infty \lambda d\lambda\,\Theta(\lambda)\times\delta'[\lambda+m^2x^2-2m\omega x(1-x)],$$

$$\sigma^{\mathrm{II}}=+\frac{a}{\omega^2}\mathbf{Q}\int_0^1 xdx\left|\frac{d}{d\lambda}[\lambda\Theta(\lambda)]\right|_{\lambda=2m\omega x(1-x)-m^2x^2},$$

$$\sigma^{\mathrm{II}}=+\frac{a}{\omega^2}\mathbf{Q}\int_0^1 xdx\,\Theta[2m\omega x(1-x)-m^2x^2].$$

For the integrand to be different from zero, we must have

$$x<2\omega(1-x)/m,\quad x<2\gamma\rho,$$

so that

$$\sigma^{\mathrm{II}}=\frac{a}{\omega^2}\mathbf{Q}\int_0^{2\gamma\rho} xdx=\frac{a}{\omega^2}\mathbf{Q}(2\gamma^2\rho^2),$$

$$\sigma^{\mathrm{II}}=2a\gamma\rho^2\bar{u}(P_eK_e+K_eP_e+2\gamma K_e^2)u,$$

which is the result given in Sec. 4.

[17] J. Mathews, Phys. Rev. **102**, 270 (1956).

APPENDIX II

The average values needed in Sec. 5 can both be determined from the tensor

$$\mathrm{Av}[\bar{u}s_{\alpha\beta}s_{\mu\nu}s_{\lambda\rho}s_{\sigma\tau}u]. \tag{A,1}$$

This tensor may only be formed out of products of the metric tensor $g_{\mu\nu}$ and the impulse vector p_μ. A simplification can be achieved by noting that an asymmetry between any pairs of indices (say $\alpha\beta$ and $\lambda\rho$) can be reduced away by using the commutation relations

$$[s_{\alpha\beta},s_{\lambda\rho}]=i(\delta_{\alpha\lambda}s_{\beta\rho}+\delta_{\beta\rho}s_{\alpha\lambda}-\delta_{\alpha\rho}s_{\beta\lambda}-\delta_{\beta\lambda}s_{\alpha\rho}).$$

In this way the tensor (A,1) is decomposed in two parts: one part which is a tensor completely symmetric in all pairs of indices (and of course antisymmetric in the components of each pair), and a second part which depends on the average of three spin matrices. The process can be continued and, when completed, we are left with a tensor having a known form but with several spin-dependent constants which remain to be determined. Such is, for example, the case with the tensors given in (11) and (12).

For the determination of the constants we must look for the invariants that may be used to characterize the irreducible representations[18] $D(k,l)$ of the Lorentz group. The quantities k and l are eigenvalues of the square of the modulus of the two vectors[19]

$$K_\rho=(i/4m)p^\lambda(\delta_\lambda{}^\mu\delta_\rho{}^\nu-\delta_\rho{}^\mu\delta_\lambda{}^\nu-\epsilon_{\lambda\rho}{}^{\mu\nu})s_{\mu\nu},$$
$$L_\rho=(i/4m)p^\lambda(\delta_\lambda{}^\mu\delta_\rho{}^\nu-\delta_\rho{}^\mu\delta_\lambda{}^\nu+\epsilon_{\lambda\rho}{}^{\mu\nu})s_{\mu\nu},$$

the eigenvalues of $K_\rho K^\rho$ being $k(k+1)$ and those of $L_\rho L^\rho$ being $l(l+1)$. For the symmetric and traceless tensors, $k=l=\frac{1}{2}s$. For the Rarita-Schwinger representation we have either $k=l+\frac{1}{2}=\frac{1}{2}s+\frac{1}{4}$ or $k=l-\frac{1}{2}=\frac{1}{2}s-\frac{1}{4}$. ($k$ and l are not good quantum numbers in this case.)

It is not difficult to see that

$$K_\rho K^\rho=\tfrac{1}{8}s_{\mu\nu}s^{\mu\nu}+\tfrac{1}{16}\epsilon^{\mu\nu\lambda\rho}s_{\mu\nu}s_{\lambda\rho},$$
$$L_\rho L^\rho=\tfrac{1}{8}s_{\mu\nu}s^{\mu\nu}-\tfrac{1}{16}\epsilon^{\mu\nu\lambda\rho}s_{\mu\nu}s_{\lambda\rho},$$

or

$$s_{\mu\nu}s^{\mu\nu}=4(K_\rho K^\rho+L_\rho L^\rho), \tag{A,2}$$
$$\epsilon^{\mu\nu\lambda\rho}s_{\mu\nu}s_{\lambda\rho}=8(K_\rho K^\rho-L_\rho L^\rho). \tag{A,3}$$

In spite of the fact that k and l are not good quantum numbers for $s=$ half an odd integer, the invariant $s_{\mu\nu}s^{\mu\nu}$ commutes with any component of the spin matrix and is a good quantum number. For integer spin

$$s_{\mu\nu}s^{\mu\nu}=4[\tfrac{1}{2}s(\tfrac{1}{2}s+1)+\tfrac{1}{2}s(\tfrac{1}{2}s+1)], \tag{A,4}$$

and for half-integer spin

$$s_{\mu\nu}s^{\mu\nu}=4[(\tfrac{1}{2}s+\tfrac{1}{4})(\tfrac{1}{2}s+5/4)+(\tfrac{1}{2}s-\tfrac{1}{4})(\tfrac{1}{2}s+\tfrac{3}{4})]. \tag{A,5}$$

(A,4) and (A,5) are particular cases of

$$s_{\mu\nu}s^{\mu\nu}=2s(s+2)+\eta, \tag{A,6}$$

where $\eta=0$ for $s=$ integer and $\eta=\frac{1}{2}$ for $s=$ half-integer.

The left-hand side of (A,3) also commutes with any component of the spin matrix. That invariant is zero for integer spin and may have two equal and opposite eigenvalues for half-integer spin. The square of that invariant is then a good quantum number.

$$(\epsilon^{\mu\nu\lambda\rho}s_{\mu\nu}s_{\lambda\rho})^2=32(s+1)^2\eta. \tag{A,7}$$

There are other numbers that are useful for the determination of the average (A,1). When all the involved indices are spatial, then

$$s_{ij}s^{ij}=2s(s+1),\quad (i,\,j=1,\,2,\,3). \tag{A,8}$$

Also, the average of any power of a component (say s_{12}) can be determined by a mere counting:

$$\mathrm{Av}[\bar{u}(s_{12})^4u]=s(s+1)[3s(s+1)-1]/15. \tag{A,9}$$

(A,6) to (A,9) are enough for the determination of the spin-dependent constants of (A,1). The so-obtained values of the constants has been checked by a direct calculation using the actual form of the projection operators and the spin matrices for the cases $s=\frac{1}{2}$, $s=1$, and $s=\frac{3}{2}$.

[18] E. M. Corson, reference 3, Chap. II, Sec. 17.
[19] Reference 3, p. 45.