Some Singularities of Scattering Amplitudes on Unphysical Sheets

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An investigation of the consequences on unphysical sheets of a Mandelstam-type representation holding for a two-particle scattering amplitude on the physical sheet is described. The domain of analyticity in the energy and momentum transfer variables is constructed and compared with perturbation theory. A theorem on multiplication of singularities for Legendre polynomial expansions used in our discussion is proved.

1. INTRODUCTION

THIS paper is a direct continuation of a previous paper[1] on unstable particles (hereafter denoted by UP). We investigate more fully the implications for our previous results of a Mandelstam-type representation holding for the two-particle scattering amplitude on the physical sheet. In Sec. 2, the domain of analyticity in both variables on the first unphysical sheet is constructed. In Sec. 3 the results are compared with perturbation theory, which is used to provide a simple picture of the nature and origin of the extra singularities appearing on the unphysical sheet. At the same time we obtain some results on the existence of singularities of perturbation theory on unphysical sheets. An important theorem on "multiplication of singularities" for Legendre polynomial expansions is proved in the Appendix, as we have not found a statement or proof of it in the standard literature.

2. TWO-PARTICLE SCATTERING AMPLITUDE ON UNPHYSICAL SHEETS

In UP, the continuation of the two-particle scattering amplitude $M(W^2,\Delta^2)$ round the lowest branch point $W^2=(m+\mu)^2$ is given by

$$M'(W^2,\Delta^2)=8\pi\frac{W}{K}\sum_{l=0}^{\infty}(2l+1)\frac{C_l(W^2)}{1-2iC_l(W^2)}P_l(\cos\theta), \quad (2.1)$$

using the same notation. For fixed W^2, the semiaxis sum $R(W^2)$ of the ellipse of convergence of this series is given by

$$R(W^2)=\liminf_{l\to\infty}\left|\frac{C_l(W^2)}{1-2iC_l(W^2)}\right|^{-1/l}$$
$$=\liminf_{l\to\infty}|C_l(W^2)|^{-1/l}, \quad (2.2)$$

which is just the expression for the ellipse of convergence on the physical sheet. This ellipse exists for all W^2 except for values of W^2 on the left-hand cuts in the partial wave amplitudes $C_l(W^2)$. As convergence is uniform on any closed subdomain of this ellipse, analyticity in $\cos\theta$ for fixed W^2 is assured inside the ellipse. However, the Mandelstam representation for $M(W^2,\Delta^2)$ implies more than this. For sufficiently large $l>L$, say, and fixed W^2, we can write (2.1) as

$$M'(W^2,\Delta^2)=8\pi\frac{W}{K}\bigg[\sum_{l=0}^{L}(2l+1)\frac{C_l(W^2)}{1-2iC_l(W^2)}P_l(\cos\theta)$$
$$+\sum_{L+1}^{\infty}(2l+1)C_l(W^2)P_l(\cos\theta)+\sum_{L+1}^{\infty}(2l+1)$$
$$\times 2iC_l^2(W^2)P_l(\cos\theta)+\cdots+\sum_{L+1}^{\infty}(2l+1)$$
$$\times\frac{(2i)^{n-1}C_l^n(W^2)}{1-2iC_l(W^2)}P_l(\cos\theta)\bigg]$$
$$=A_0+A_1+A_2+\cdots+A_{n-1}+B_n, \text{ say.} \quad (2.3)$$

The domains of analyticity in $\cos\theta$ of each term of this expansion can be determined from the known singularities of the Mandelstam representation and an application of theorem 1 of the Appendix. For example, the singularities of A_2 are found at the points $\{\alpha\alpha'+(\alpha^2-1)^{\frac{1}{2}}(\alpha'^2-1)^{\frac{1}{2}}\}$, where α and α' are two singularities of A_1, which is essentially the amplitude on the physical sheet. Repeated applications of the theorem give the singularities of the other terms. The remainder B_n is analytic inside a large ellipse of semiaxis sum

$$\liminf_{l\to\infty}\left|\frac{(2i)^{n-1}C_l^n(W^2)}{1-2iC_l(W^2)}\right|^{-1/l}$$
$$=[\liminf_{l\to\infty}|C_l(W^2)|^{-1/l}]^n=R^n(W^2), \quad (2.4)$$

which becomes arbitrarily large as $n\to\infty$. That all the extra singularities are definitely present follows from the fact that the multiplied coefficients in this case are simply integral powers of the original ones.

If we consider a path from a point $W^2<0$ on the physical sheet going round the branch point $W^2=(m+\mu)^2$ and back to the same point on the first unphysical sheet, then initially we have cut plane analyticity in $\cos\theta$ with branch points determined by the physical thresholds

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[1] J. Gunson and J. G. Taylor, Phys. Rev. **119**, 1121 (1960).

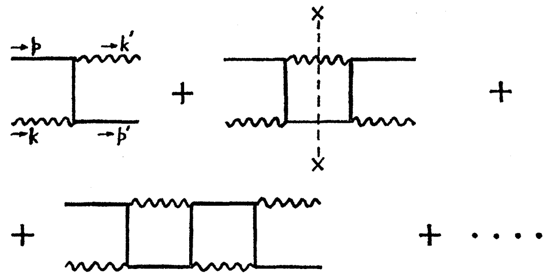

FIG. 1. Subset of perturbation series satisfying unitarity condition for $(m+\mu)^2<W^2<(m+2\mu)^2$; $2\mu<m$. The line $\times\times$ shows a transverse dissection into lower order diagrams.

$\Delta^2=(2\mu)^2$, $(3\mu)^2$, etc., and for the crossed processes. This condition holds until we reach the boundary of the physical sheet at $(m+\mu)^2<W^2<(m+2\mu)^2$ when we still have cut plane analyticity in $\cos\theta$, but with all the extra branch points now appearing on the normal cuts. As W^2 becomes complex in the unphysical sheet, these extra branch points swing out from the original cuts into the complex $\cos\theta$ plane and appear in most directions from the origin for sufficiently large $\cos\theta$.

From these observations it is possible to construct the domain of analyticity of $M'(W^2,\Delta^2)$ in both variables. A point (W^2,Δ^2) is in this domain if M' is analytic in a neighborhood of this point in each variable separately.[2] M' is thus analytic in the usual domain of the Mandelstam representation except for the poles at $1-2iC_l(W^2)=0$, the kinematic cuts in W^2 which appear in the partial wave amplitudes and the complex cuts in the $\cos\theta$ planes described in the previous paragraph.

3. COMPARISON WITH PERTURBATION THEORY

The Mandelstam representation has been verified for some orders of perturbation theory,[3,4,4a] and the singularities of the fourth order graphs have been worked out in detail. However, it is of little value to compare analyticity regions on unphysical sheets of the fourth order amplitude directly with our results, as a single order of perturbation theory does not satisfy the unitarity condition as used in UP. In effect we need to consider an infinite subset of the perturbation series, containing at least the ladder diagrams (Fig. 1). Let the formal sum of this series be

$$T(W^2,\Delta^2)=\sum_{n=1}^{\infty} g^{2n}T^{(2n)}(W^2,\Delta^2), \tag{3.1}$$

where g is the (unique) coupling constant. For $(m+\mu)^2<W^2<(m+2\mu)^2$ the unitarity condition implies $\mathrm{Im}T_l(W^2)=|T_l(W^2)|^2$ for the lth partial wave. Substituting 3.1 and comparing coefficients, we get

$$\mathrm{Im}T_l^{(2n)}(W^2)=\sum_{r=1}^{n-1} T_l^{*[2(n-r)]}(W^2)T_l^{(2r)}(W^2). \tag{3.2}$$

In particular, the fourth order graph (Fig. 1) has the imaginary part $T_l^{*(2)}(W^2)T_l^{(2)}(W^2)$, where the second order amplitudes are just those obtained by cutting the graph with the line $\times\times$. The analysis of Tarski[4] gives the possible singularity spectrum for real values of the invariant variables as shown in Fig. 2, using his notation. For convenience we have taken $m=2\mu$, when the only singularities on the physical sheet are the branch points at $y_{13}=-1$ and $y_{24}=-1$. Continuation into the first unphysical sheet corresponds in this case to continuation around $y_{24}=-1$, and the continued function is obtained simply by adding the jump across the cut $y_{24}<-1$, which is just $2i\,\mathrm{Im}T^{(4)}$, where

$$\begin{aligned}\mathrm{Im}T^{(4)}(W^2,\Delta^2)&\\ &=8\pi\frac{W}{K}\sum_{l=0}^{\infty}(2l+1)T_l^{*(2)}(W^2)T_l^{(2)}(W^2)P_l(\cos\theta)\\ &=\frac{g^4}{32\pi WK^3}\sum_{l=0}^{\infty}(2l+1)Q_l(z_0)Q_l(z_0)P_l(\cos\theta),\end{aligned} \tag{3.3}$$

where

$$z_0=1+(m^2+2\mu^2-W^2)/2K^2.$$

Summing,[5] we get,

$$\mathrm{Im}T^{(4)}(W^2,\Delta^2)=\frac{g^4}{64\pi WK^3N(z_0,\cos\theta)}\times\ln\left[\frac{\cos\theta-z_0^2+N(z_0,\cos\theta)}{\cos\theta-z_0^2+N(z_0,\cos\theta)}\right], \tag{3.4}$$

where

$$N(z_0,\cos\theta)=\{\cos^2\theta+2z_0^2-1-2z_0^2\cos\theta\}^{\frac{1}{2}}.$$

This expression has been calculated using different methods by Mandelstam,[3] who discusses the analytic properties in $\cos\theta$. The logarithmic singularities of the Legendre functions $Q_l(z_0)$ at $z_0=\pm1$ give rise to logarithmic branch points of $T^{(4)}(y_{13},y_{24})$ in the unphysical sheet at $y_{24}=-\mu/2m$ and $y_{24}=(\mu/2m)(3-\mu^2/m^2)$, agreeing with the $L_1^{\pm}$ and $L_3^{\pm}$ of Tarski (Fig. 2).

The right-hand side of (3.4) is analytic in $\cos\theta$, except for a cut running from the value z_1 to ∞, where

$$z_1=2z_0^2-1. \tag{3.5}$$

The jump across this cut for real $W^2>(m+\mu)^2$ is just the spectral function appearing in the Mandelstam representation for the fourth order amplitude. This singularity corresponds to one of the "multiplied" singularities appearing in the general case of Sec. 2 arising from

[2] S. Bochner and W. T. Martin, *Several Complex Variables* (Princeton University Press, Princeton, New Jersey, 1948), p. 140.
[3] S. Mandelstam, Phys. Rev. **115**, 1741 and 1752 (1959).
[4] J. Tarski, J. Math. Phys. **1**, 154 (1960).
[4a] *Note added in proof.* This has now been extended to all orders of perturbation theory in preprints of papers by R. J. Eden and by P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor.

[5] P. Henrici, J. Ration. Mech. Analysis 4, 983 (1955), Eq. 98.

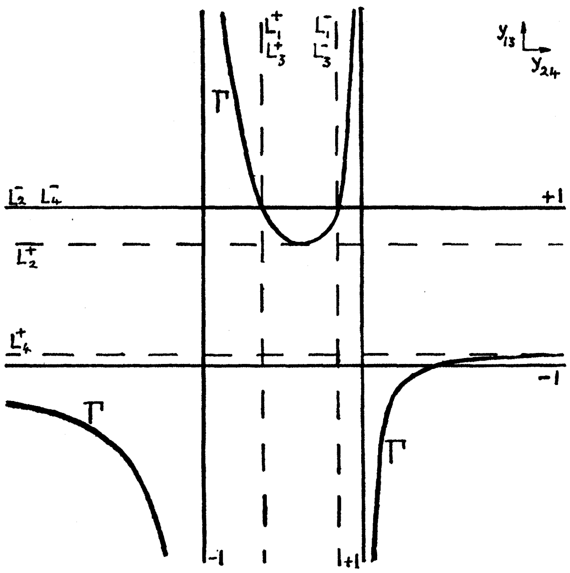

FIG. 2. Spectrum of possible singularities of fourth order amplitude of Fig. 1. The variables are defined by $y_{24}=(m^2+\mu^2-W^2)/2m\mu$ and $y_{13}=1-(\Delta^2/2m^2)$. The two branches of Γ are joined by a surface of singularities for complex values of y_{13}, y_{24}.

the single-particle pole at $\overline{W}^2=(p-k')^2=m^2$, which is the source of the value $\cos\theta=z_0$ in (3.3). The curve Γ in Fig. 2 is determined also by (3.5).

We see that the source of all the singularities appearing in unphysical sheets of the fourth order graphs, apart from the usual physical cuts, can be traced back to the singularities of the second order amplitudes on the physical sheet. An analogous statement will hold for all the ladder diagrams of Fig. 1, in that if we have the Mandelstam representation holding on the physical sheets,[6] then the extra singularities on the first unphysical sheet arise from singularities on physical sheets of the lower order amplitudes obtained by cutting the graph transversely across any pair of internal lines. There are two types of singularity arising in this way. Singularities of the first type depend only on W^2 and appear in the separate partial wave amplitudes of the lower order diagrams, whereas those of the second type appear only on summation over all partial waves. The singularity spectrum produced by the totality of these graphs is then identical with that obtained independently of perturbation theory in Sec. 2. This now includes the unstable particle poles obtained in UP, as the arguments presented there apply unchanged to the sum of the ladder graph contributions.

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[6] This has been proved in the single-meson approximation for all the ladder diagrams by Mandelstam in (3).

APPENDIX

The theorem for "multiplication of singularities" of Legendre series is an analog of Hadamard's classic theorem for power series expansions.[7] It may be expressed in the following form:

Theorem 1. Let $a(z)$ and $b(z)$ be two functions regular on $[-1,+1]$ and such that the Legendre expansions,

$$a(z)=\sum_{n=0}^{\infty}a_nP_n(z),\quad b(z)=\sum_{n=0}^{\infty}b_nP_n(z),\tag{A.1}$$

converge inside the largest ellipses E_a and E_b with foci ±1, inside which the functions are analytic. Let $\{\alpha\}$ and $\{\beta\}$ denote the sets of singular and external points of $a(z)$ and $b(z)$. Then the function

$$f(z)=\sum_{n=0}^{\infty}a_nb_nP_n(z)\tag{A.2}$$

is regular on $[-1,+1]$ and has an analytic continuation along any finite path which does not pass through any point of the set $\{\alpha\beta+(\alpha^2-1)^{\frac{1}{2}}(\beta^2-1)^{\frac{1}{2}}\}$ and does not return to $[-1,+1]$. Stated otherwise, the singularities and external points of $f(z)$ are to be found among the points $\{\alpha\beta+(\alpha^2-1)^{\frac{1}{2}}(\beta^2-1)^{\frac{1}{2}}\}$, where initially the principal values of the roots are taken.

The proof is based on the following lemma:

Lemma. The singularities and external points of $A(h)=\sum_{n=0}^{\infty}a_nh^n$ are the points $\{\alpha+(\alpha^2-1)^{\frac{1}{2}}\}$, under the conditions of the above theorem.

The Cauchy-Hadamard formula

$$\liminf_{n\to\infty}|a_n|^{-1/n}=R_a>1,\tag{A.3}$$

where R_a is the sum of the semiaxes of the ellipse E_a, implies that the radius of convergence of the above expansion of $A(h)$ is just R_a. We set

$$A(z,h)=\sum_{n=0}^{\infty}a_nh^nP_n(z),\quad \begin{array}{l}A(1,h)=A(h),\\ A(z,1)=a(z),\end{array}\tag{A.4}$$

which converges uniformly in h for $|h|<R_a|z+(z^2-1)^{\frac{1}{2}}|^{-1}-\epsilon$.

The coefficients a_n are determined by

$$a_n=\frac{2n+1}{2}\int_L P_n(z)a(z)dz,$$

where L is an arc connecting $z=-1$ and $z=+1$ and lying initially inside the ellipse E_a. So

$$A(h)=\frac{1}{2}\int_L dz\,a(z)\sum_{n=0}^{\infty}(2n+1)h^nP_n(z),$$

on interchanging the summation and integration as

[7] See, for example, E. C. Titchmarsh, *Theory of Functions* (Oxford University Press, New York, 1939), p. 157.

permitted by the uniform convergence of the sum in h and z for small enough h. This gives

$$A(h)=\frac{1}{2}\int_L dz\, a(z)\frac{1-h^2}{(1-2zh+h^2)^{\frac{3}{2}}}, \tag{A.5}$$

where we choose L so as to avoid singularities arising from the denominator. Using standard techniques of continuation,[8] we see that $A(h)$ can be continued along any finite path starting from $h=0$ and not passing through any of the points $\{\alpha+(\alpha^2-1)^{\frac{1}{2}}\}$, or $h=1$, or returning to the origin. The initial coincidence of singularities for $h=\{\alpha-(\alpha^2-1)^{\frac{1}{2}}\}$ is harmless as the arc L is not pinched. It is also easy to see that $h=1$ is not actually a singularity of $A(h)$, as the radius of convergence of $\sum a_n h^n$ is $R_a>1$.

The main theorem follows from the above lemma if we express $f(z)$ in the form

$$f(z)=\frac{1}{(2\pi i)^2}\int_C dh\int_{C'} dh'\frac{A(h)B(h')}{[(hh')^2-2zhh'+1]^{\frac{1}{2}}}, \tag{A.6}$$

which follows on substituting the expressions

$$a_n=\frac{1}{2\pi i}\int_C \frac{dh}{h^{(n+1)}}A(h),$$

and likewise for b_n, into (A.2) and summing over n. C is a circle centered on the origin and of radius any value in between $R_a-\epsilon$ and $1+\epsilon$ ($\epsilon>0$), if we keep z initially on the interval $[-1,+1]$. C' is defined in a similar manner. The integration in (A.6) is over the distinguished surface $C\times C'$ of a circular bicylinder. This can always be deformed into a general bicylinder when we attempt to continue in z, so as to avoid singularities of the integrand, unless it is pinched by a triple coincidence of the singularities

$$h=\alpha+(\alpha^2-1)^{\frac{1}{2}},\quad h'=\beta+(\beta^2-1)^{\frac{1}{2}},\quad (hh')^2-2zhh'+1=0,$$

which gives

$$z=\alpha\beta+(\alpha^2-1)^{\frac{1}{2}}(\beta^2-1)^{\frac{1}{2}}. \tag{A.7}$$

(A.6) thus gives the continuation of $f(z)$ required for the theorem. The singularities of $[(hh')^2-2zhh'+1]^{-\frac{1}{2}}$ are in general branch points at $hh'=z\pm(z^2-1)^{\frac{1}{2}}$ joined by a cut. This cut has usually to be deformed when continuing in z and gives no trouble unless we attempt to continue back to $z=\pm1$ on another sheet of the function, as the ends of the cut coincide to form a simple pole at these points. Thus we cannot exclude the possibility of singularities at $z=\pm1$ on other sheets of its Riemann surface, as well as those of the type arising from $h=\alpha-(\alpha^2-1)^{\frac{1}{2}}$.

[8] These are fully described in (4) and by J. C. Polkinghorne and G. R. Screaton, Nuovo cimento **15**, 289 (1960).

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Structure of the S Matrix in the Presence of a Bound State*

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It is shown that the phase factor associated with the "orthogonality phase shift" due to a bound state should be factored out of the S matrix. A crucial test of this statement is found in a study of the final state interaction of an inelastic process which ends in a channel involving the bound state. If we assume that the sum of the Born series for the S matrix gives a right answer after we separate the effect of the bound state in terms of the orthogonality phase shift, an agreement with Watson's result obtains only when the S matrix has the factored structure.

1. INTRODUCTION

RECENTLY it has been stated by Nishijima,[1] Zimmermann,[2] and Haag[3] that there is no difference between a composite particle and an elementary particle as far as the theory of scattering is concerned. This is true to the extent that it is possible to have an initial or a final state in which the composite particle moves as a single entity at a distance from all other particles. However, it is not quite arbitrary to regard a particle as elementary or as composite, since there is some experimental indication even in scattering when two "elementary" particles form a stable "composite" particle. Thus, a positive scattering length, when an attractive force acts between two colliding "elementary" particles, suggests that there is a bound state, i.e., a "composite" particle formed of the two "elementary" particles, of not too large a binding energy.[4]

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[1] K. Nishijima, Progr. Theoret. Phys. (Kyoto) **17**, 765 (1957); Phys. Rev. **111**, 995 (1958).
[2] W. Zimmermann, Nuovo cimento **10**, 597 (1958).
[3] R. Haag, Phys. Rev. **112**, 669 (1958).
[4] See, for instance, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 68.